THE ART OF MOVES ON GRAPHS AND THE EXTENDED COVARIANT FUNCTORIALITY OF GRAPH ALGEBRAS

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Based on joint work with
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We unravel $*$-homomorphisms between graph C*-algebras, including $*$-isomorphisms given by moves, in terms of morphisms of graphs rendering the construction of graph C*-algebras a covariant or a contravariant functor:

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However, to functorially induce the inverses of these $*$-isomorphisms, we need to extend the category of graphs and path homomorphisms. It turns out that in order to cover all isomorphisms given by moves one would need to work with more general constructions such as (extended) relation morphisms or go even further beyond.

Directed graphs and finite paths

A directed graph $E := (E^0, E^1, s_E, t_E)$ is a quadruple consisting of the set of vertices E^0, the set of edges E^1, and the source and target maps $s_E, t_E \colon E^1 \to E^0$ assigning to each edge its source and target vertex respectively. A vertex is called regular iff it emits at least one and at most finitely many edges. The set of all regular vertices in E is denoted by $\mathrm{reg}(E)$.

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A finite path in E is a vertex or a finite sequence of edges $p_n := (e_1, \ldots, e_n)$ satisfying

$$t_E(e_1) = s_E(e_2), \quad t_E(e_2) = s_E(e_3), \quad \ldots, \quad t_E(e_{n-1}) = s_E(e_n).$$

We denote the set of all finite paths in E by $\mathrm{FP}(E)$. The beginning $s_{PE}(p_n)$ of p_n is $s(e_1)$ and the end $t_{PE}(p_n)$ of p_n is $t_E(e_n)$. The beginning and the end of a vertex is the vertex itself. A path is called a cycle when its beginning coincides with its end. A loop is a cycle of length 1.

Morphisms of graphs

Definition

A morphism of graphs is a map $f\colon \mathrm{FP}(E) \to \mathrm{FP}(F)$. If it satisfies

1. $f(E^0) \subseteq F^0$,
2. $s_{PF} \circ f = f \circ s_{PE}\,, \quad t_{PF} \circ f = f \circ t_{PE}\,,$
3. $\forall\, p, q \in \mathrm{FP}(E)$ s.t. $t_{PE}(p) = s_{PE}(q)\colon f(pq) = f(p)f(q)$,

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Additionally, we say that two graph homomorphisms $f\colon F \to E, f'\colon F' \to E$ are isomorphic over E if there exists an isomorphism $\alpha\colon F \to F'$ such that $f = f' \circ \alpha$.

Standard graph homomorphisms

We start the story with (contravariantly) admissible graph homomorphisms, with out-split being the example of our biggest interest. It turns out that we can classify these homomorphisms nicely and state when they come from out-splits up to an isomorphism of graph homomorphisms.

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Let $\pi\colon E_\pi \to E$ be a graph homomorphism. We say that π is a standard graph homomorphism if the following conditions hold:

1. The set $E^1_\pi = E^1 \times_{E^0} E^0_\pi$ is the set-theoretic pullback of $E^1 \xrightarrow{t_E} E^0 \xleftarrow{\pi_0} E^0_\pi$.
2. The map $\pi_1 = \mathrm{pr}_1\colon E^1_\pi \to E^1$ is the canonical projection onto the first factor.
3. The map $t_\pi\colon E^1_\pi \to E^0_\pi$ is the canonical projection onto the second factor.

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1. Proper, i.e. both f_0 and f_1 are finite-to-one.
2. Target bijective, i.e. for $e \in F^1$ the restricted target map
$$t_E\colon f^{-1}(e) \to f^{-1}(t_F(e))$$
is bijective.
3. Regular, i.e. $f^{-1}(\mathrm{reg}(F)) \subseteq \mathrm{reg}(E)$.

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Theorem (Hajac, Katsura, Tobolski)

There is a contravariant functor from the category of graphs and admissible graph homomorphisms to the category of $U(1)$-C-algebras and $U(1)$-equivariant $*$-homomorphisms. This functor maps a graph E to its graph C*-algebra and a map f to the pullback f^*.*

A standard graph homomorphism $\pi\colon E_\pi \to E$ is, by definition, target bijective. Thus admissible graph homomorphisms are essentially special cases of standard graph homomorphisms, up to an isomorphism of graph homomorphisms. So the question is, which standard graph homomorphisms are admissible. This is answered in the following proposition:

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Proposition

Let $\pi\colon E_\pi \to E$ be a standard graph homomorphism. Then it is admissible if and only if π_0 is proper and it satisfies

$$\pi_0^{-1}(\mathrm{reg}(E)) \subseteq s_\pi(E_\pi^1).$$

Moreover, any admissible graph homomorphism $f\colon F \to E$ is isomorphic to (at least) one such $\pi\colon E_\pi \to E$ over E.

The out-split move

Definition (Bates, Maszczyk, Pask)

An out-split graph homomorphism into E is a graph homomorphism that is isomorphic over E to a surjective standard graph homomorphism $\pi\colon E_\pi \to E$ such that for all $v \in E^0$ we have

$$|\pi_0^{-1}(v) \cap (E_\pi^0 \setminus \mathrm{reg}(E_\pi))| \leq 1$$

and such that there exists $s'\colon E^1 \to E_\pi^0$ rendering the following diagram commutative:

$$\begin{tikzcd} E_\pi^1 \arrow[r, "s_\pi"] \arrow[d, "\mathrm{pr}_1"] & E_\pi^0 \arrow[d, "\pi_0"] \\ E^1 \arrow[r, "s_E"] \arrow[ur, "s'"] & E^0 \end{tikzcd}$$

Some comments about this definition:

1. If π_0 has trivial fibres outside of a single vertex w, then we recover the definition of the usual out-split of E at w. In that case the out-split is maximal if and only if $s'|_{s_E^{-1}(w)}$ is injective.

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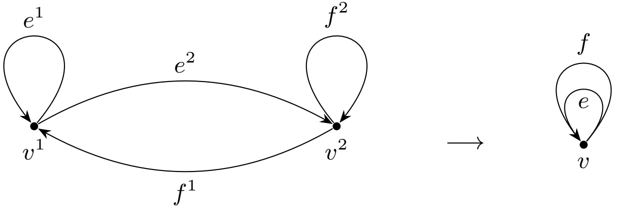
2. On the other hand, if for every $v \in \mathrm{reg}(E)$ the restrictions $s'|_{s_E^{-1}(v)}$ are all injective, then we recover the notion of the total out-split of E.
3. The class of out-split graph homomorphisms is not closed under composition, but instead it enjoys the following cancellation property: Suppose that we have a commutative triangle:

$$\begin{tikzcd} E \arrow[rr, "\rho"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\tau"] \\ & F & \end{tikzcd}$$

where π and τ are out-split graph homomorphisms and ρ is just a surjective graph homomorphism. Then ρ is also an out-split graph homomorphism.

The Cuntz algebra $\mathcal{O}_2$



Here the maximal out-split move $E_\pi \to E$ is given by

$$v^1, v^2 \longmapsto v, \quad e^1, e^2 \longmapsto e, \quad f^1, f^2 \longmapsto f.$$

Note that $e^i := (e, v^i)$, $f^i := (f, v^i)$, $i \in \{1, 2\}$, so

$$t_\pi(e^i) = v^i = t_\pi(f^i), \quad i \in \{1, 2\}.$$

Also, $s'(e) := v^1$ and $s'(f) := v^2$, so

$$s_\pi(e^i) = v^1, \quad s_\pi(f^i) = v^2, \quad i \in \{1, 2\}.$$

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- We have just talked about graph homomorphism that induce a map between graph C*-algebras contravariantly.
- The other side of the story is about path homomorphism that induce a map between graph C*-algebras covariantly.
- These are the admissible path homomorphisms and, more generally, the (regular) extended path homomorphisms defined via graph inverse semigroups.

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2. It is monotonic, i.e. if $f(e)$ extends $f(e')$ for two edges $e, e' \in E$, then $e = e'$.
3. It is regular, i.e. for any $v \in \mathrm{reg}(E)$ that emits at least one edge that is not a loop $f|_{s_E^{-1}(v)}$ is injective, and a positive path p lies in $f(s_E^{-1}(v))$ if and only if all of the below hold
   - If a product pq is in $f(s_E^{-1}(v))$, then q must be a vertex.
   - For a vertex $w \neq t_{PE}(p)$ that p runs through let p' be the subpath of p that starts at the same vertex and ends at w. Then for any $e \in s_E^{-1}(w)$ there exists $r \in \mathrm{FP}(E)$ such that $p'r \in f(s_E^{-1}(v))$.

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4. For any $v \in \operatorname{reg}(E)$ that only emits a loop we require that either the above condition holds or $f(s_E^{-1}(v)) = f(v)$.

Theorem

Let APH *be the category of graphs and admissible path homomorphisms. Then there exists a covariant functor from* APH *to graph C*-algebras that assigns to a map* $\varphi\colon \mathrm{FP}(E) \to \mathrm{FP}(E)$ *the pushforward* $\varphi_*\colon C^*(E) \to C^*(F)$ *defined as*

$$\varphi_*(S_\alpha) = S_{\varphi(\alpha)}.$$

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Let E be a graph. The graph inverse semigroup $S(E)$ is the set

$$S(E) := \{(\alpha, \beta) \in \mathrm{FP}(E) \times \mathrm{FP}(E) \mid t_{PE}(\alpha) = t_{PE}(\beta)\} \cup \{0\}$$

equipped with the multiplication defined by

$$(\alpha, \beta)(\gamma, \delta) := \begin{cases} (\alpha\gamma', \delta) & \text{if } \gamma = \beta\gamma' \\ (\alpha, \delta\beta') & \text{if } \beta = \gamma\beta' \\ 0 & \text{otherwise,} \end{cases}$$

and the requirement that 0 times anything is 0.

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- If $t \leq (\gamma, \gamma)$ for some $t \in S(E)$ and a path γ, then $t = (\alpha, \alpha)$ for a path α that extends γ.

Extended path homomorphisms

Definition

Let E and F be graphs. An extended path homomorphism from E to F is a semigroup homomorphism $\varphi\colon S(E) \to S(F)$ that preserves 0. We say that an extended path homomorphism is regular whenever $u \in \operatorname{reg}(E)$, $t \in S(F) \setminus \{0\}$ and $t \leq \varphi(u,u)$ imply that there exists $e \in s_E^{-1}(u)$ such that $t\varphi(e,e) \neq 0$.

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Lemma

Directed graphs and regular extended path homomorphisms form a category, which we denote by REG. *The category* APH *of directed graphs and admissible path homomorphisms is a subcategory of* REG.

Extending the covariant functor

Theorem

Let $\varphi\colon S(E) \to S(F)$ be a regular extended path homomorphism. Then the formula

$$\boxed{\forall\, p \in E^0 \cup E^1 \colon \varphi_*(S_p) := S_\alpha S_\beta^*,\ (\alpha, \beta) := \varphi(p, t_E(p))},$$

defines a $*$-homomorphism $\varphi_*\colon C^*(E) \to C^*(F)$, and $\varphi \mapsto \varphi_*$ yields a covariant functor from REG *to* CAL *extending the covariant functor from* APH *to* CAL.

Unital reduction

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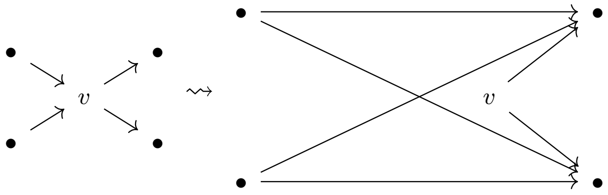

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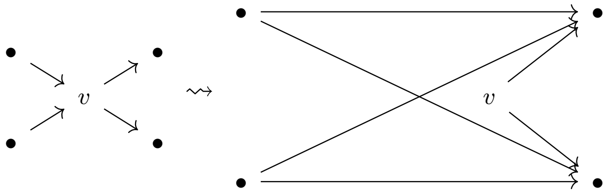

We propose a generalization of this move to the following construction:

Our initial data consists of:

- A regular vertex $v \in \mathrm{reg}(E)$, which can support a loop.
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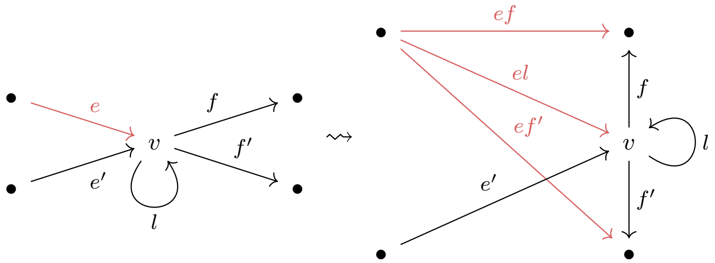

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- We obtain an admissible path homomorphism from the unital reduction of E into E that is still an isomorphism after passing to graph C*-algebras.

There is a closely-related notion of a reduction of a graph E at a suitable vertex v, which can also be generalized in a similar manner. Interestingly, our construction yields an admissible path homomorphism that is not a Morita equivalence in general (in the case of the classical reduction move it is a Morita equivalence).

Shift move

The classical shift move is defined using the following input data:

- Two vertices $v \in \mathrm{reg}(E)$ and $w \in E^0$.
- A commutative diagram:

$$\begin{tikzcd} s_E^{-1}(v) \arrow[rr, hook, "\theta"] \arrow[dr, "t_E"'] & & s_E^{-1}(w) \arrow[dl, "t_E"] \\ & E^0 & \end{tikzcd}$$

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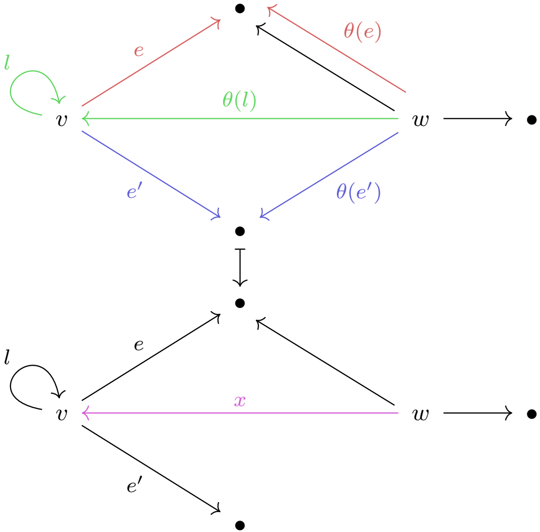
This yields an admissible path homomorphism from E to the shifted graph that becomes an isomorphism after passing to C*-algebras.



l
e
$\theta(e)$
$\theta(l)$
v
w
e'
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l
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x
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 & & \bullet & & & & \bullet & & & & \bullet \\
w \arrow[r] & v \arrow[ur] \arrow[dr] & & \rightsquigarrow & w \arrow[urr] \arrow[drr] & v \arrow[ur] \arrow[dr] & & \rightsquigarrow & w \arrow[r] & v \arrow[ur] \arrow[dr] & \\
 & & \bullet & & & & \bullet & & & & \bullet
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$$

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$$\begin{tikzcd}
 & & \bullet & & & & \bullet & & & & \bullet \\
w \arrow[r] & v \arrow[ur] \arrow[dr] & & \rightsquigarrow & w \arrow[urr] \arrow[drr] & v \arrow[ur] \arrow[dr] & & \rightsquigarrow & w \arrow[r] & v \arrow[ur] \arrow[dr] & \\
 & & \bullet & & & & \bullet & & & & \bullet
\end{tikzcd}$$

In order to accommodate for all shifts, one can arrive at a generalization for the unital reduction. Our generalization is even still more general.

Generalizing the shift move

Motivation: Classical shift and unital reduction moves sometimes undo each other:

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We define the generalized shift moves as the moves that undo (generalized) unital reductions.

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Definition

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Of course, these new shift moves also enjoy an explicit construction.

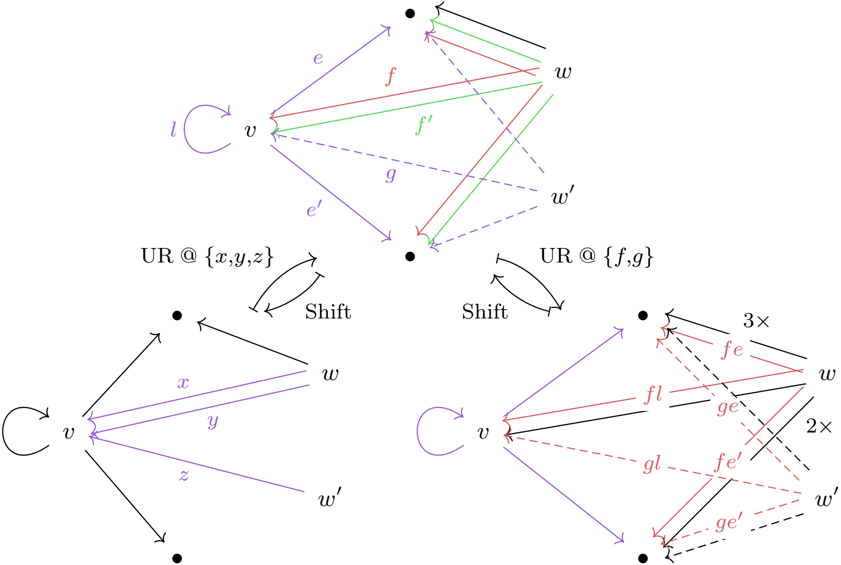
The new shift move of a graph E requires the following input data:

- A vertex $v \in \operatorname{reg}(E)$ and a set of vertices $W \subseteq E^0 \setminus \{v\}$.
- For each $w \in W$ a collection of commutative diagrams:

$$\begin{tikzcd} s_E^{-1}(v) \arrow[rr, hook, bend left, "\{\theta_w^i\}_{i\in I_w}"] \arrow[rr, hook, bend right] \arrow[rr, phantom, "\vdots"] \arrow[dr, "t_E"'] & & s_E^{-1}(w) \arrow[dl, "t_E"] \\ & E^0 & \end{tikzcd}$$

  We additionally require that the images of θ_w^i are pair-wise disjoint for different i's.

The construction is the same as for the classical shift, but we apply shifts for every instance of θ_w^i, for every i and w.



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UR @ $\{f,g\}$
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Shift
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v
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Realizing isomorphisms: Out-split

Let $f\colon E_O \to E$ be an out-split graph homomorphism, i.e. we have two squares with the left one being a pullback diagram:

$$\begin{tikzcd}
E^0_\pi & E^1_\pi \arrow[l, "\mathrm{pr}_2"'] \arrow[r, "s_\pi"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dl, phantom, "\llcorner", very near start] & E^0_\pi \arrow[d, "\pi_0"'] \\
E^0 \arrow[from=u, "\pi_0"'] & E^1 \arrow[l, "t_E"'] \arrow[r, "s_E"] \arrow[ur, "s'"] & E^0
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It induces a $*$-isomorphism $f^*\colon C^*(E) \to C^*(E_O)$ contravariantly. It was already known what $(f^*)^{-1}$ is on the C*-level:

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It induces a $*$-isomorphism $f^*\colon C^*(E) \to C^*(E_O)$ contravariantly. It was already known what $(f^*)^{-1}$ is on the C*-level: Let $w \in E_O^0, v = \pi_0(w)$, $(\epsilon, u) \in E_O^1$ and we have:

$$(f^*)^{-1}(S_w) = \begin{cases} S_v, & (s')^{-1}(w) = \emptyset \\ \sum_{e \in (s')^{-1}(w)} S_e S_e^*, & |(s')^{-1}(w)| < \infty \\ S_v - \sum_{e \in s_E^{-1}(v) \setminus (s')^{-1}(w)} S_e S_e^*, & |(s')^{-1}(w)| = \infty \end{cases}$$

$$(f^*)^{-1}(S_{(\epsilon,u)}) = S_\epsilon \cdot (f^*)^{-1}(S_u).$$

Theorem

Let E be a graph, $X \subseteq \mathrm{reg}(E)$ and $E_{\widehat{\mathcal{O}}}$ be the maximal out-split of E at X. Then the out-split graph homomorphism $f\colon E_{\widehat{\mathcal{O}}} \to E$ induces a $*$-isomorphism $f^*\colon C^*(E) \to C^*(E_{\widehat{\mathcal{O}}})$ and its inverse (in the category of C-algebras) is given by a covariantly induced map $(f^*)^{-1} = \varphi_*$ for a regular extended path homomorphism $\varphi\colon E_{\widehat{\mathcal{O}}} \to E$.*

Theorem

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Corollary

Let E be a graph, E_O be an out-split at a subset $X \subseteq \operatorname{reg}(E)$ and $E_{\widehat{O}}$ be the maximal out-split at the same subset X. If $f\colon E_O \to E$ is the out-split homomorphism, then $(f^*)^{-1}$ can be realized as:

$$(f^*)^{-1} = \varphi_* \circ g^*,$$

where $\varphi\colon E_{\widehat{O}} \to E$ is a certain regular extended path homomorphism and $g\colon E_{\widehat{O}} \to E_O$ is the out-split homomorphism.

Realizing isomorphisms: Shift

Let E be a graph, $v \in \mathrm{reg}(E)$, $W \subseteq E^0 \setminus \{v\}$ and $\{\theta_w^i \colon s_E^{-1}(v) \to s_E^{-1}(w)\}_{w \in W, i \in I_w}$ be a collection of injections defined over E so that we can form the shifted graph $E_{(v,W)}$.

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There is a canonical admissible path homomorphism $f \colon E \to E_{(v,W)}$ that maps

$$f(v) = v, \qquad f(e) = \begin{cases} e & e \in E^1 \setminus \bigcup_{w,i} \mathrm{im}\, \theta_w^i, \\ x_{w,v}^i (\theta_w^i)^{-1}(e) & e \in \mathrm{im}\, \theta_w^i \end{cases}$$

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It is classically known that this is an isomorphism after passing to C*-algebras. Its inverse is given by

$$(f_*)^{-1}(S_v) = S_v, \quad (f_*)^{-1}(e) = \begin{cases} S_e, & e \notin \{x_{w,v}^i\}_{w,i} \\ \sum_{\epsilon \in s_E^{-1}(v)} S_{\theta_w^i(\epsilon)} S_\epsilon^*, & e = x_{w,v}^i. \end{cases}$$

Theorem

Let E be a graph and $E_{(v,W)}$ be a graph obtained from E via a shift move at $W \subseteq E^0 \setminus \{v\}$. Under the hypothesis that $s_E^{-1}(v)$ is a single edge that is not a loop, then the inverse $(f_*)^{-1} = \varphi_*$ of the $*$-isomorphism induced by $f\colon E \to E_{(v,W)}$ is again covariantly induced, by a certain regular extended graph homomorphism φ.

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Corollary

Let E be a graph and $E_{(v,W)}$ be a graph obtained from E via a shift move at $W \subseteq E^0 \setminus \{v\}$. Under the hypothesis that $v \in \operatorname{reg}(E)$ does not emit a loop, then the inverse of the covariantly induced $*$-isomorphism from the shift map $f\colon E \to E_{(v,W)}$ decomposes as

$$(f_*)^{-1} = \varphi_* \circ g^*,$$

where g is an out-split graph homomorphism and φ is a certain regular extended graph homomorphism.

Limitations and future work

- It is clear that the notion of regular extended graph homomorphism lets us find inverses of isomorphisms given by moves only in very special cases (maximal out-splits at regular vertices, shifts to vertices that emit only one edge).

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Limitations and future work

- It is clear that the notion of regular extended graph homomorphism lets us find inverses of isomorphisms given by moves only in very special cases (maximal out-splits at regular vertices, shifts to vertices that emit only one edge).
- One remedy is to use both the covariant and contravariant functor, which fits nicely in the framework of (extended) relation morphisms.
- This is still not enough to describe inverses of, e.g. out-splits at infinite emitters. Our next hope is that the formalism of groupoids (cf. Gilles' talk) may be enough for all isomorphisms given by moves.

Thank you for your attention!

Merci pour votre attention!