Spectral Dimension of a Cantor Set

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Outline

Motivation and Objectives

- Noncommutative metric geometry of fractals.
- Spectral triples for Cantor-type sets.
- Dimension spectrum and residues.
- Spectral metric and spectral torsion functionals.

Main questions:

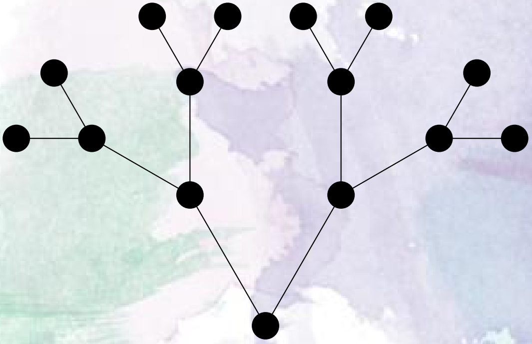
What are the spectral properties of some objects, which are commutative but not described by the Riemannian differential geometry ?
What meaning have spectral functionals for objects of non-integer dimension?

Weighted Rooted Trees

Definition

Let T be a rooted tree, by $\mathcal{V}$ we denote the vertices. A *level* of a vertex v, denoted $|v|$, is the distance from the root (length of path from the root).
For $0 < \rho < 1$, we define:

$$\omega(v) = \rho^{|v|}.$$



The ultrametric

Definition

The boundary of the tree, ∂T, is the space of infinite paths beginning at the root. For two paths $\eta, \kappa \in \partial T$ we define $\eta \wedge \kappa$ as the vertex at which these paths separate.

Lemma

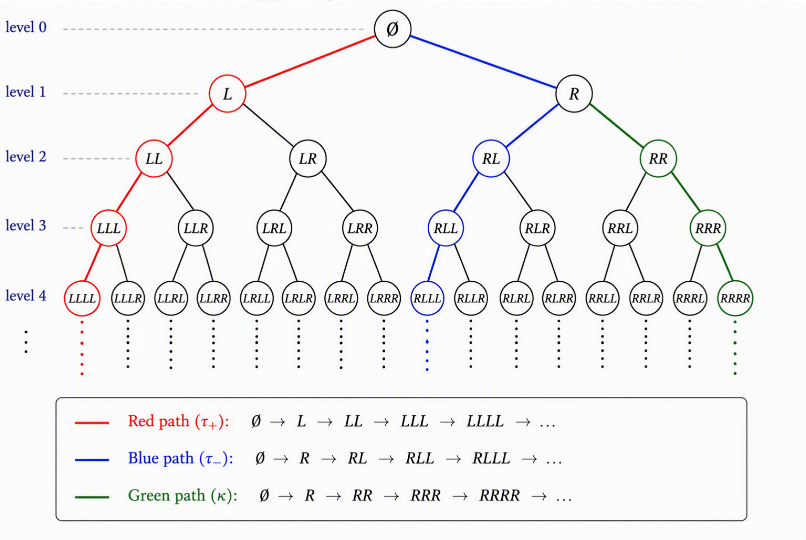
The space ∂T has an **ultrametric** *defined as follows:*

$$d(\eta, \kappa) = \omega(\eta \wedge \kappa).$$

The strong triangle inequality holds, $\eta, \kappa, \lambda \in \partial T$:

$$d(\eta, \lambda) \leq \max\{d(\eta, \kappa),\, d(\lambda, \kappa)\}$$

For the binary tree, ∂T, is an ultrametric Cantor set.


level 0
level 1
level 2
level 3
level 4
Ø
L
R
LL
LR
RL
RR
LLL
LLR
LRL
LRR
RLL
RLR
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LLLL
LLLR
LLRL
LLRR
LRLL
LRLR
LRRL
LRRR
RLLL
RLLR
RLRL
RLRR
RRLL
RRLR
RRRL
RRRR
Red path (τ_+): $\emptyset \to L \to LL \to LLL \to LLLL \to \dots$
Blue path (τ_-): $\emptyset \to R \to RL \to RLL \to RLLL \to \dots$
Green path (κ): $\emptyset \to R \to RR \to RRR \to RRRR \to \dots$

Choice Functions

Definition

Choice functions: a pair of functions, (τ_+, τ_-) such that

$$\tau_\pm : \mathscr{V} \to \partial T,$$

and

$$v \in \tau_\pm \pm (v), \qquad d(\tau_+(v), \tau_-(v)) = \omega(v).$$

A typical choice function for the binary tree is for a vertex v such that it has path P from the root:

$$\tau_+(v) = PLLL \cdots , \qquad \tau_-(v) = PRRR \cdots .$$

Hilbert Space and Representation

We define (after Pearson & Bellisard: *Noncommutative Riemannian Geometry and Diffusion on Ultrametric Cantor Sets*):

$$\mathcal{H} = \ell^2(\mathcal{V}) \otimes \mathbb{C}^2,$$

and the following representation of the algebra of functions on ∂T:

$$\pi_\tau(f)\phi(v) = \mathrm{diag}\big(f(\tau_+(v)), f(\tau_-(v))\big)\phi(v).$$

We define $C_{Lip}(\partial T)$ to be the algebra of **Lipschitz** functions with respect to the ultrametric:

$$f \in C_{Lip}(\partial T) \Leftrightarrow \exists_K : |f(\kappa) - f(\eta)| \leq K d(\kappa, \eta).$$

Definition

We define the following unbounded operator on $\mathscr{H}$:

$$(D\phi)(v) = \omega(v)^{-1}\sigma_1\phi(v).$$

where

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

on the domain:

$$\sum_v \omega(v)^{-2}\|\phi(v)\|^2 < \infty.$$

Bounded commutators:

$$([D, \pi_\tau(f)]\phi)(v) = \frac{f(\tau_+(v)) - f(\tau_-(v))}{\omega(v)} \sigma_1 \phi(v).$$

for all $f \in C_{Lip}(\partial T)$.

$\mathbb{Z}_2$**-grading** :

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Regularity:

$$[D^2, f] = [|D|, f] = 0.$$

Finite Summability and zeta function

The spectral triple is finitely summable provided that the tree satisfies certain conditions: the number of children is bounded.
We concentrate here on the **binary tree**:

Theorem

For the binary tree the Dirac operator is s_0 summable, where

$$s_0 = -\frac{\ln 2}{\ln \rho},$$

and the zeta function of D is:

$$\zeta_D(s) = \frac{1}{1 - 2\rho^s}.$$

Spectral dimension

For the binary tree, there are 2^n vertices at level n and $\omega(v) = \rho^n$.

Proof:

$$\zeta_D(s) = \frac{1}{2}\mathrm{Tr}(|D|^{-s}) = \sum_{v \in \mathscr{V}} \omega(v)^s = \sum_{n=0}^{\infty} 2^n \rho^{ns} = \sum_{n=0}^{\infty} (2\rho^s)^n.$$

Theorem

The dimension spectrum is

$$Sd(D) = \left\{ s_0 + \frac{2\pi i k}{\ln \rho} : k \in \mathbb{Z} \right\},$$

where $s_0 = -\frac{\ln 2}{\ln \rho}$, *every pole is simple.*

Spectral functionals

Definition

The dimension spectrum is, however, defined as a set of poles of meromorphic extensions of all zeta functions:

$$Tr(b|D|^{-z}),$$

for all $b \in C_{Lip}(\partial T)$.

Moreover, one can define further zeta functions using differential forms:

$$\zeta_{g(u,v)}(z) = \mathrm{Tr}(uv|D|^{-z}).$$

for $u, v \in \Omega^1_D$.

Psedodifferential calculus and Wodzicki residue

Connes-Moscovici

For regular, finitely summable spectral triples one can define a generalization of pseudodifferential operators and a functional, which is a trace on this algebra (noncommutative residue).

Then using Wodzicki residue one can recover geometric information, for example for a manifold of dimension n and a function b:

$$\mathrm{Wres}(b|D|^{-n}) = \int b\sqrt{g}, \qquad \mathrm{Wres}(bD^2|D|^{-n}) = \int bR(g)\sqrt{g},$$

whereas using differential forms u, v, w:

$$\mathrm{Wres}(uv|D|^{-n}) = \int g(u,v)\sqrt{g}, \qquad \mathrm{Wres}(u\{D,v\}D|D|^{-n}) = \int G(u,v)\sqrt{g},$$

$$\mathrm{Wres}(uvwD|D|^{-n}) = \int T(u,v,w)\sqrt{g}.$$

Julien–Savinien Functions

We need to restrict ourselves to some special class of functions.

Definition

Let β be a bounded non-negative function on vertices, $\beta : \mathscr{V} \to \mathbb{R}_+$. Then:

$$F_\beta(\kappa) = \sum_{j=0}^{\infty} \beta(\kappa_j)\rho^j.$$

is a Lipschitz function.

Using different β we shall define three classes of Lipschitz functions.

Classes of Lipschitz Functions

Characteristic functions of a vertex

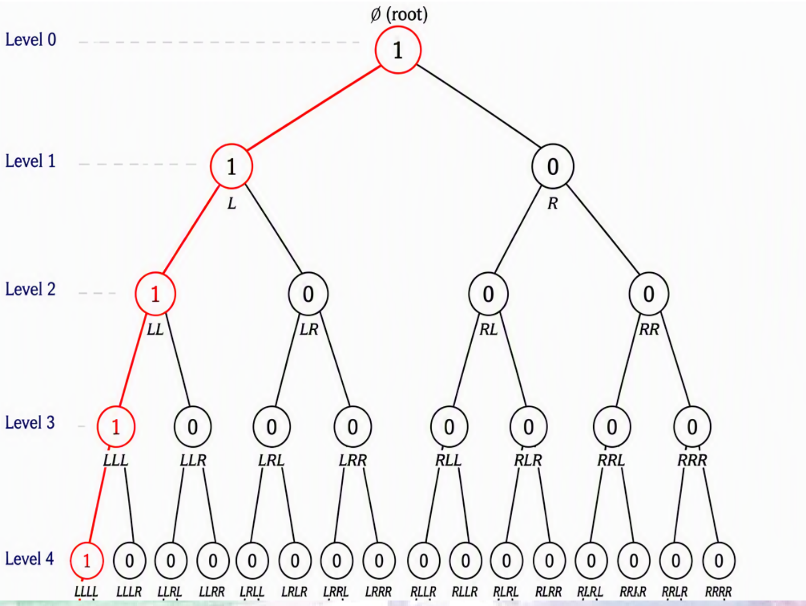
$$\beta_v(w) = \begin{cases} 1, & \exists \kappa : w, v \in \kappa \\ 0, & \text{otherwise} \end{cases}, \qquad F_\beta(\kappa) = \chi_v(\kappa) = \begin{cases} 1, & v \in \kappa \\ 0, & v \notin \kappa \end{cases}$$

These generate a dense algebra.

Ultrametric distance from a chosen path

Let us fix v and a path $\tau_+(v)$.

$$\beta_v(w) = \begin{cases} 1 & w \in \tau_+(v), \\ 0 & \text{otherwise.} \end{cases}, \qquad F_\beta(\kappa) = Cd(\kappa, \tau_+(v)).$$



∅ (root)
Level 0
Level 1
Level 2
Level 3
Level 4
1
1
0
L
R
1
0
0
0
LL
LR
RL
RR
1
0
0
0
0
0
0
0
LLL
LLR
LRL
LRR
RLL
RLR
RRL
RRR
1
0
0
0
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LLLL
LLLR
LLRL
LLRR
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We can also consider:

Regular distance function

Define:

$$\beta(v) = \begin{cases} 0 & v \in \tau_-(v^p), \\ 1 & v \in \tau_+(v^p), \end{cases}$$

where v^p is the direct predecessor (parent) of v. Then we call $F_\beta(\kappa)$ the metric distance.

Remark

All of the above functions generate algebras, which are dense in the continuous functions over ∂T.

Spectral dimension for characteristic functions

Theorem

The dimension spectrum is unchanged by characteristic functions:

$$\zeta_{\chi_v, D}(s) = \rho^{-|v|}\zeta_D(s).$$

Moreover, for each v the commutators $[D, \chi_v]$ are finite rank operators, therefore the respective zeta functions involving differential forms over characteristic functions have no poles.

Remark

The only geometric quantity provided by the spectral triple is volume density.

Spectral Dimension over Ultrametric Distance Functions

Theorem

For $f_p(\kappa) = d(\kappa, 0_t)^{p+1}$, where $p > 0$, poles of $\zeta_{f_p,D}(z)$ occur at

$$s = s_0 + \frac{2\pi i m}{\ln \rho}, \qquad s = -p + \frac{2\pi i m}{\ln \rho}.$$

Moreover if

$$W = \pi(f_{p_0})[D, \pi(f_{p_1})] \cdots [D, \pi(f_{p_k})],$$

then the associated zeta function,

$$\zeta_{W,D}(s) = \frac{1}{2} Tr\left(W|D|^{-s}\right)$$

has poles at $s = -P + \frac{2\pi}{\ln \rho} ik$, $k \in \mathbb{Z}$. where $P = \sum_{j=0}^{k} p_j$.

Theorem

Let H_p be a metric distance function for ρ^p replacing ρ in the definition of the β function. Then, the zeta function $\zeta_{H_p,D}(z)$ has poles at:

$$s = s_0 + \frac{2\pi i m}{\ln \rho}, \qquad s = s_0 - p + \frac{2\pi i m}{\ln \rho}.$$

Moreover, the zeta functional over the Clifford algebra:

$$\zeta_{[D,H_p]^m,D}(z) = \frac{1}{2} Tr\big([D, H_p]^m |D|^{-z}\big),$$

has poles at $z = 1 - pm + \frac{2\pi i m}{\ln \rho}$, $m \in \mathbb{Z}$.

Conclusions and Outlook

- Spectral dimension depends on the choice of algebra and functions.
- For some algebras the spectrum **is continuous** ?
- For characteristic functions almost all spectral functionals vanish.
- s_0 is always in the dimension spectrum
- Does a Dirac operator such that $[|D|, b] \neq 0$ exist?
- Higher-dimensional constructions?
- Links to other approaches to spectral triples on fractals?

Thank You