On Graded Irreducible Representations of Leavitt Path Algebras
(Joint work with Ashish Srivastava)

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Dedicated to Professor Paul Baum celebrating his 90th Birthday

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- Examples are constructed illustrating the results.

Irreducible representations

- Let k be a field. Given a k-algebra A, a **representation** of A consists of a k-vector space V together with a k-algebra homomorphism $f : A \to End_k(V)$. If W is a k-vector subspace of V, we say that W is f-**invariant** if $f(a)(W) \subseteq W$ for all $a \in A$, and we say that f is an **irreducible representation** when the only f-invariant subspaces of V are $\{0\}$ and V.

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- Observe that if $f : A \to End_k(V)$ is a representation of A, then V becomes a left A-module by defining $a \cdot x := f(a)(x)$ for $a \in A$ and $x \in V$. This representation f is **irreducible** if and only if, under the above module operation, V becomes a **simple** A-module (i.e., V has no submodules other than $\{0\}$ and V).

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- Hereafter, we will use the term **simple modules instead of irreducible representations**.

Graph Preliminaries

- A (directed) graph $E = (E^0, E^1, r, s)$ consists of a set E^0 of elements called **vertices** and a set E^1 of elements called **edges**, together with maps $s, r : E^1 \longrightarrow E^0$. If e is an edge, say, $\bullet_u \xrightarrow{e} \bullet_v$, then $s(e) = u$ is called the **source** of e, and $r(e) = v$ is called the **range** of e.

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- A **path** of length n is a finite sequence of edges $\mu = e_1 e_2 \cdots e_n$ with $n \geq 1$, where $r(e_i) = s(e_{i+1})$ for all $1 \leq i \leq n-1$. We write $|\mu|$ to denote the length of μ. A vertex is considered a path of length 0.

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- A path $\mu = e_1 \ldots e_n$ in E is **closed** if $r(e_n) = s(e_1)$. The closed path μ is called a **cycle** if $s(e_i) \neq s(e_j)$ for every $i \neq j$.

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- A non-empty subset D of vertices is said to be **downward directed**, if for any $u, v \in D$, there exists a $w \in D$ such that $u \geq w$ and $v \geq w$.

A **Leavitt path algebra** $L_K(E)$ of an arbitrary graph E over a field K is the free K-algebra generated by the set $\{v : v \in E^0\}$ and the set $\{e, e^* : e \in E^1\}$ subject to the following conditions

(1) For $u, v \in E^0$, $u \cdot u = u$ and $u \cdot v = 0$, if $u \neq v$.

(2) $s(e)e = e = er(e)$ for all $e \in E^1$.

(3) $r(e)e^* = e^* = e^*s(e)$ for all $e \in E^1$.

(4) (The "CK-1 relations") For all $e, f \in E^1$, $e^*e = r(e)$ and $e^*f = 0$ if $e \neq f$.

(5) (The "CK-2 relations") For every regular vertex $v \in E^0$,
$v = \sum_{e \in E^1, s(e)=v} ee^*$

Every element $a \in L_K(E)$ can be written in the form $a = \sum_{i=1}^{i=n} \alpha_i \beta_i^*$, where α_i, β_i are paths with $r(\alpha_i) = r(\beta_i)$.

Graded rings

Every Leavitt path algebra $L_K(E)$ is a $\mathbb{Z}$*-graded algebra*, in which vertices have degree 0, for any edge e, $\deg(e) = 1, \deg(e^*) = -1$, and $L_K(E) = \bigoplus_{n\in\mathbb{Z}} L_n$ where the L_n are abelian subgroups called **homogeneous components** whose elements are called **homogeneous elements** and they satisfy $L_m L_n \subseteq L_{m+n}$ for all $m, n \in \mathbb{Z}$. The L_n are given by

$$L_n = \{\sum \alpha_i \beta_i^* : \text{ where } \alpha_i, \beta_i \text{ are paths in } E \text{ and } |\alpha| - |\beta| = n\}$$

An ideal I is called a **graded ideal** if $I = \bigoplus_{n\in\mathbb{Z}} (I \cap L_n)$.

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- **Corollary 2:** Every graded one-sided finitely generated ideal of $L_K(E)$ is a direct summand.
- **Proposition 3**: Every one-sided ideal of $L_K(E)$ is a graded one sided ideal if and only the graph E contains no cycles.

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- We say a vertex v has **bifurcation** or v is a **bifurcating vertex** if v emits two or more edges.

$$\begin{tikzcd} \bullet & \bullet \\ \bullet_v \arrow[u] \arrow[ur] \arrow[r] & \bullet \end{tikzcd}$$

Graded-simple left/right ideals - Two types of vertices

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- We say a vertex v has **bifurcation** or v is a **bifurcating vertex** if v emits two or more edges. $\begin{matrix} \bullet & & \bullet \\ \uparrow & \nearrow & \\ \bullet_v & \longrightarrow & \bullet \end{matrix}$
- A vertex v is called a **line point** if $T(v)$ contains no bifurcating vertices and no cycles. Thus $T(v)$ becomes, (when we add all the edges between adjacent vertices in $T(v)$), a straight line path like $\cdot_v \to \cdot_{v_1} \to \cdot_{v_2} \cdot \to \cdots$ which becomes a finite path if $T(v)$ contains a sink. In particular, a sink itself is a line point.

.A vertex v is called a **Laurent vertex** if $T(v)$ is the set of all vertices on a single path $\gamma = \mu c$ with $s(\gamma) = v$, where μ is a path without bifurcations and c is a cycle without exits.

$$\begin{tikzcd}
 & & & \cdot \arrow[r] & \cdot \arrow[dr] & \\
\cdot_v \arrow[r] & \cdot \arrow[r] & \cdot \arrow[ur] & & & \cdot \arrow[dl] \\
 & & & \cdot \arrow[ul] & \cdot \arrow[l] &
\end{tikzcd}$$

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- A graded-simple module need not be a simple module. For example, $K[x, x^{-1}] = \bigoplus_{n \in \mathbb{Z}} Kx^n$ is a commutative $\mathbb{Z}$-graded ring in which x^n is invertible for all n. If $0 \neq I$ is a graded ideal, then $I = K[x, x^{-1}]$, as $I = \bigoplus_{n \in \mathbb{Z}} (I \cap Kx^n)$.implies that the invertible element $x^n \in I$ for some n. Hence, $K[x, x^{-1}]$ is a graded-simple module. But $K[x, x^{-1}]$ is not a simple $K[x, x^{-1}]$-module, since it has infinitely many proper ideals generated by polynomialls with non-zero constant term..

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- Likewise, a simple module need not be graded-simple. For example, for any irreducible polynomial $p(x) \in K[x, x^{-1}]$, $K[x, x^{-1}]/ < p(x) >$ is a simple $K[x, x^{-1}]$-module, but it is not graded-simple, it is not even a graded $K[x, x^{-1}]$-module.

- **Lemma 4**: Given a vertex $v \in E$, the left (right) ideal $L_K(E)v$ ($vL_K(E)$) is a graded-simple left (right) ideal of $L_K(E)$, if and only if v is a line point or a Laurent vertex

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- **Theorem 5:** (Hazrat-K.M.R.) Let E be an arbitrary graph, K be any field and $L = L_K(E)$. Then the graded socle $Soc_{gr}(L)$ is the two-sided ideal generated by the set consisting of all the line points and all the Laurent vertices (on cycles without exits) in E

Constructing graded-simple modules

- Two infinite paths $p = e_1 e_2 \cdots e_n \cdots$ and $q = f_1 f_2 \cdots f_n \cdots$ in a graph E are said to be **tail-equivalent**, if there exist positive integers m, n such that $e_{m+i} = f_{n+i}$ for all $i \geq 0$.

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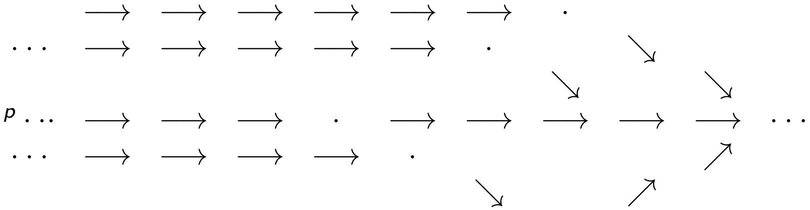
- An infinite path which is not rational is called an **irrational path**.

Tail-equivalence of infinite paths

- 

Constructing Simple modules using paths

- Let p be an infinite path in the graph E. Consider $[p]$. Let $V_{[p]}$ denote the K-vector space having as a basis $\{q : q \in [p]\}$.

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- (ii) For all edges e in E, $e \cdot q = eq$ if $r(e) = s(q)$, otherwise $e \cdot q = 0$; and $e^* \cdot q = q_1$ if $q = eq_1$, otherwise, $e^* \cdot q = 0$.

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- Given two infinite paths p, q, **the simple modules** $V_{[p]} \cong V_{[q]}$ **if and only if** p, q **are tail equivalent.**

Which Chen Simple modules are graded?

- Recall, an infinite path p is called a **rational path** if p is tail-equivalent to the infinite path $c^\infty = ccc\cdots$, for cycle c.

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- (ii) If w is a sink, then the vector space N_w having the set of all paths that end at w as a basis becomes a graded-simple module under the same actions indicated by Chen above.

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- $\Longrightarrow$ L must be a graded-simple ring, that is, $\{0\}$ and L are the only graded ideals of L (Because, if $I \neq 0$ is a graded ideal of L and if T is a graded-simple L/I-module, then T is a graded-simple L-module so $ann_L(T) \supseteqq I \neq \{0\}$, contradicting $\{0\}$ is the only annihilator).

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- $\Longrightarrow$ The graph E is downward directed and E^0 and the empty set Φ are the only hereditary saturated sets of vertices in E

Constructing infinite number of graded-simple modules

- **Lemma 8:** If the graph E contains two cycles c, d having a common vertex, then there are uncountably many irrational paths in E which are pair-wise not tail-equivalent. There are uncountably many non-isomorphic graded-simple left $L_K(E)$-modules.

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- $$\begin{tikzcd}
 & \bullet \arrow[dr] & & \bullet \arrow[dr] & & \\
\bullet \arrow[ur] & & \bullet \arrow[dl] \arrow[ur] & & \bullet \arrow[d] & . \\
\mathbf{c} & \bullet \arrow[ul] & & \bullet \arrow[ul] & \bullet \arrow[l] & \mathbf{d}
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- $$\begin{array}{cccccccccc}
 & & \bullet & & & & \bullet & & & \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & & \\
\bullet & & & & \bullet & & & & \bullet & . \\
 & \nwarrow & & \swarrow & & \nwarrow & & & \downarrow & \\
\mathbf{c} & & \bullet & & & & \bullet & \longleftarrow & \bullet & \mathbf{d}
\end{array}$$
- Now the paths cd, dc are well-defined. For each infinite subset S of $\mathbb{N}$, define the infinite **irrational path** $p_S = d_1 d_2 d_3 \cdots$ where $d_i = c$, if $i \in S$ and $d_i = d$ if $i \notin S$. If $T \neq S$ is another infinite subset of $\mathbb{N}$, then p_T is not tail-equivalent to p_S and $V_{[p_S]} \ncong_{gr} V_{[p_T]}$. Thus, we get uncountably many infinite irrational paths no two of which are tail-equivalent

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- **Corollary 10**: If $L_K(E)$ has, up to graded isomorphism, only one graded-simple module, then the graph E is row-finite, that is, E has no infinite emitters.
- **Outline of the proof**: Suppose, on the contrary, E contains an infinite emitter w. Then, for each of the infinitely many edges e_i with $s(e_i) = w$, by Lemma 9, there is a path from $r(e_i)$ to w, thus giving rise to a closed path c_i. Thus we get infinitely many closed path c_i all sharing a common vertex w. By Lemma 8 $L_K(E)$ has infinitely many graded-simple modules which are not graded isomorphic, a contradiction.

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- *(c) L is a graded-simple ring with $L = Soc_{gr}(L)$.*

LPAs with countably many graded-simple modules

- **Lemma 12:** If $L = L_K(E)$ has at most countably many graded-simple left modules, no two of which are graded-isomorphic, then we have the following.

(i) E contains line points or Laurent vertices or both.

(ii) Distinct cycles in E are disjoint, that is, they share no common vertex.

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- *(a) There are at most countably many isomorphism classes of graded-simple left L-modules.*
- *(b) L is "graded semi-artinian", being the union of a smooth well-ordered chain of graded ideals*

$$0 = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_\alpha \subseteq I_{\alpha+1} \subseteq \cdots \qquad (\alpha < \tau)$$

where τ is a finite or countable ordinal and for each $0 \leq \alpha < \tau$, $I_{\alpha+1}/I_\alpha = Soc_{gr}(L/I_\alpha)$.

EXAMPLE-1

The graph E_1 below satisfies the conditions of Theorem 11: E_1 is downward directed, v_{11} is a Laurent vertex and its hereditary saturated closure is $(E_1)^0 = \{v_{11}, v_{12}, v_{13}\}$

$$\begin{tikzcd}
 & & & \cdot v_{14} \arrow[d] \\
\cdot v_{11} \arrow[r] & \cdot v_{12} \arrow[r] & \cdot v_{13} \arrow[ur] & \cdot v5 \arrow[l]
\end{tikzcd}$$

Now $L_K(E_1) =< v_{11>}$. By Theorem 12, $L_K(E_1) = Soc_{gr}(L_K(E_1)) \cong_{gr} M_3(K[x, x^{-1}])$, a graded direct sum of three isomorphic copies of $K[x, x^{-1}]$. There is exactly one graded-isomorphism class of graded-simple $L_K(E_1)$-modules.

EXAMPLE-2

The Leavitt path algebra of the following graph E_2 has exactly two graded-isomorphism classes of graded-simple $L_K(E_2)$-modules
There are paths connecting the vertices v_{21}, v_{22}, v_{23} to the vertex v_{11}.

$$
\begin{array}{ccccccc}
 & & & & & & \cdot v_{15} \\
 & & & & & \nearrow & \downarrow \\
\cdot v_{11} & \longrightarrow & \cdot v_{12} & \longrightarrow & \cdot v_{13} & \longleftarrow & \cdot v_{16} \\
\uparrow & \nwarrow & \nwarrow & & & & \cdot v_{25} \\
\uparrow & & \nwarrow & \nwarrow & & \nearrow & \downarrow \\
\cdot v_{21} & \longrightarrow & \cdot v_{22} & \longrightarrow & \cdot v_{23} & \longleftarrow & \cdot v_{26}
\end{array}
$$

Here v_{11} is a Laurent vertex. By Theorem 12, $I_1 =< v_{11} >$ is the graded-socle and $L_K(E_2))/I_1 \cong L_K(E_1)$.. Then we have the chain of graded ideals $0 = I_0 \subseteq I_1 \subseteq I_2 = L_K(E)$. Here $I_1 = soc_{gr}(L_K(E_2)$ and $L_K(E_2))/I_1 \cong L_K(E_1)$. So $Soc_{gr}(L_K(E_2)/I_1 = L_K(E_2)/I_1$.

EXAMPLE-3

The Leavitt path algebra of the following graph E_3 has exactly three graded-isomorphism classes of graded-simple $L_K(E_3)$-module

$$
\begin{array}{ccccccc}
 & & & & & & \cdot \\
 & & & & & \nearrow & \downarrow \\
\cdot_{v_{11}} & \longrightarrow & \cdot & \longrightarrow & \cdot & \longleftarrow & \cdot \\
\uparrow & \nwarrow & \nwarrow & & & & \cdot \\
\uparrow & & \nwarrow & \nwarrow & & \nearrow & \downarrow \\
\cdot_{v_{21}} & \longrightarrow & \cdot & \longrightarrow & \cdot & \longleftarrow & \cdot \\
\uparrow & \nwarrow & \nwarrow & & & & \cdot \\
\uparrow & & \nwarrow & \nwarrow & & \nearrow & \downarrow \\
\cdot_{v_{31}} & \longrightarrow & \cdot & \longrightarrow & \cdot & \longleftarrow & \cdot
\end{array}.
$$

Example-4

Continue the above process, to obtain an ascending chain of graphs $E_1 \subsetneqq E_2 \subsetneqq E_3 \subsetneqq \cdots \subsetneqq E_n \subsetneqq E_{n+1} \subsetneqq \cdots$ by identifying E_1 with the first row of E_2, identifying E_2 with first two rows of E_3,..., E_n with the first n rows of $E_{n+1}, \ldots$. Then, for each $n \geq 1$, the Leavitt path algebra $L_K(E_n)$ has exactly n distinct graded-isomorphism classes of graded-simple modules. If we define $E_\omega = \cup_{n\geq 1} E_n$, then $L_K(E_\omega)$ will have a countable number of graded-simple modules, no two of which are graded-isomorphic.

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THANK YOU

Matrix grading

Let Γ be an additive abelian group and $A = \oplus_{\lambda\in\Gamma} A_\lambda$ be a Γ-graded ring. Fix an n-tuple $(\delta_1, \cdots, \delta_n)$, where $\delta_i \in \Gamma$. Then the matrix ring $M_n(A)$ can be made a Γ-graded ring $M_n(A) = \oplus_{\lambda\in\Gamma} M_n(A)_\lambda(\delta_1, \cdots, \delta_n)$ by defining, for each $\lambda \in \Gamma$, its λ-homogeneous component as the following additive subgroup $M_n(A)_\lambda(\delta_1, \cdots, \delta_n)$ of $M_n(A)$ consisting of $n \times n$ matrices $M_n(A)_\lambda(\delta_1, \cdots \delta_n) =$

$$\begin{pmatrix} A_{\lambda+\delta_1-\delta_1} & A_{\lambda+\delta_2-\delta_1} & \cdot \quad \cdot \quad \cdot & A_{\lambda+\delta_n-\delta_1} \\ A_{\lambda+\delta_2-\delta_2} & A_{\lambda+\delta_2-\delta 2} & & A_{\lambda+\delta_n-\delta_2} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ A_{\lambda+\delta_1-\delta_n} & A_{\lambda+\delta_2-\delta_n} & & A_{\lambda+\delta_n-\delta_n} \end{pmatrix} \tag{1}$$

This shows that for each homogeneous element $x \in A$, the degree of the matrix $e_{ij}(x)$ with x in the ij-position and with every other entry 0 is given by

$$\deg(e_{ij}(x)) = \deg(x) + \delta_i - \delta_j. \tag{2}$$

Next, let A be a Γ-graded ring and let I be an arbitrary infinite index set. Consider the ring $M_I(A)$ consisting of square matrices whose rows and columns are indexed by I such that at most finitely many entries are non-zero. Now fix a "vector" $\bar{\delta} = (\cdots, \delta_i, \cdots)_{i \in I}$ where $\delta_i \in \Gamma$. Following the same way used for grading matrices of finite order, $M_I(A)$ can be made a Γ-graded ring. As before, for each $(i, j) \in I \times I$ and homogeneous element $x \in A$, $e_{ij}(x)$ denotes the matrix having x at the ij position and every other entry being 0. Under the stated grading, $\deg(e_{ij}(x)) = \deg(x) + \delta_i - \delta_j$. This makes $M_I(A)$ a Γ-graded ring which we denote by $M_I(A)(\bar{\delta})$.