Skew Products: Coactions One Can See

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Abstract

Given a left-cancellative small category $\mathcal{C}$, a discrete group G, and a functor $\eta\colon \mathcal{C} \to G$, we construct a *skew product category* $\mathcal{C} \times_\eta G$. This represents a "coaction one can see" in the sense that the Cuntz-Krieger algebra $\mathcal{O}(\mathcal{C} \times_\eta G)$ of the skew product is isomorphic to a crossed product $\mathcal{O}(\mathcal{C}) \rtimes_\delta G$ by a coaction of G. Moreover, the skew product carries a natural free action of G that corresponds to the dual action on the crossed product under this isomorphism, and that allows us to recover $\mathcal{C}$ as the quotient category $(\mathcal{C} \times_\eta G)/G$. As a sort of converse, we also have a "Gross-Tucker"-type theorem that says that any LCSC that carries a free action of G can be realized as a skew product category. This theory fits into a broader context that goes back over 25 years to ideas of Kumjian, Pask, and Raeburn for graph C^*-algebras.

In this talk, I'll present these results in as elementary a fashion as possible, and say what I can about their proofs. Time permitting, I'll also try to explain how we can use our results to study Cuntz-Krieger algebras arising from free actions of groups on finitely-aligned LCSCs, and to construct coactions of groups on Exel-Pardo algebras. The talk is based on joint work with Erik Bédos and John Quigg.

Epigraph

Como todo poseedor de una biblioteca, Aureliano se sabía culpable de no conocerla hasta el fin. – Jorge Luis Borges, Los teólogos

Like all those possessing a library, Aurelian was aware that he was guilty of not knowing his in its entirety.

References

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KQR S. Kaliszewski, J. Quigg and I. Raeburn, *Skew products and crossed products by coactions*, J. Operator Theory **46** (2001)

Rae I. Raeburn, *Graph algebras*, CBMS Regional Conference Series in Mathematics **103** (2005)

KQ S. Kaliszewski and J. Quigg, *Coactions and skew products of topological graphs*, Integral Equations Operator Theory **75** (2013)

BKQ E. Bédos, S. Kaliszewski and J. Quigg, *Skew products of finitely aligned left cancellative small categories and Cuntz-Krieger algebras*, Münster J. Math. **14** (2021).

ABD P. Ara, A. Buss and A. Dalla Costa, *Free actions of groups on separated graph C^*-algebras*, Trans. Amer. Math. Soc. **376** (2023)

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Skew-Product Graphs

Let $E = (E^0, E^1, r, s)$ be a directed graph, let G be a discrete group, and let $c \colon E^1 \to G$ be a *labeling* of the edges of E by elements of G.

The *skew product graph* is $E \times_c G = (E^0 \times G, E^1 \times G, r, s)$, with[1]

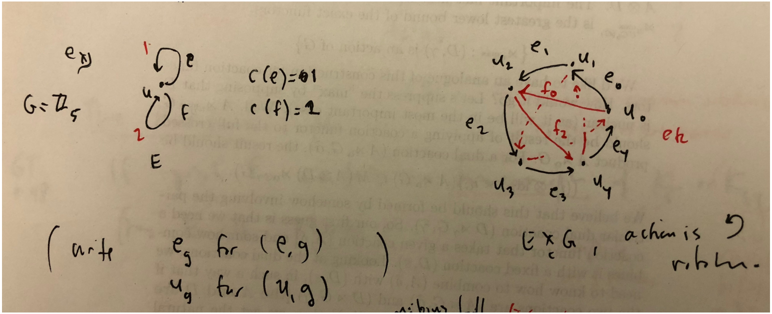
$$r(e,g) = (r(e), g) \quad \text{and} \quad s(e,g) = (s(e), c(e)g)$$

Gross and Tucker (and others) call the pair (E, c) a *voltage graph*, and $E \times_c G$ is the *derived graph*.

[1]Or, $s(e,g) = (s(e), g)$ and r(e,g)=(r(e),gc(e)).

Skew-Product Graphs

Example: K_5 [2]



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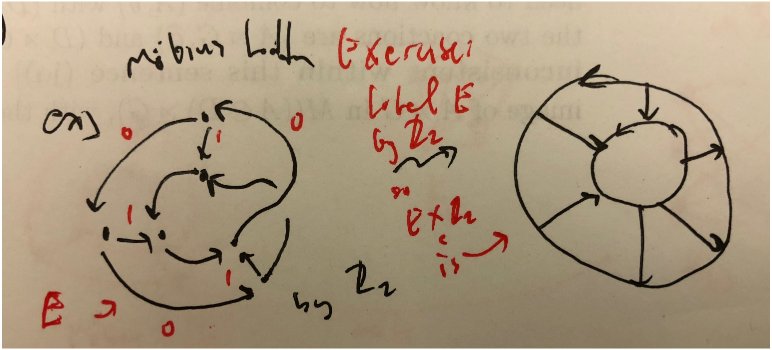
[2] AI tools were used to find and check references for this talk, but not to generate the diagrams in these slides. `https://leidendeclaration.ai/`

Skew-Product Graphs

Example: $K_{3,3}$



Skew-Product Graphs

Every skew product graph $E \times_c G$ has a *free* action of G defined by

$$h \cdot (x, g) = (x, gh^{-1})$$

and then $(E \times_c G)/G \cong E$. Conversely:

Theorem: (Gross-Tucker) If G acts freely on a graph F, then there is a labeling c of F/G by G such that

$$(F/G) \times_c G \cong F$$

G-equivariantly.

Proof: Explicitly (carefully) define c and check that it works.

Graph C^*-Algebras

Theorem: (Kumjian-Pask) If G acts freely on a (row-finite) graph F, then

$$C^*(F) \rtimes_\alpha G \overset{\mathrm{M}}{\sim} C^*(F/G)$$

where α is the induced action on $C^*(F)$.

Compare with Green's Theorem:

$$C_0(G/H) \rtimes_\tau G \overset{\mathrm{M}}{\sim} C^*(H)$$

and Mansfield's:

$$C_0(G) \rtimes_\tau N \overset{\mathrm{M}}{\sim} C_0(G/N)$$

Duality?

Alternating the graph constructions, given a labeling c of E by G,

$$E \mapsto E \times_c G \mapsto (E \times_c G)/G \cong E$$

(although the original labeling c is lost); given a free action of G on F,

$$F \mapsto (F/G, c) \mapsto (F/G) \times_c G \cong F$$

(here the isomorphism intertwines the free G-actions).

So the Kumjian-Pask theorem

$$C^*(F) \rtimes_\alpha G \overset{\mathrm{M}}{\sim} C^*(F/G)$$

implies that for any labeling c of a graph E by G,

$$C^*(E \times_c G) \rtimes_\alpha G \overset{\mathrm{M}}{\sim} C^*(E).$$

Comparing this with *Katayama duality*, $C^*(E \times_c G)$ looks like a crossed product of $C^*(E)$ by a *coaction* of G whose dual action is α...

Takesaki-Takai Duality

Suppose G is (discrete) *abelian*, and $\alpha\colon G \to Aut(A)$ is an action. There is a *dual action* $\hat{\alpha}$ of the (compact) Pontryagin dual $\hat{G}$ on $A \rtimes_\alpha G$ given for $f \in C_c(G, A)$ by

$$\hat{\alpha}_\chi(f)(g) = \chi(g)f(g).$$

Theorem: In this situation,

$$(A \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G} \overset{\mathrm{M}}{\sim} A$$

equivariantly for the actions $\hat{\hat{\alpha}}$ and α of G.

Coactions

Now suppose $\beta\colon \hat{G} \to Aut(B)$ is any (strongly continuous) action of $\hat{G}$. We get a $*$-homomorphism $\delta_\beta\colon B \to B \otimes C(\hat{G}) \cong C(\hat{G}, B)$ by

$$\delta_\beta(b)(\chi) = \beta_\chi(b)$$

The algebraic properties of β imply that

$$\delta_\beta(\beta_\eta(b))(\chi) = \delta_\beta(b)(\chi\eta),$$

which can be expressed as an identity of maps $B \to B \otimes C(\hat{G}) \otimes C(\hat{G})$:

$$(\delta_\beta \otimes \mathrm{id}) \circ \delta_\beta = (\mathrm{id} \otimes \delta_{\hat{G}}) \circ \delta_\beta$$

Here $\delta_{\hat{G}}\colon C(\hat{G}) \to C(\hat{G} \times \hat{G}) \cong C(\hat{G}) \otimes C(\hat{G})$ by $\delta_{\hat{G}}(f)(\chi, \eta) = f(\chi\eta)$.

Conversely, a map $\delta\colon B \to B \otimes C(\hat{G})$ with the above structure determines an action of $\hat{G}$ on B: thus *actions of $\hat{G}$ correspond to coactions of G*...

Coactions

Using the isomorphism $C(\hat{G}) \cong C^*(G)$ we can remove any mention of $\hat{G}$, and thus remove the need for G to be abelian.

Definition: A coaction of a discrete group G on a C^*-algebra B is a $*$-homomorphism[3] $\delta\colon B \to B \otimes C^*(G)$ satisfying the *coaction identity*

$$(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta_G) \circ \delta$$

and *nondegenerate* in the sense that $\overline{\delta(B)(1 \otimes C^*(G))} = B \otimes C^*(G)$.[4]

Here $\delta_G\colon C^*(G) \to C^*(G) \otimes C^*(G)$ is the *comultiplication* induced by the homomorphism $g \mapsto u(g) \otimes u(g)$.

Example: δ_G is itself a coaction of G on $C^*(G)$.

[3] By assumption, our $*$-homomorphisms are nondegenerate and $\otimes$ denotes the minimal tensor product.

[4] These two conditions force δ to be injective.

Imai-Takai Duality

Example: Let $\alpha\colon G \to Aut(A)$ be an action of (discrete) G, with canonical covariant representation (i_A, i_G) of (A, G) in the crossed product $A \rtimes_\alpha G$. The *dual coaction* of G on $A \rtimes_\alpha G$ is the integrated form

$$\hat{\alpha} = (i_A \otimes 1) \times (i_G \otimes u)\colon A \rtimes_\alpha G \to (A \rtimes_\alpha G) \otimes C^*(G).$$

Theorem: In this situation,

$$(A \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G} \overset{\mathrm{M}}{\sim} A$$

equivariantly for the actions $\hat{\hat{\alpha}}$ and α of G.

Crossed Products by Coactions

For a coaction δ of G on B, we define the *spectral subspaces*

$$B_g = \{b \in B \mid \delta(b) = b \otimes g\}$$

for each $g \in G$. We write $b = b_g$ when $b \in B_g$.

A *covariant representation* (π, μ) of δ in a C^*-algebra C consists of nondegenerate $*$-homomorphisms

$$\pi \colon B \to M(C) \quad \text{and} \quad \mu \colon C_0(G) \to M(C)$$

such that

$$\pi(b_g)\mu(\chi_h) = \mu(\chi_{gh})\pi(b_g)$$

for $g, h \in G$ (and $b_g \in B_g$).

There always exists a *crossed product C^*-algebra* $B \rtimes_\delta G$, with a *universal covariant representation* (j_B, j_G) of δ in $M(B \rtimes_\delta G)$.

The Quigg calculus

For $g, h \in G$, we identify

$$j_B(b_g) j_G(\chi_h) \in B \rtimes_\delta G \quad \text{with} \quad (b_g, h) \in B_g \times G.$$

Then

$$B \rtimes_\delta G = \overline{\operatorname{span}}\{(b_g, h) \mid g, h \in G, b_g \in B_g\}$$

with

$$(b_g, h)^* = (b_g^*, gh)$$

and

$$(b_g, h)(c_k, \ell) = \begin{cases} (b_g c_k, \ell) & \text{if } h = k\ell \\ 0 & \text{otherwise.} \end{cases}$$

Katayama Duality

Suppose δ is a coaction of (discrete) G on B. There is a *dual action* $\hat{\delta}$ of G on $B \rtimes_\delta G$ given (in the Quigg calculus) by

$$\hat{\delta}_h(b_g, k) = (b_g, kh^{-1}).$$

Theorem: If δ is maximal in this situation, then

$$(B \rtimes_\delta G) \rtimes_{\hat{\delta}} G \overset{\mathrm{M}}{\sim} B$$

equivariantly for the coactions $\hat{\hat{\delta}}$ and δ of G.

Again, compare this with the Kumjian-Pask theorem: for any labeling c of a graph E by G,

$$C^*(E \times_c G) \rtimes_\alpha G \overset{\mathrm{M}}{\sim} C^*(E).$$

Skew-Product Graphs and Coactions

Recall that the graph algebra $C^*(E)$ is the C^*-algebra generated by a universal *Cuntz-Krieger E-family*

$$\{s_e, p_v \mid e \in E^1, v \in E^0\}$$

Theorem: [*KQR*] For any labeling c of a (row-finite) graph E by G, there exists a coaction of G on $C^*(E)$ determined by

$$s_e \mapsto s_e \otimes c(e) \quad \text{and} \quad p_v \mapsto p_v \otimes 1$$

where $\{s_e, p_v\}$ is the canonical Cuntz-Krieger E-family in $C^*(E)$, and then there is a G-equivariant isomorphism

$$C^*(E \times_c G) \cong C^*(E) \rtimes_\delta G$$

Skew-Product Graphs and Coactions

Proof: To construct $\delta\colon C^*(E) \to C^*(E) \otimes C^*(G)$, observe that

$$\{s_e \otimes c(e), p_v \otimes 1\}$$

is a Cuntz-Krieger E-family in $C^*(E) \otimes C^*(G)$, and use the universal property of $C^*(E)$. Check the coaction identity on generators of $C^*(E)$.

For the isomorphism $C^*(E \times_c G) \cong C^*(E) \rtimes_\delta G$, check that

$$t_{(e,g)} = (s_e, g) \quad \text{and} \quad q_{(v,g)} = (p_v, g)$$

defines a Cuntz-Krieger $E \times_c G$-family $\{t_{(e,g)}, q_{(v,g)}\}$ in $C^*(E) \rtimes_\delta G$.

The induced map $\Phi\colon C^*(E \times_c G) \to C^*(E) \rtimes_\delta G$ is evidently surjective (its range contains a generating set), and can be shown to be injective using the gauge action of $\mathbb{T}$ on $C^*(E)$...

Skew-Product Graphs

We recover the Kumjian-Pask result: If G acts freely on F, then there is c such that

$$F \cong (F/G) \times_c G$$

equivariantly...

So there is a maximal[5] coaction δ such that

$$C^*(F) \cong C^*((F/G) \times_c G) \cong C^*(F/G) \rtimes_\delta G$$

equivariantly...

So by Katayama duality,

$$C^*(F) \rtimes_{\alpha,r} G \cong C^*(F/G) \rtimes_\delta G \rtimes_{\hat{\delta},r} G \overset{M}{\sim} C^*(F/G).$$

[5] There is an issue with full *vs* reduced crossed products. In [*KQR*] we proved that the action on $C^*(E \times_c G)$ is always amenable (Corollary 3.2); later we realized δ is maximal.

Left-Cancellative Small Categories

A *small category* may be defined as a set $\mathcal{C}$, a subset $\mathcal{C}^0 \subseteq \mathcal{C}$, maps $r, s \colon \mathcal{C} \to \mathcal{C}^0$, and a partially-defined multiplication

$$(\alpha, \beta) \mapsto \alpha\beta \quad \text{when } s(\alpha) = r(\beta)$$

such that if $s(\alpha) = r(\beta)$ and $s(\beta) = r(\gamma)$, then

1. $r(\alpha\beta) = r(\alpha)$ and $s(\alpha\beta) = s(\beta)$
2. $\alpha(\beta\gamma) = (\alpha\beta)\gamma$
3. $r(v) = s(v) = v$ for all $v \in \mathcal{C}^0$
4. $r(\alpha)\alpha = \alpha = \alpha s(\alpha)$

$\mathcal{C}$ is *left-cancellative* if

$$\alpha\beta = \alpha\gamma \quad \text{implies} \quad \beta = \gamma$$

A *left-cancellative small category* (LCSC) is a small category that is left-cancellative.

Left-Cancellative Small Categories

Examples:

- The path category of a directed graph
- Higher-rank graphs
- Quasi-lattice ordered groups

(These are all actually "categories of paths" in Spielberg's sense.)

A LCSC $\mathcal{C}$ is *finitely-aligned* if for all $\alpha, \beta \in \mathcal{C}$ there is a (possibly empty) finite set $F \subseteq \mathcal{C}$ such that

$$\alpha\mathcal{C} \cap \beta\mathcal{C} = \bigcup_{\gamma \in F} \gamma\mathcal{C}$$

"This is a strong assumption that leads to very significant simplifications in the structure of the C^*-algebras. For example, it implies that the 'usual generators and relations' can be written in a particularly simple form."

Left-Cancellative Small Categories

A *representation* of $\mathcal{C}$ in a C^*-algebra B is a map $T\colon \mathcal{C} \to B$ such that for all $\alpha, \beta \in \mathcal{C}$,

1. $T_\alpha^* T_\alpha = T_{s(\alpha)}$
2. $T_\alpha T_\beta = T_{\alpha\beta}$ if $s(\alpha) = r(\beta)$
3. $T_\alpha T_\alpha^* T_\beta T_\beta^* = \displaystyle\bigvee_{\gamma \in \alpha \vee \beta} T_\gamma T_\gamma^*$.

A representation T is *covariant* if

4. $T_v = \displaystyle\bigvee_{\alpha \in F} T_\alpha T_\alpha^*$ for all $v \in \mathcal{C}^0$ and every finite exhaustive set F at v.

Without defining "$\bigvee$", the point is: these are all algebraic relations that can be explicitly checked.

If T is a representation then:

- $T_\alpha^* T_\beta = 0$ when $s(\alpha) \neq s(\beta)$
- if α is invertible then $T_{\alpha^{-1}} = T_\alpha^*$ and $T_\alpha T_\alpha^* = T_{r(\alpha)}$
- if also $\beta = \gamma\alpha$ then $T_\beta T_\beta^* = T_\gamma T_\gamma^*$.

Left-Cancellative Small Categories

The *Cuntz-Krieger algebra* $\mathcal{O}(\mathcal{C})$ of $\mathcal{C}$ is the C^*-algebra generated by a universal covariant representation $t\colon \mathcal{C} \to \mathcal{O}(\mathcal{C})$.

The universal property means that for every covariant representation $T\colon \mathcal{C} \to B$ there is unique homomorphism $\phi\colon \mathcal{O}(\mathcal{C}) \to B$, called the *integrated form* of T, such that $\phi \circ t = T$.

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "t"] \arrow[dr, "T"'] & \mathcal{O}(\mathcal{C}) \arrow[d, dashed, "\phi"'] \\
& B
\end{tikzcd}
$$

Skew-Product LCSCs

Let $\mathcal{C} = (\mathcal{C}^0, \mathcal{C}, r, s)$ be a LCSC, let G be a discrete group. A G-valued *cocycle* on $\mathcal{C}$ is a functor $\eta\colon \mathcal{C} \to G$.

The *skew product* LCSC is $\mathcal{C} \times_\eta G = (\mathcal{C}^0 \times G, \mathcal{C} \times G, r, s)$, with

$$r(\alpha, g) = (r(\alpha), \eta(\alpha)g) \quad \text{and} \quad s(\alpha, g) = (s(\alpha), g)$$

and

$$(\alpha, g)(\beta, h) = (\alpha\beta, h) \qquad \text{when } s(\alpha) = r(\beta) \text{ and } g = \eta(\beta)h$$

Fact: $\mathcal{C} \times_\eta G$ will be finitely aligned whenever $\mathcal{C}$ is.

Example: If $\mathcal{C}$ is the path category of a directed graph E, any labeling $c\colon E^0 \to G$ naturally gives rise to a cocycle $\eta\colon \mathcal{C} \to G$...

Skew-Product LCSCs

$\mathcal{C} \times_\eta G$ has a free action of G defined by

$$h \cdot (\alpha, g) = (\alpha, gh^{-1})$$

and then $(\mathcal{C} \times_\eta G)/G \cong \mathcal{C}$.

Theorem: [BKQ] If G acts freely on a LCSC $\mathcal{D}$, then $\mathcal{D}/G$ is a LCSC, and there is cocycle $\eta \colon \mathcal{D}/G \to G$ such that

$$(\mathcal{D}/G) \times_\eta G \cong \mathcal{D}$$

G-equivariantly.

Proof: Explicitly (carefully) define η and check that it works.

Skew-Product LCSCs and Coactions

Theorem: [BKQ] For any G-valued cocycle η on a (finitely-aligned) LCSC $\mathcal{C}$, there is a coaction δ of G on $\mathcal{O}(\mathcal{C})$ determined by

$$t_\alpha \mapsto t_\alpha \otimes \eta(\alpha)$$

where $t\colon \mathcal{C} \to \mathcal{O}(\mathcal{C})$ is the canonical universal representation, and then there is a G-equivariant isomorphism

$$\mathcal{O}(\mathcal{C} \times_\eta G) \cong \mathcal{O}(\mathcal{C}) \rtimes_\delta G$$

Skew-Product LCSCs and Coactions

Proof: To construct $\delta\colon \mathcal{O}(\mathcal{C}) \to \mathcal{O}(\mathcal{C}) \otimes C^*(G)$, observe that

$$\alpha \mapsto t_\alpha \otimes \eta(\alpha)$$

defines a covariant representation of $\mathcal{C}$ in $\mathcal{O}(\mathcal{C}) \otimes C^*(G)$, and use the universal property of $\mathcal{O}(\mathcal{C})$. Check the coaction identity on generators of $\mathcal{O}(\mathcal{C})$.

For the isomorphism $\mathcal{O}(\mathcal{C} \times_\eta G) \cong \mathcal{O}(\mathcal{C}) \rtimes_\delta G$, check that

$$T_{(\alpha,g)} = (t_\alpha, g)$$

defines a covariant representation T of $\mathcal{C} \times_\eta G$ in $\mathcal{O}(\mathcal{C}) \rtimes_\delta G$, and (inversely) that there is a covariant representation (π, ϕ) of δ in $\mathcal{O}(\mathcal{C} \times_\eta G)$ such that

$$\pi(t_\alpha)\phi(\chi_g) = t_{(\alpha,g)}$$

Skew-Product LCSCs and Coactions

Interestingly, δ is always maximal, but *not always normal*. For graphs — even higher-rank graphs — δ is both maximal and normal.

Example: Any discrete group G is a finitely-aligned LCSC (viewed as usual as a category with one object), and $\mathcal{O}(G)$ is the (full) group C^*-algebra $C^*(G)$. If $\eta \colon G \to G$ is the identity map, then the associated coaction is the comultiplication δ_G on $C^*(G)$, and δ_G is known to be normal if and only if G is amenable.

Application: Free Actions on LCSCs

Theorem:[*BKQ*] If G acts freely on a LCSC $\mathcal{D}$ and $\mathcal{D}/G$ is finitely aligned, then $\mathcal{D}$ is finitely aligned, and

$$\mathcal{O}(\mathcal{D}) \rtimes_\beta G \overset{\mathrm{M}}{\sim} \mathcal{O}(\mathcal{D}/G)$$

where β is the induced action on $\mathcal{O}(\mathcal{D})$.

Further Work:

Ara-Buss-DallaCosta (TAMS 2023): *Separated graphs* generalize directed graphs.

- Skew products of separated graphs
- Gross-Tucker Theorem for separated graphs
- C^*-algebras of separated graphs
- skew products are coaction crossed products:

$$C^*(E, C) \rtimes_\delta G \overset{\mathrm{M}}{\sim} C^*(E \times_c G, C \times_c G)$$

- A Kumjian-Pask-type theorem:

$$C^*(F, D) \rtimes_\theta G \overset{\mathrm{M}}{\sim} C^*(F/G, D/G)$$

Future Work?

Schäfer-Tobolski (arxiv 2026): *Voltage quantum graphs*

- derived (skew-product) quantum graphs
- quantum Gross-Tucker theorem for *finite abelian* G...