Leavitt Path Algebra Functor on Out-split Convolution Groupoid

July 29, 2026

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The Plan

1. Quivers and Leavitt path algebras
2. Morphisms of quivers
3. Bates–Pask out-splits as morphisms
4. Hypercategory of Quivers
5. Leavitt path algebra functor

Noncommutative Hauptvermutung

- We show that a combinatorial recipe to define the Bates–Pask (proper) outsplit is equivalent to a morphism of quivers satisfying some distinguished categorical properties.
- As a formal consequence, we reveal a canonical convolution type hypercategory structure containing proper Bates–Pask out-split moves on quivers which is provided with a Leavitt path algebra functor to the category of graded *-algebras.
- It implies also that the class of Bates–Pask (proper) out-splits has all formal properties of the class of (proper) weak equivalences in a Quillen-model category.

Quivers

Definition

A quiver $Q = (V, E, s, t)$ consists of a set of vertices V, a set of edges E, and the *source* and *target* maps s and t

$$V \xleftarrow{t} E \xrightarrow{s} V.$$

Definition

A vertex of a quiver is *regular* if $s^{-1}(v)$ is finite and nonempty. A vertex is called a *sink* if $s^{-1}(v)$ is empty, and an *infinite emitter* if $s^{-1}(v)$ is infinite.

Leavitt path algebras

Definition

Let Q be a row-finite quiver and K a commutative ring with involution. The Leavitt path algebra $L_K(Q)$ is the universal graded $*$-K-algebra generated by

- vertices $v \in V$ (as pairwise orthogonal idempotents), $\deg(v) = 0$
- edges $e \in E$, $\deg(e) = 1$
- ghost edges e^*, for every $e \in E$, $\deg(e^*) = -1$

subject to the following Cuntz–Krieger relations:

(CK0) $s(e)e = e = et(e), \quad t(e)e^* = e^* = e^*s(e),$

(CK1) $e^*f = \delta_{e,f}t(e),$

(CK2) $v = \sum_{s(e)=v} ee^*$ for every regular vertex.

Bates-Pask out-splits

Definition (Bates–Pask)

A *Bates–Pask out-split* of a quiver $Q = (V, E, s, t)$ is a datum consisting of

- for each vertex $v \in V$, an index set

$$I_v = \begin{cases} \{0\} & \text{if } v \text{ is a sink,} \\ \{1, \ldots, m_v\} & \text{if } v \text{ is an emitter,} \end{cases}$$

  and a decomposition

$$s^{-1}(v) = \coprod_{i_v \in I_v} E_v^{i_v},$$

such that for all vertices v and indices $i_v \in I_v$

$$\begin{cases} E_v^{i_v} = \emptyset & \text{if } v \text{ is a sink,} \\ E_v^{i_v} \neq \emptyset & \text{if } v \text{ is an emitter,} \end{cases}$$

- the out-split quiver $\widetilde{Q}$ defined by new sets of vertices and edges

$$\widetilde{V} = \coprod_{v \in V} I_v \times \{v\}, \qquad \widetilde{E} = \coprod_{e \in E} I_{t(e)} \times \{e\},$$

  equipped with the new target and source maps defined for every index $i_{t(e)} = 0, \ldots, m_{t(e)}$ as follows:

$$\widetilde{t}(i_{t(e)}, e) = (i_{t(e)}, t(e)),$$

  and for $e \in E_{s(e)}^{i_{s(e)}}$

$$\widetilde{s}(i_{t(e)}, e) = (i_{s(e)}, s(e)).$$

Properness

Definition (Bates–Pask)

The Bates–Pask out-split is called *proper* if for every infinite emitter v there is at most finite number m_v of blocks $E_v^{i_v}$ and exactly one block is infinite.

Theorem (Bates–Pask)

*Any proper Bates–Pask out-split induces an isomorphism of Leavitt path graded *-algebras uniquely defined on standard generators as follows*

$$v \mapsto \sum_{i_v \in I_v} (i_v, v), \qquad e \mapsto \sum_{i_{t(e)} \in I_{t(e)}} (i_{t(e)}, e).$$

Morphisms of quivers

Definition

A morphism of quivers $\pi : \widetilde{Q} \to Q$ is a pair of maps (π_V, π_E) making the diagram

$$\begin{tikzcd}
\widetilde{V} \arrow[d, "\pi_V"'] & \widetilde{E} \arrow[l, "\widetilde{t}"'] \arrow[r, "\widetilde{s}"] \arrow[d, "\pi_E"] & \widetilde{V} \arrow[d, "\pi_V"] \\
V & E \arrow[l, "t"'] \arrow[r, "s"] & V
\end{tikzcd}$$

commute, i.e.

$$s \circ \pi_E = \pi_V \circ \widetilde{s}, \qquad t \circ \pi_E = \pi_V \circ \widetilde{t}.$$

We call such a morphism π of quivers a *cover* if maps π_V and π_E are surjective. We say that π is proper if π_V and π_E are proper, equivalently, they have finite fibers. We say that a morphism π is *sink-preserving* if the image of any sink is also a sink, or equivalently, *emitter-reflecting* if any element of the pre-image of an emitter is an emitter as well. If π induces a bijection on sinks (resp. infinite emitters), we say that π is bijective on sinks (resp. infinite emitters). Note that if π is bijective on sinks, it obviously must be sink-preserving.

Target-Cartesian and Source-split squares

If the left square of a morphism of quivers is a Cartesian square (which is denoted below by a litle box inside), we say that the morphism π is *target-Cartesian*. If the right square of a quiver morphism π is split (i.e. admits a diagonal filler, or a lift, denoted below by a dashed arrow below), we say that the quiver morphism is *source-split*.

$$\begin{tikzcd}
\widetilde{V} \arrow[d, "\pi_V"'] & \widetilde{E} \arrow[l, "\tilde{t}"'] \arrow[r, "\tilde{s}"] \arrow[d, "\pi_E"] \arrow[dl, phantom, "\square"] & \widetilde{V} \arrow[d, "\pi_V"] \\
V & E \arrow[l, "t"'] \arrow[r, "s"] \arrow[ur, dashed] & V.
\end{tikzcd}$$

The Bates–Pask out-split as a morphism

The first conceptual result is the following theorem.

Theorem

1. *Any Bates–Pask out-split is equivalent to a cover which is target-Cartesian, source-split and bijective on sinks.*
2. *Under this equivalence, a Bates–Pask out-split is proper if and only if the equivalent cover is in addition proper and bijective on infinite emitters.*

Leavitt path algebras and out-splits

Corollary

Any proper cover $\pi_Q : \widetilde{Q} \to Q$ *which is target-Cartesian, source-split and bijective on sinks and infinite emitters defines an isomorphism of Leavitt path graded *-algebras*

$$\pi_Q^* : L_K(Q) \to L_K(\widetilde{Q}),$$

uniquely determined on standard generators as follows

$$v \mapsto \sum_{\widetilde{v} \in \pi_V^{-1}(v)} \widetilde{v}, \qquad e \mapsto \sum_{\widetilde{e} \in \pi_E^{-1}(e)} \widetilde{e}.$$

It is tempting to believe that it is a functor but first we have to resolve the following conundrum.

The conundrum

It is well known that the classes

- of out-splits
- of split squares

are not closed under the obvious vertical composition.

There are at least two standard ways to bypass this difficulty

- embed out-splits into more flexible morphisms of some category and compose them externally anyway
- reveal a hidden structure on out-splits allowing an internal, although maybe not so obvious, composition.

The second way: Hypercategories

We introduce the following notion of a hypercategory (e.g. a category enriched in the Cartesian-monoidal category of sets and relations).

Definition

A *hypercategory* is a class M of *morphisms* equipped with the following structure whose constituents are subject to the following conditions prescribed below:

- a distinguished sub-class $I \subseteq M$ of *identities*,
- an *associative concatenation* of morphisms, generated by a two-argument operation with the class or results denoted by

$$\frac{M}{M},$$

and *unital* with respect to the class I of identities, in the sense that for every $m \in M$, there are unique two concatenable pairs of the form

$$\frac{i}{m} \in \frac{I}{M} \quad \text{and} \quad \frac{m}{i} \in \frac{M}{I},$$

with i over the bar called a *domain*, and i below the bar a *codomain* of m, in particular, the maps

$$M \to \frac{I}{M} \quad \text{and} \quad M \to \frac{M}{I},$$
$$m \mapsto \frac{i}{m} \quad \text{and} \quad m \mapsto \frac{m}{i},$$

are well defined,

- a *relation of composition*

$$\frac{M}{M} \mathrel{\succ\!\!\!-} M,$$

  which is

  1. *associative* in the sense that the two compositions of relations

$$\begin{array}{ccccc} M & & & & \\ \cline{1-1} M & \mathrel{\succ\!\!\!-} & M & & \\ \cline{1-3} M & = & M & \mathrel{\succ\!\!\!-} & M \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} M & = & M & \mathrel{\succ\!\!\!-} & M \\ \cline{1-3} M & \mathrel{\succ\!\!\!-} & M & & \\ \cline{1-1} M & & & & \end{array},$$

are equal as relations between $\dfrac{\dfrac{M}{M}}{M}$ and M,

- and *unital* with respect to the sub-class I of identities in the sense that $m' \in M$ satisfies

  $$\frac{i}{m} \succ m' \quad \text{or} \quad \frac{m}{i} \succ m'$$

  if and only if $m' = m$,

implying, in particular, that both compositions of relations

$$M \to \frac{I}{M} \frac{\subseteq}{=} \frac{M}{M} \rightarrowtail M \quad \text{and} \quad M \to \frac{M}{I} \frac{=}{\subseteq} \frac{M}{M} \rightarrowtail M$$

where the first arrows are well defined using uniqueness of the domain and the codomain, are equal to the equality relation $M = M$.

Motivating example

Theorem

Spans of commutative squares in the category of sets where the left one is Cartesian and the right one is split form a hypercategory.

On the Cartesian left side composition is nothing but usual vertical pasting of squares. The split right side is more interesting: it is a relation which is the transpose of the canonical decomposition.

Warning. This shouldn't be confused with composition by vertical pasting under which the class of split squares is not closed. However, in terms of vertical arrows, both behave the same.

Canonical decomposition of split squares

$$
\begin{tikzcd}
\widetilde{\widetilde{E}} \arrow[r, "\widetilde{\widetilde{s}}"] \arrow[d, "\pi_{\widetilde{E}}"'] & \widetilde{\widetilde{V}} \arrow[d, "\pi_{\widetilde{V}}"] \\
\widetilde{E} \arrow[r, "\widetilde{s}"] \arrow[d, "\pi_E"'] \arrow[ur, dashed, "\widetilde{\lambda}"] & \widetilde{V} \arrow[d, "\pi_V"] \\
E \arrow[r, "s"'] \arrow[ur, dashed, "\lambda"] & V
\end{tikzcd}
\succ
\begin{tikzcd}
\widetilde{\widetilde{E}} \arrow[r, "\widetilde{\widetilde{s}}"] \arrow[d, "\widetilde{\pi_E}"'] & \widetilde{\widetilde{V}} \arrow[d, "\widetilde{\pi_V}"] \\
E \arrow[r, "s"'] \arrow[ur, dashed, "\lambda^L"] & V
\end{tikzcd}
$$

- Take the right hand side diagram satisfying

$$\begin{aligned}\widetilde{\pi_V} \circ \lambda^L &= s,\\ \lambda^L \circ \widetilde{\pi_E} &= \widetilde{\widetilde{s}},\end{aligned}$$

- decompose its vertical arrows as

$$\begin{aligned}\widetilde{\pi_V} &= \pi_V \circ \pi_{\widetilde{V}},\\ \widetilde{\pi_E} &= \pi_E \circ \pi_{\widetilde{E}}\end{aligned}$$

- define the remaining arrows on the left hand side as

$$\begin{aligned}\widetilde{s} &:= \pi_{\widetilde{V}} \circ \lambda^L \circ \pi_E,\\ \lambda &:= \pi_{\widetilde{V}} \circ \lambda^L,\\ \widetilde{\lambda} &:= \lambda^L \circ \pi_E.\end{aligned}$$

Vertical isomorphisms

Lemma

Every commutative square with invertible vertical arrows is Cartesian.

Lemma

Every commutative square with invertible vertical arrows is split in a unique way.

Stability under retracts

Lemma

The class of Cartesian squares is closed under retracts.

Lemma

The class of split commutative squares of sets is closed under retracts.

2-out-of-3 property

Lemma

Among vertically surjective squares, the class of vertically surjective Cartesian squares has the 2-out-of-3 property.

Lemma

Among vertically surjective squares, the class of vertically surjective split squares has the 2-out-of-3 property.

Lemma

The class of vertically surjective sink-bijective squares has the 2-ot-of-3 property with respect to $\rightarrowtail$.

Lemma

The class of vertically surjective proper infinite-emitter-bijective squares has the 2-ot-of-3 property with respect to $\rightarrowtail$.

Combinatorial formal weak equivalences

Theorem

In the hypercategory of proper regular target-Cartesian and source-split morphisms of quivers, the class of proper covers being bijective on sinks and infinite emitters contains isomorphisms, is stable under proper retracts and has the 2-out-of-3 property.

Since the Leavitt path algebra $L_K(Q)$ construction uses only vertical (proper) arrows, we have the following.

Corollary

*The Leavitt path algebra is a functor from our hypercategory to the category of graded *-algebras, mapping the class of proper Bates–Pask out-splits into isomorphisms.*

Combinatorial homotopy theory of quivers?

All this means that the class of proper Bates–Pask out-splits has all formal properties of the class of (proper) weak equivalences in a Quillen-model category, and suggests that the Leavitt path algebra could be an invariant of the corresponding combinatorial formal homotopy type.

In making this idea rigorous, a hyper-versions of

- the Dugger theorem, providing a universal model category for our hypercategory of quivers with proper Bates–Pask out-splits as "weak equivalences to be", and
- the left Bousfield localization,

would be very much helpful.

Thank You / Merci