A groupoid approach to the functoriality of graph algebras

Gilles Gonçalves de Castro
(Joint work with F. D'Andrea, P. M. Hajac, R. Meyer)

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Directed graphs

A (directed) graph E is a quadruple (E^0, E^1, s_E, t_E), where E^0 and E^1 are sets and $s_E : E^1 \to E^0$ and $t_E : E^1 \to E^0$ are functions. We call E^0 the set of vertices, E^1 the set of edges, s_E the source map, t_E the target map. A vertex v is called regular if $s_E^{-1}(v)$ if finite and non-empty. The set of regular vertices of a graph E is denoted by $\mathrm{reg}(E)$.

Paths on graphs

A path in E of length $n \geq 1$ is a sequence $e_1 e_2 \cdots e_n$ of edges such that $t_E(e_i) = s_E(e_{i+1}) \ \forall\, 1 \leq i < n$. We define its source and target by $s_{PE}(e_1 \cdots e_n) := s_E(e_1)$ and $t_{PE}(e_1 \cdots e_n) := t_E(e_n)$. We think of a vertex $v \in E^0$ as a path of length 0, and define its source and target by $s_{PE}(v) = t_{PE}(v) = v$. We denote by $FP(E)$ the set of all finite paths in E. An infinite path is an infinite sequence $e_1 e_2 \cdots$ such that $t_E(e_i) = s_E(e_{i+1}) \ \forall\, 1 \leq i < n$. Its source is $s_{PE}(e_1 e_2 \cdots) := s_E(e_1)$. We denote by $IP(E)$ the set of infinite paths in E.

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If $x, x' \in FP(E)$ satisfy $t_{PE}(x) = s_{PE}(x')$, we denote by xx' their concatenation, and say that x is an initial subpath of xx' (if x resp. x' is a vertex, we put $xx' = x'$ resp. $xx' = x$).

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We define a partial order $\preceq$ by letting $x \preceq x'$ if x is an initial subpath of x'. We write $x \prec x'$ to mean that $x \preceq x'$ and $x \neq x'$. Finally we write $x \sim x'$, and say that x and x' are comparable, if either $x \preceq x'$ or $x' \preceq x$.

Graph Algebras

Fix a graph E and a field K.

Definition

The Leavitt path algebra $L_K(E)$ is the universal K-algebra generated by disjoint sets $\{S_v\}_{v\in E^0}$, $\{S_e\}_{e\in E^1}$ and $\{S_e^*\}_{e\in E^1}$ with relations:

(V) $S_vS_{v'} = \delta_{v,v'}S_v$ for all $v, v' \in E^0$;

(E1) $S_{s_E(e)}S_e = S_e = S_eS_{t_E(e)}$ for all $e \in E^1$;

(E2) $S_{t_E(e)}S_e^* = S_e^* = S_e^*S_{s_E(e)}$ for all $e \in E^1$;

(CK1) $S_f^*S_e = \delta_{e,f}S_{t_E(e)}$ for all $e \in E^1$;

(CK2) $S_v = \sum_{e\in s_E^{-1}(v)} S_eS_e^*$ for every $v \in \mathrm{reg}(E)$.

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The graph C*-algebra of E is the universal C*-algebra of $L_{\mathbb{C}}(E)$ (with the natural involution) and is denoted by $C^*(E)$.

Functoriality of graph algebras

Goal

To define a notion of morphisms between graphs in such a way that the construction of graph C*-algebras and Leavitt path algebras are functorial.

We want the morphisms to be simple enough that we can use the combinatorics of the graphs, but broad enough that they include examples of interest.

Homomorphism of graphs

Definition

A homomorphism of graphs $\varphi : E \to F$ is a pair of maps

$$(\varphi^0 : E^0 \to F^0, \quad \varphi^1 : E^1 \to F^1)$$

such that $s_F \circ \varphi^1 = \varphi^0 \circ s_E$ and $t_F \circ \varphi^1 = \varphi^0 \circ t_E$. The category of graphs and homomorphisms of graphs is denoted by OG.

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Theorem (Katsura, Hajac-Tobolski)

There is a subcategory CRTBPOG of OG such that if $\varphi \in CRTBPOG(E, F)$ there is homomorphism $\varphi^* : L_K(F) \to L_K(E)$ (or $*$-homomorphism $\varphi^* : C^*(F) \to C^*(E)$) such that

$$\varphi^*(P_v) = \sum_{u \in (\varphi^0)^{-1}(v)} P_u, \qquad \varphi^*(S_f) = \sum_{e \in (\varphi^1)^{-1}(f)} S_e$$

for all $v \in F^0$ and $f \in F^1$. Moreover, these associations are functorial.

Path homomorphism of graphs

Definition

A path homomorphism of graphs $\vartheta : E \to F$ is a map $\vartheta : FP(E) \to FP(F)$ such that satisfying

(i) $\vartheta(E^0) \subseteq F^0$,

(ii) $s_{PF} \circ \vartheta = \vartheta \circ s_{PE}, \; t_{PF} \circ \vartheta = \vartheta \circ t_{PE}$,

(iii) $\vartheta(xx') = \vartheta(x)\vartheta(x') \; \forall \; x, x' \in FP(E)$ such that $t_{PE}(x) = s_{PE}(x')$.

We denote the category of graphs and path homomorphisms of graphs by PG.

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Theorem (Hajac-Tobolski)

There is a subcategory RMIPG of PG such that if $\vartheta \in RMIPG(E, F)$ there is unique homomorphism $\vartheta_* : L_K(E) \to L_K(F)$ (or $*$-homomorphism $\vartheta_* : C^*(E) \to C^*(F)$) such that $\vartheta_*(S_x) = S_{\vartheta(x)}$ for all $x \in FP(E)$. Moreover, these association are functorial.

Relation morphism of directed graphs

Question

From the theorems above, we obtain:

- a contravariant functor CRTBPOG $\to$ K-Alg,
- a covariant functor RMIPG $\to$ K-Alg,

where K-Alg is the category of K-algebras with homomorphisms of K-algebras as morphisms. How to unify them?

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Using relations!

Relation morphism of directed graphs

Definition

Let E and F be graphs. A relation morphism $R : E \to F$ is a relation

$$R \subseteq FP(E) \times FP(F)$$

such that

(1) if $(x, y) \in R$ and $y \in F^0$, then $x \in E^0$ (that is $R^{-1}(F^0) \subseteq E^0$);

(2) if $(x, y) \in R$, then $(s_{PE}(x), s_{PF}(y)) \in R$ (source preserving);

(3) if $(x, y) \in R$, then $(t_{PE}(x), t_{PF}(y)) \in R$ (target preserving).

The category of graphs and relation morphisms is denoted by RG.

OG and PG as subcategories of RG

For $\varphi \in OG(E, F)$, we define $R_\varphi \in RG(E, F)$ by

$$R_\varphi = \{(x, \varphi(x)) \mid x \in FP(E)\}.$$

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For $\vartheta \in PG(F, E)$, we define $R^\vartheta \in RG(F, E)$

$$R^\vartheta = \{(\vartheta(y), y) \mid y \in FP(F)\}.$$

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For $\vartheta \in PG(F, E)$, we define $R^\vartheta \in RG(F, E)$

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Proposition (de C.-D'Andrea-Hajac)

There is an embedding of the categories OG *and* $\mathrm{PG}^{\mathrm{op}}$ *in* RG *given as follows. It is the identity on objects. A morphism* $\varphi : FP(E) \to FP(F)$ *in* OG *is mapped to the relation morphism* $R_\varphi : E \to F$*, and a morphism* $\vartheta : FP(F) \to FP(E)$ *in* $\mathrm{PG}^{\mathrm{op}}$ *is mapped to the relation morphism* $R^\vartheta : E \to F$*.*

Admissible relation of graphs

Definition

A relation morphism $R : E \to F$ is said to be admissible if:

(1) it is multiplicative, that is, $(x, y), (x', y') \in R$, $s_{PE}(x') = t_{PE}(x)$, $s_{PF}(y') = t_{PF}(y)$ imply that $(xx', yy') \in R$;

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(2) it is decomposable, that is, $y, y' \in FP(F)$, $s_{PF}(y') = t_{PF}(y)$, $(\overline{x}, yy') \in R$ imply that $\exists\, x \in R^{-1}(y)$, $x' \in R^{-1}(y') : xx' = \overline{x}$;

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(6) it is monotone, that is, $(x, f), (x', f') \in R^1$, $x \preceq x'$ imply $(x, f) = (x', f')$,

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(6) it is monotone, that is, $(x,f),(x',f') \in R^1$, $x \preceq x'$ imply $(x,f) = (x',f')$,

(7) it is regular, that is, $v \in \mathrm{reg}(F)$, $u \in R^{-1}(v)$ and $x \in s_{PE}^{-1}(u)$ imply that there is $(x', f) \in R^1$ such that $s_F(f) = v$ and $x \sim x'$.

Proposition

Admissible relation morphisms define a subcategory, denoted by ARG of RG whose objects are all directed graphs. Moreover,

$$CRTBPOG = OG \cap ARG, \quad \text{and} \quad RMIPG^{op} = PG^{op} \cap ARG.$$

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Theorem (de C.-D'Andrea-Hajac)

There is a contravariant functor from ARG to K-Alg (or C^*-Alg) that sends a graph to its Leavitt path algebra and an admissible relation morphism $R : E \to F$ to the homomorphism $R^* : L_K(F) \to L_K(E)$ ($*$-homomorphism $R^* : C^*(F) \to C^*(E)$) uniquely defined by

$$R^*(S_y) = \sum_{x \in R^{-1}(y)} S_x$$

for all $y \in FP(F)$.

The groupoid picture

A groupoid $\mathcal{G}$ is a small category such that every morphisms is an isomorphism. We can think it as a set with a partially defined multiplication and an inverse satisfying axioms similar to that of groups. We identify an object with its identity morphism, and call it a unit. The set of units is denoted by $\mathcal{G}^0$.

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We assume that $\mathcal{G}$ has a topology and that the operations are continuous with respect to this topology. For the construction of the algebras, we assume that $\mathcal{G}^0$ is a locally compact Hausdorff space.

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We define maps $d, r : \mathcal{G} \to \mathcal{G}^0$ by $d(\gamma) = \gamma^{-1}\gamma$ and $r(\gamma) = \gamma\gamma^{-1}$ for every $\gamma \in G$.

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Given $A \subseteq \mathcal{G}^0$, the reduction of $\mathcal{G}$ by A is the groupoid $\mathcal{G}|_A := \{\gamma \in \mathcal{G} : d(\gamma), r(\gamma) \in A\}$.

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An étale groupoid is a topological groupoid with a basis of open bisection. An ample groupoid is an étale groupoid such that $\mathcal{G}^0$ is a locally compact totally disconnected Hausdorff space.

Steinberg Algebras

Let $\mathcal{G}$ be a Hausdorff ample groupoid and K a field. Given a function $f : \mathcal{G} \to K$, its support is the set

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The Steinberg algebra of $\mathcal{G}$ is the vector space

$$\mathcal{A}_K(\mathcal{G}) := \{f : \mathcal{G} \to K : f \text{ is locally constant and } \mathrm{supp}(f) \text{ is compact}\},$$

with multiplication given by

$$f * g(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta).$$

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(There is a construction where only $\mathcal{G}^0$ is assumed to be Hausdorff.)

Groupoid C*-algebras

Let $\mathcal{G}$ be a Hausdorff étale groupoid. Given a continuous function $f : \mathcal{G} \to \mathbb{C}$, its support is the set

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There is a notion of regular representation and universal representation. The completion with respect to the norms coming from these representations are the reduced and full groupoid C*-algebra denoted by $C_r^*(\mathcal{G})$ and $C^*(\mathcal{G})$ respectively. (There is a construction where only $\mathcal{G}^0$ is assumed to be Hausdorff.)

Groupoid actions

Let $\mathcal{G}$ an étale groupoid X be a topological space. A right action of $\mathcal{G}$ on X is a pair of maps $d : X \to \mathcal{G}^0$, called the anchor map, and $\cdot : X \times_{d,r} \mathcal{G} \to X$, where $X \times_{d,r} \mathcal{G} := \{(x, \gamma) \in X \times \mathcal{G} : d(x) = r(\gamma)\}$, satisfying

(i) $d(x \cdot \gamma) = d(\gamma)$ for all $(x, \gamma) \in X \times_{d,r} \mathcal{G}$,

(ii) $x \cdot (\gamma_1\gamma_2) = (x \cdot \gamma_1) \cdot \gamma_2$ for all $(x, \gamma_1) \in X \times_{d,r} \mathcal{G}$ and all $\gamma_2 \in \mathcal{G}$ such that $d(\gamma_1) = r(\gamma_2)$,

(iii) $x \cdot d(x) = x$ for all $x \in X$.

The orbit space $X/\mathcal{G}$ is the quotient of X by the equivalence relation saying that $x \sim y$ if there exists $\gamma \in \mathcal{G}$ such that $d(x) = r(\gamma)$ and $x \cdot \gamma = y$ for $x, y \in X$.

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(ii) $x \cdot (\gamma_1 \gamma_2) = (x \cdot \gamma_1) \cdot \gamma_2$ for all $(x, \gamma_1) \in X \times_{d,r} \mathcal{G}$ and all $\gamma_2 \in \mathcal{G}$ such that $d(\gamma_1) = r(\gamma_2)$,

(iii) $x \cdot d(x) = x$ for all $x \in X$.

The orbit space $X/\mathcal{G}$ is the quotient of X by the equivalence relation saying that $x \sim y$ if there exists $\gamma \in \mathcal{G}$ such that $d(x) = r(\gamma)$ and $x \cdot \gamma = y$ for $x, y \in X$.

Left actions are defined similarly.

Groupoid actions

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Left actions are defined similarly.

A groupoid acts on itself on the left and on the right by translation. In both cases the orbit space is homeomorphic to the unit space.

Groupoid actors

Let $\mathcal{G}$ and $\mathcal{H}$ be étale groupoids. An actor $\mathcal{G} \to \mathcal{H}$ is a left action of $\mathcal{G}$ on $\mathcal{H}$ that commutes with the right translation action on $\mathcal{H}$. The actor is called proper if the map $\mathcal{H}/\mathcal{H} \cong \mathcal{H}^0 \to \mathcal{G}^0$ induced by the anchor map $r : \mathcal{H} \to \mathcal{G}^0$ is proper.

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Proposition (de C.-Meyer)

Given actors $\mathcal{G} \to \mathcal{H}$ and $\mathcal{H} \to \mathcal{K}$, there exists a unique left action of $\mathcal{G}$ on $\mathcal{K}$ such that $(g \cdot h) \cdot k = g \cdot (h \cdot k)$ for all $g \in \mathcal{G}$, $h \in \mathcal{H}$ and $k \in \mathcal{K}$ such that $d(g) = r(h)$ and $d(h) = r(k)$. Moreover, this action commutes with the right translation action on K.

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The above proposition gives the composition of actors. The identity actor is given by the left translation action of a groupoid on itself. This way, we obtain a category whose objects are étale groupoids and morphisms are actors.

Remark

The notion of actor goes back to the work of Zakrzewski and Buneci-Strachura.

Groupoid actors

Proposition (Buneci-Strachura, de C.-Meyer)

*An actor $\mathcal{G} \to \mathcal{H}$ induces a nondegenerate *-homomorphism from $C^*(\mathcal{G})$ to the multiplier algebra of $C^*(\mathcal{H})$. This takes values in $C^*(\mathcal{H})$ if and only if the actor is proper. The *-homomorphism maps the subalgebra $C_0\left(\mathcal{G}^0\right) \subseteq C^*(\mathcal{G})$ to $C_b\left(\mathcal{H}^0\right)$, and to $C_0\left(\mathcal{H}^0\right)$ in the proper case.*

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Corollary

There is a functor from the category of étale groupoid with proper actors as morphisms to the category of C-algebras with *-homomorphisms.*

The graph groupoid

Given a graph E, its boundary path space is the set

$$\partial E = \{x \in FP(E) \mid t_{PE}(x) \notin \mathrm{reg}(E)\} \cup IP(E).$$

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The boundary path groupoid E is the set

$$\mathcal{G}(E) := \{(xz, |x| - |y|, yz) \in \partial E \times \mathbb{Z} \times \partial E \mid x, y \in FP(E),\ z \in \partial E\}$$

with operations given by

$$(x, k, y)(y, l, z) = (x, k + l, z)$$

and

$$(x, k, y)^{-1} = (y, -k, x).$$

The graph groupoid

Given a graph E, let ∂E its boundary path space and $\mathcal{G}(E)$ its boundary path groupoid. For each $x, y \in FP(E)$ such that $r_{PE}(x) = r_{PE}(y)$, we define

$$Z_x = \{z \in \partial E \mid z = xz' \text{ for some } z' \in \partial E\}$$

and

$$Z(x, y) = \{(xz, |x| - |y|, yz) \in \mathcal{G}(E) \mid z \in \partial\Gamma,\ s_{PE}(z) = r_{PE}(x)\}.$$

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Moreover given $F \subseteq s_e^{-1}(r_{PE}(x))$ finite, we let

$$Z_{x,F} = Z_x \setminus \bigcup_{e \in F} Z_{xe}$$

and

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These sets give basis for topologies on ∂E and $\mathcal{G}(E)$ respectively.

The graph groupoid

Theorem

Let E be a graph and K a field. Then $C_r^*(\mathcal{G}(E)) \cong C^*(\mathcal{G}(E)) \cong C^*(E)$, and $\mathcal{A}_K(\mathcal{G}(E)) \cong L_K(E)$.

Dynamical Cuntz-Krieger family

Definition

A dynamical Cuntz-Krieger E-family in a groupoid $\mathcal{H}$ is a family of compact-open subsets $\Omega_v \subseteq \mathcal{H}^0$ for $v \in E^0$ and compact-open bisections $T_e \subseteq \mathcal{H}$ for $e \in E^1$ such that

(i) $\Omega_v \cap \Omega_w = \emptyset$ for $v, w \in E^0$ with $v \neq w$;

(ii) for all $e \in E$, $d(T_e) = \Omega_{t_E(e)}$ and $r(T_e) \subseteq \Omega_{s_E(e)}$;

(iii) $r(T_e) \cap r(T_f) = \emptyset$ for $e, f \in E$ with $e \neq f$;

(iv) $\bigsqcup_{s_E(e)=v} r(T_e) = \Omega_v$ for each regular vertex $v \in \mathrm{reg}(E)$.

It is called nondegenerate if $\mathcal{H}^0 = \bigcup_{v \in E^0} \Omega_v$.

Dynamical Cuntz-Krieger family

Theorem (de C.-Meyer)

Let E be a graph and $\mathcal{H}$ an étale groupoid. There is a canonical bijection between proper actors $\mathcal{G}(E) \to \mathcal{H}$ and nondegenerated dynamical Cuntz-Krieger E-families in $\mathcal{H}$.

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Remark

If the dynamical Cuntz-Krieger E-family is not nondegerate, we can use the reduction of $\mathcal{H}$ to the open set $\Omega := \bigcup_{v \in E^0} \omega_v$. We have that $C^*(\mathcal{H}|_\Omega)$ is canonically isomorphic to a subalgebra of $C^*(\mathcal{H})$. Therefore any dynamical Cuntz-Krieger E-family in $\mathcal{H}$ gives rise to a *-homomorphism $C^*(E) \to C^*(\mathcal{H})$ that maps

$$S_v \mapsto \mathbf{1}_{\Omega_v} \text{ for every } v \in E^0,$$
$$S_e \mapsto \mathbf{1}_{T_e} \text{ for every } e \in E^1,$$

where **1** represents the characteristic function.

Theorem (de C.-Meyer)

Let E and F be graphs, and $R : E \to F$ be an admissible relation morphism. For each $v \in F^0$, set

$$\Omega_v = \bigcup_{u \in R^{-1}(v)} Z_u,$$

and for each $f \in F^1$, set

$$T_f = \bigcup_{x \in R^{-1}(f)} Z(x, t_E(x)).$$

Then the families $\{\Omega_v\}_{v \in F^1}$ and $\{T_f\}_{f \in F^1}$ form a dynamical Cuntz-Krieger F-family in $\mathcal{G}(E)$ satisfying

(a) for all $u \in E^0$ and $v \in F^0$, either $Z_u \Omega_v = \emptyset$ or $Z_u \Omega_v = Z_u$,

(b) for all $u \in E^0$ and $f \in E_2$, either $T_f Z_u = \emptyset$ or there is $x \in FP(\Gamma_1)$ such that $T_f Z_u = Z(x, t_E(x))$.

Moreover, every Cuntz-Krieger F-family in $\mathcal{G}(E)$ satisfying the above conditions arises from a relation morphism.

What is next?

- Extended path homomorphisms (joint with P. M. Hajac, M. Lowiel E. A. Pacheco).
- Pullback of graph C*-algebras using relations (joint P. M. Hajac and M. Tobolski).
- Pullback of groupoid C*-algebras coming from reductions of groupoid (joint E. J. Kang)