Expanders and K-theory for group C^* algebras

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EXPANDERS AND MORITA-COMPATIBLE EXACT CROSSED PRODUCTS

— OR —

THERE ARE NO KNOWN COUNTER-EXAMPLES TO BC

An expander or expander family is a sequence of finite graphs $X_1, X_2, X_3, \ldots$ which is efficiently connected. A discrete group G which contains an expander as a sub-graph of its Cayley graph is a counter-example to the Baum-Connes (BC) conjecture with coefficients. Some care must be taken with the definition of "contains". It is sufficient that there exists a coarse embedding of an expander into the Cayley graph of G. M. Gromov outlined a method for constructing such a group. G. Arjantseva and T. Delzant completed the construction. Any group so obtained is known as a Gromov group (or Gromov monster). More recently examples have been constructed by Damian Osajda which truly have an expander as a sub-graph of the Cayley graph. These more recent examples and the Gromov groups are the only known examples of non-exact groups.

The left side of BC with coefficients “sees” any group as if the group were exact. This talk will indicate how to make a change in the right side of BC with coefficients so that the right side also “sees” any group as if the group were exact. This corrected form of BC with coefficients uses the unique minimal exact and Morita-compatible intermediate crossed product. For exact groups (i.e. all groups except the Gromov groups and the more recent Osajda examples) the corrected conjecture coincides with the original conjecture — i.e. for any exact group the unique minimal exact and Morita-compatible intermediate crossed product is the reduced crossed product.

In the corrected form of BC with coefficients any Gromov group or Osajda group acting on the coefficient algebra obtained from its expander is not a counter-example.

Thus at the present time (July, 2026) there is no known counter-example to the corrected form of BC with coefficients.

The above is joint work with E. Guentner and R. Willett.
This work is based on — and inspired by — a result of R. Willett and G. Yu, and is very closely connected to results in the thesis of M. Finn-Sell.

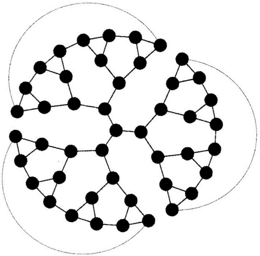
A discrete group Γ which "contains" an expander in its Cayley graph is a counter-example to the usual (i.e. uncorrected) BC conjecture with coefficients.

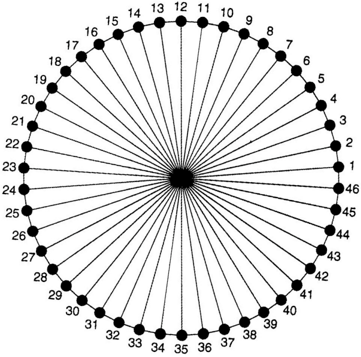
Some care must be taken with the definition of "contains".

An expander or expander family is a sequence of finite graphs

$$X_1, X_2, X_3, \ldots$$

which is *efficiently connected*.





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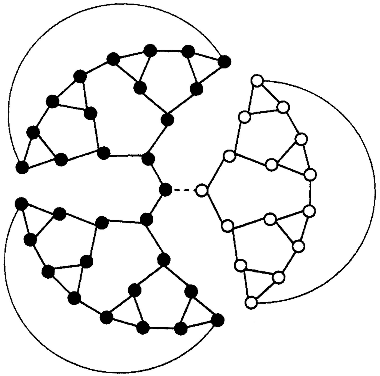

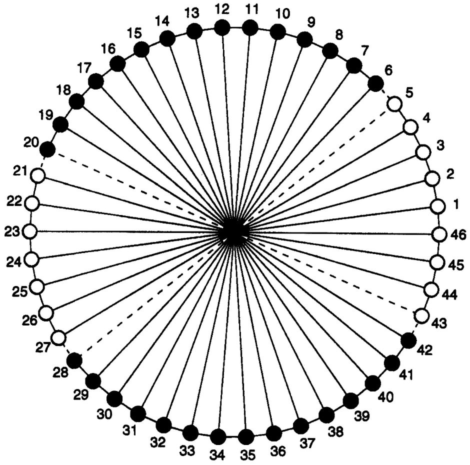
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The isoperimetric constant $h(X)$

For a finite graph X with vertex set V and $|V|$ vertices

$$h(X) =: min\{\frac{|\partial F|}{|F|} \quad | \quad F \subset V \text{ and } |F| \leq \frac{|V|}{2}\}$$

$|F|$ = number of vertices in F. $\quad F \subset V$.
$|\partial F|$ = number of edges in X having one vertex in F and one vetex in $V - F$.

Definition of expander

An expander is a sequence of finite graphs

$$X_1, X_2, X_3, \ldots$$

such that

- Each X_j is conected.
- $\exists$ a positive integer d such that all the X_j are d-regular.
- $|X_n| \to \infty$ as $n \to \infty$.
- $\exists$ a positive real number $\epsilon > 0$ with
  $h(X_j) \geq \epsilon > 0 \qquad \forall j = 1, 2, 3, \ldots.$

Definition

A Banach algebra is an algebra A over $\mathbb{C}$ with a given norm $\| \ \|$

$$\| \ \| : A \to \{t \in \mathbb{R} \mid t \geqq 0\}$$

such that A is a complete normed algebra:

$\|\lambda a\| = |\lambda| \|a\| \quad \lambda \in \mathbb{C}, \quad a \in A$

$\|a + b\| \leq \|a\| + \|b\| \quad a, b \in A$

$\|ab\| \leq \|a\| \|b\| \qquad a, b \in A$

$\|a\| = 0 \iff a = 0$

Every Cauchy sequence is convergent in A (with respect to the metric $\|a - b\|$).

$A \quad C^*$ algebra $A = (A, \| \, \|, *)$ $\qquad a \mapsto a^* \qquad *\colon A \to A$

$(A, \| \, \|)$ is a Banach algebra

$(a^*)^* = a$

$(a+b)^* = a^* + b^*$

$(ab)^* = b^* a^*$

$(\lambda a)^* = \overline{\lambda} a^* \quad a, b \in A, \quad \lambda \in \mathbb{C}$

$\|aa^*\| = \|a\|^2 = \|a^*\|^2$

A $*$-homomorphism is an algebra homomorphism $\varphi\colon A \to B$ such that $\varphi(a^*) = (\varphi(a))^* \ \forall a \in A$.

Lemma

If $\varphi\colon A \to B$ is a $*$-homomorphism, then $\|\varphi(a)\| \leqq \|a\| \ \forall a \in A$.

EXAMPLES OF C^* ALGEBRAS

example X topological space, Hausdorff, locally compact

$$\begin{aligned}
X^+ &= \text{ one-point compactification of } X \\
&= X \cup \{p_\infty\} \\
C_0(X) &= \big\{\alpha\colon X^+ \to \mathbb{C} \mid \alpha \text{ continuous}, \alpha(p_\infty) = 0\big\} \\
\|\alpha\| &= \sup_{p \in X} |\alpha(p)| \\
\alpha^*(p) &= \overline{\alpha(p)} \\
(\alpha + \beta)(p) &= \alpha(p) + \beta(p) \quad p \in X \\
(\alpha\beta)(p) &= \alpha(p)\beta(p) \\
(\lambda\alpha)(p) &= \lambda(\alpha(p)) \quad \lambda \in \mathbb{C}
\end{aligned}$$

If X is compact Hausdorff, then

$$C_0(X) = C(X) = \{\alpha\colon X \to \mathbb{C} \mid \alpha \text{ continuous}\}$$

example

H Hilbert space

$$\mathcal{L}(H) = \{\text{bounded operators } T\colon H \to H\}$$

$$\|T\| = \sup_{\substack{u\in H \\ \|u\|=1}} \|Tu\| \qquad \underset{\|u\|=\langle u,u\rangle^{1/2}}{\text{operator norm}}$$

$$T^* = \text{ adjoint of } T \qquad \underset{u,v\in H}{\langle Tu, v\rangle = \langle u, T^*v\rangle}$$

$$(T+S)u = Tu + Su$$

$$(TS)u = T(Su)$$

$$(\lambda T)u = \lambda(Tu) \qquad \lambda \in \mathbb{C}$$

G topological group

G is assumed to be :
locally compact, Hausdorff, and second countable.

(second countable $=$ The topology of G has a countable base.)

Examples

Lie groups	$\mathsf{SL}(n, \mathbb{R})$
p-adic groups	$\mathsf{SL}(n, \mathbb{Q}_p)$
adelic groups	$\mathsf{SL}(n, \mathbb{A})$
discrete groups	$\mathsf{SL}(n, \mathbb{Z})$

G topological group

locally compact, Hausdorff, and second countable

example C_r^*G, the reduced C^* algebra of G

Fix a left-invariant Haar measure dg for G

"left-invariant" $=$ whenever $f\colon G \to \mathbb{C}$ is continuous and compactly supported

$$\int_G f(\gamma g)dg = \int_G f(g)dg \qquad \forall \gamma \in G$$

L^2G Hilbert space

$L^2G = \{u\colon G \to \mathbb{C} \mid \int_G |u(g)|^2 dg < \infty\}$

$\langle u, v\rangle = \int_G \overline{u(g)} v(g) dg \qquad u, v \in L^2G$

$\mathcal{L}(L^2G) = C^*$ algebra of all bounded operators $T\colon L^2G \to L^2G$

$C_cG = \{f\colon G \to \mathbb{C} \mid f$ is continuous and f has compact support$\}$

C_cG is an algebra

$(\lambda f)g = \lambda(fg) \qquad \lambda \in \mathbb{C} \quad g \in G$

$(f+h)g = fg + hg$

Multiplication in C_cG is convolution

$$(f * h)g_0 = \int_G f(g)h(g^{-1}g_0)dg \qquad g_0 \in G$$

$0 \to C_c G \to \mathcal{L}(L^2 G)$

Injection of algebras

$f \mapsto T_f$

$T_f(u) = f * u \qquad u \in L^2 G$

$(f * u) g_0 = \int_G f(g) u(g^{-1} g_0) dg \qquad g_0 \in G$

$C_r^* G \subset \mathcal{L}(L^2 G)$

$C_r^* G = \overline{C_c G} =$ closure of $C_c G$ in the operator norm

$C_r^* G$ is a sub C^* algebra of $\mathcal{L}(L^2 G)$

A C^* algebra (or a Banach algebra) with unit 1_A.

Define abelian groups $K_1A, K_2A, K_3A, ...$ as follows :

$\mathrm{GL}(n, A)$ is a topological group.

The norm $\| \;\|$ of A topologizes $\mathrm{GL}(n, A)$.

$\mathrm{GL}(n, A)$ embeds into $\mathrm{GL}(n+1, A)$.

$$\mathrm{GL}(n, A) \hookrightarrow \mathrm{GL}(n+1, A)$$

$$\begin{bmatrix} a_{11} & ... & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & ... & a_{nn} \end{bmatrix} \mapsto \begin{bmatrix} a_{11} & ... & a_{1n} & 0 \\ \vdots & & \vdots & \vdots \\ a_{n1} & ... & a_{nn} & 0 \\ 0 & ... & 0 & 1_A \end{bmatrix}$$

$$\mathrm{GL}\,A = \lim_{n\to\infty} \mathrm{GL}(n, A) = \bigcup_{n=1}^{\infty} \mathrm{GL}(n, A)$$

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Give $\mathrm{GL}\,A$ the direct limit topology.

This is the topology in which a set $U \subset \mathrm{GL}\,A$ is open if and only if $U \cap \mathrm{GL}(n, A)$ is open in $\mathrm{GL}(n, A)$ for all $n = 1, 2, 3, \ldots$

A C^* algebra (or a Banach algebra) with unit 1_A

$K_1A, K_2A, K_3A, \ldots$

Definition

$K_jA := \pi_{j-1}(\mathrm{GL}\,A)$ $\qquad$ $j = 1, 2, 3, \ldots$

$\Omega^2\,\mathrm{GL}\,A \sim \mathrm{GL}\,A$ $\qquad$ Bott Periodicity

$K_jA \cong K_{j+2}A$ $\qquad$ $j = 0, 1, 2, \ldots$

$K_0A \qquad K_1A$

A C^* algebra (or a Banach algebra) with unit 1_A

$A = (A, \| \|, *)$

For K_0A forget $\| \|$ and $*$. View A as a ring with unit.

$K_0A = K_0^{alg}A =$ Grothendieck group of finitely generated (left) projective A-modules

For K_1A cannot forget $\| \|$ and $*$.

$K_0A \qquad\qquad K_1A$

A C^* algebra (or a Banach algebra) with unit 1_A

The Bott periodicity isomorphism

$K_0A = \widehat{J(A)} \longrightarrow K_2A = \pi_1 GLA$

assigns to $\alpha \in P_n(A)$ the loop of $n \times n$ invertible matrices

$t \mapsto I + (e^{2\pi it} - 1)\alpha \qquad\qquad t \in [0,1]$

$I =$ the $n \times n$ identity matrix

A $\quad C^*$ algebra (or a Banach algebra)

If A is not unital, adjoin a unit.

$0 \longrightarrow A \longrightarrow \tilde{A} \longrightarrow \mathbb{C} \longrightarrow 0$

Define: $K_j A = K_j \tilde{A}$ $\qquad j = 1, 3, 5, \ldots$

$K_j A =$ Kernel$(K_j \tilde{A} \longrightarrow K_j \mathbb{C})$ $\qquad j = 0, 2, 4, \ldots$

$K_j A \cong K_{j+2} A$ $\qquad j = 0, 1, 2, \ldots$

$K_0 A \qquad K_1 A$

FUNCTORIALITY OF K-THEORY

A, B $\quad\quad$ C^* algebras

$\varphi : A \longrightarrow B$ $\quad\quad$ $*$- homomorphism

$\varphi_* : K_j A \longrightarrow K_j B$ $\quad\quad$ $j = 0, 1$

K-theory can be applied to classification problems for C^* algebras. See results of G. Elliott and others.

Ordinary BC and BC with coefficients are for topological groups G which are locally compact, Hausdorff, and second countable.

$\underline{E}G$ denotes the universal example for proper actions of G.

EXAMPLE. If Γ is a (countable) discrete group, then $\underline{E}\Gamma$ can be taken to be the convex hull of Γ within $l^2(\mathit{\Gamma})$.

Example

Give Γ the measure in which each $\gamma \in \Gamma$ has mass one.
Consider the Hilbert space $l^2(\Gamma)$.
Γ acts on $l^2(\Gamma)$ via the (left) regular representation of Γ.
Γ embeds into $l^2(\Gamma)$ $\quad \Gamma \hookrightarrow l^2(\Gamma)$
$\gamma \in \Gamma \quad \gamma \mapsto [\gamma]$ where $[\gamma]$ is the Dirac function at γ.
Within $l^2(\Gamma)$ let Convex-Hull(Γ) be the smallest convex set which contains Γ. The points of Convex-Hull(Γ) are all the finite sums

$$t_0[\gamma_0] + t_1[\gamma_1] + \cdots + t_n[\gamma_n]$$

$$\text{with} \quad t_j \in [0,1] \quad j = 0,1,\ldots,n \quad \text{and} \quad t_0 + t_1 + \cdots + t_n = 1$$

The action of Γ on $l^2(\Gamma)$ preserves Convex-Hull(Γ).
$\Gamma \times$Convex-Hull(Γ) $\longrightarrow$Convex-Hull(Γ)
$\underline{E}\Gamma$ can be taken to be Convex-Hull(Γ) with this action of Γ.

Let X be a paracompact Hausdorff topological space with a given continuous action of G on X.

$$G \times X \longrightarrow X$$

The action of G on X is proper if :

The quotient space X/G (with the quotient topology) is paracompact Hausdorff and

For each $x \in X, \exists$ a triple (U, H, φ) such that:

- U is an open set of X with $x \in U$ and with $gp \in U$ whenever $g \in G$ and $p \in U$.
- H is a compact subgroup of G.
- $\varphi\colon U \to G/H$ is a continuous G-equivariant map from U to G/H.

$\underline{E}G$ is a paracompact Hausdorff topological space with a given proper action of G :

$$G \times \underline{E}G \longrightarrow \underline{E}G$$

such that whenever X is a paracompact Hausdorff topological space with a given proper action of G on X

- $\exists$ a continuous G-equivariant map $f\colon X \to \underline{E}G$.
- Any two continuous G-equivariant maps $f_0\colon X \to \underline{E}G$, $f_1\colon X \to \underline{E}G$ are homotopic through continuous G-equivariant maps.

$K_j^G(\underline{E}G)$ denotes the Kasparov equivariant K-homology — with G-compact supports — of $\underline{E}G$.

Definition

A closed subset Δ of $\underline{E}G$ is G-compact if:

1. The action of G on $\underline{E}G$ preserves Δ.
and
2. The quotient space Δ/G (with the quotient space topology) is compact.

Definition

$$K_j^G(\underline{E}G) = \varinjlim_{\substack{\Delta \subset \underline{E}G \\ \Delta\ G\text{-compact}}} KK_G^j(C_0(\Delta), \mathbb{C}).$$

The direct limit is taken over all G-compact subsets Δ of $\underline{E}G$.

$K_j^G(\underline{E}G)$ is the Kasparov equivariant K-homology of $\underline{E}G$ with G-compact supports.

Ordinary BC

Conjecture

For any G which is locally compact, Hausdorff and second countable

$$\mathrm{K}_j^G(\underline{E}G) \to \mathrm{K}_j(C_r^* G) \qquad j = 0, 1$$

is an isomorphism

Corollaries of BC

Novikov conjecture $=$ homotopy invariance of higher signatures

Stable Gromov Lawson Rosenberg conjecture (Hanke + Schick)

Idempotent conjecture

Kadison Kaplansky conjecture

Mackey analogy (Higson)

Exhaustion of the discrete series via Dirac induction
(Parthasarathy, Atiyah + Schmid, V. Lafforgue)

Homotopy invariance of ρ-invariants
(Keswani, Piazza + Schick)

G topological group
locally compact, Hausdorff, second countable

Examples

Lie groups ($\pi_0(G)$ finite)	SL$(n,\mathbb{R})$ OK✓
p-adic groups	SL$(n,\mathbb{Q}_p)$OK✓
adelic groups	SL$(n,\mathbb{A})$OK✓
discrete groups	SL$(n,\mathbb{Z})$

Let A be a $G - C^*$ algebra i.e. a C^* algebra with a given continuous action of G by automorphisms.

$$G \times A \longrightarrow A$$

BC with coefficients

Conjecture

For any G which is locally compact, Hausdorff, and second countable and any $G - C^*$ algebra A

$$\mathrm{K}_j^G(\underline{E}G, A) \to \mathrm{K}_j(C_r^*(G, A)) \qquad j = 0, 1$$

is an isomorphism.

Definition

$$K_j^G(\underline{E}G, A) = \varinjlim_{\substack{\Delta\subset\underline{E}G \\ \Delta\ G\text{-compact}}} KK_G^j(C_0(\Delta)\,,\,A).$$

The direct limit is taken over all G-compact subsets Δ of $\underline{E}G$.

$K_j^G(\underline{E}G, A)$ is the Kasparov equivariant K-homology of $\underline{E}G$ with G-compact supports and with coefficient algebra A.

THEOREM [N. Higson + G. Kasparov] Let Γ be a discrete (countable) group which is amenable or a-t-menable, and let A be any $\Gamma - C^*$algebra. Then

$$\mu\colon K_j^{\Gamma}(\underline{E}\Gamma, A) \to K_j C_r^*(\Gamma, A)$$

is an isomorphism. $j = 0, 1$

THEOREM [V. Lafforgue] Let Γ be a discrete (countable) group which is hyperbolic (in Gromov's sense), and let A be any $\Gamma - C^*$algebra. Then

$$\mu\colon K_j^{\Gamma}(\underline{E}\Gamma, A)) \to K_j C_r^*(\Gamma, A)$$

is an isomorphism. $j = 0, 1$

$\mathrm{SL}(3,\mathbb{Z})$??????

Basic property of C^* algebra K-theory

SIX TERM EXACT SEQUENCE

Let

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

be a short exact sequence of C^* algebras.

Then there is a six term exact sequence of abelian groups

$$\begin{tikzcd}
K_0 I \arrow[r] & K_0 A \arrow[r] & K_0 B \arrow[d] \\
K_1 B \arrow[u] & K_1 A \arrow[l] & K_1 I \arrow[l]
\end{tikzcd}$$

DEFINITION. G is *exact* if whenever

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

is an exact sequence of $G - C^*$ algebras, then

$$0 \longrightarrow C_r^*(G, I) \longrightarrow C_r^*(G, A) \longrightarrow C_r^*(G, B) \longrightarrow 0$$

is an exact sequence of C^* algebras.

LEMMA. Let

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

be an exact sequence of $G - C^*$ algebras. Assume that G is exact. Then there is a six term exact sequence of abelian groups

$$\begin{tikzcd}
K_0C_r^*(G,I) \arrow[r] & K_0C_r^*(G,A) \arrow[r] & K_0C_r^*(G,B) \arrow[d] \\
K_1C_r^*(G,B) \arrow[u] & K_1C_r^*(G,A) \arrow[l] & K_1C_r^*(G,I) \arrow[l]
\end{tikzcd}$$

The left side of BC with coefficients “sees” any G as if G were exact.

LEMMA. For any locally compact Hausdorff second countable topological group G and any exact sequence

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

of $G - C^*$ algebras,
there is a six term exact sequence of abelian groups

$$\begin{tikzcd}
K_0^G(\underline{E}G, I) \arrow[r] & K_0^G(\underline{E}G, A) \arrow[r] & K_0^G(\underline{E}G, B) \arrow[d] \\
K_1^G(\underline{E}G, B) \arrow[u] & K_1^G(\underline{E}G, A) \arrow[l] & K_1^G(\underline{E}G, I) \arrow[l]
\end{tikzcd}$$

With G fixed, $\{G - C^*$ algebras$\}$ denotes the category whose objects are all the $G - C^*$ algebras.

Morphisms in $\{G - C^*$ algebras$\}$ are $*$-homomorphisms which are G-equivariant.

$\{C^*$ algebras$\}$ denotes the category of C^* algebras.
Morphisms in $\{C^*$ algebras$\}$ are $*$-homomorphisms.

A crossed-product is a functor , denoted $A \mapsto C^*_\tau(G, A)$
from $\{G - C^*$ algebras$\}$ to $\{C^*$ algebras$\}$

$$C^*_\tau \colon \{G - C^*\text{algebras}\} \longrightarrow \{C^*\text{algebras}\}$$

“intermediate” $=$ “between the max and the reduced crossed-product”

For an intermediate crossed-product C^*_τ there are surjections:

$$C^*_{max}(G,A) \longrightarrow C^*_\tau(G,A) \longrightarrow C^*_r(G,A)$$

Denote by $\tau(G,A)$ the kernel of the surjection $C^*_{max}(G,A) \longrightarrow C^*_\tau(G,A)$

$$0 \longrightarrow \tau(G,A) \longrightarrow C^*_{max}(G,A) \longrightarrow C^*_\tau(G,A) \longrightarrow 0$$

is exact.

Denote by $\epsilon(G, A)$ the kernel of $C^*_{max}(G, A) \longrightarrow C^*_r(G, A)$.

$$0 \longrightarrow \epsilon(G, A) \longrightarrow C^*_{max}(G, A) \longrightarrow C^*_r(G, A) \longrightarrow 0$$

is exact.

An intermediate crossed-product C^*_τ is then a function τ which assigns to each $G - C^*$ algebra A a norm closed ideal $\tau(G, A)$ in $C^*_{max}(G, A)$ such that :

- For each $G - C^*$ algebra A, $\tau(G, A) \subseteq \epsilon(G, A)$.
- For each morphism $A \to B$ in $\{G - C^*$ algebras$\}$ the resulting $*$-homomorphism $C^*_{max}(G, A) \to C^*_{max}(G, B)$ maps $\tau(G, A)$ to $\tau(G, B)$.

$$C^*_\tau(G, A) = C^*_{max}(G, A) \,/\, \tau(G, A)$$

An intermediate crossed-product C^*_τ is exact if whenever

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is an exact sequence in $\{G - C^* \text{ algebras}\}$ the resulting sequence in $\{C^* \text{ algebras}\}$

$$0 \longrightarrow C^*_\tau(G, A) \longrightarrow C^*_\tau(G, B) \longrightarrow C^*_\tau(G, C) \longrightarrow 0$$

is exact.

Equivalently :
An intermediate crossed-product C^*_τ is exact if whenever

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is an exact sequence in $\{G - C^* \text{ algebras}\}$ the resulting sequence in $\{C^* \text{ algebras}\}$

$$0 \longrightarrow \tau(G, A) \longrightarrow \tau(G, B) \longrightarrow \tau(G, C) \longrightarrow 0$$

is exact.

Set $H_G = L^2(G) \oplus L^2(G) \oplus \ldots$ $\qquad$ $\mathcal{K}_G = \mathcal{K}(H_G)$

An intermediate crossed product C^*_τ is Morita-compatible if for any $G - C^*$ algebra A the natural isomorphism of C^* algebras

$$C^*_{max}(G, A \otimes \mathcal{K}_G) = C^*_{max}(G, A) \otimes \mathcal{K}_G$$

descends to give an isomorphism of C^* algebras

$$C^*_\tau(G, A \otimes \mathcal{K}_G) = C^*_\tau(G, A) \otimes \mathcal{K}_G$$

QUESTION. Given G, does there exist a unique minimal intermediate crossed product which is exact and Morita-compatible?

PROPOSITION. (E. Kirchberg, P.Baum& E.Guentner& R.Willett) For any locally compact Hausdorff second countable topological group G there exists a unique minimal intermediate crossed product which is exact and Morita-compatible.

Denote the unique minimal intermediate exact and Morita-compatible crossed-product by C^*_{exact}.

Reformulation of BC with coefficients.

CONJECTURE. Let G be a locally compact Hausdorff second countable topological group, and let A be a $G - C^*$ algebra, then

$$K_j^G(\underline{E}G, A) \longrightarrow K_j(C^*_{exact}(G, A)) \qquad\qquad j = 0, 1$$

is an isomorphism.

Theorem (PB and E. Guentner and R. Willett)

Let Γ be a Gromov group and let A be the Γ - C^* algebra obtained by mapping an expander to the Cayley graph of Γ. Then

$$\mathrm{K}_j^{\Gamma}(\underline{E}\Gamma, A) \to \mathrm{K}_j(C^*_{exact}(\Gamma, A)) \qquad j = 0, 1$$

is an isomorphism.

Implications of the corrected conjecture ????

Novikov (homotopy invariance of higher signatures) and stable Gromov-Lawson-Rosenberg are implied by the corrected conjecture.✓

Kadison-Kaplansky : If Γ is a torsion-free discrete group, then in $C_r^*\Gamma$ there are no idempotent elements (other than 0 and 1).

Kadison-Kaplansky is not implied by the corrected conjecture.

Implied by validity of the corrected conjecture:

If Γ is a torsion-free discrete group, then in the Banach algebra $l^1\Gamma$ there are no idempotent elements (other than 0 and 1).

$l^1\Gamma \subset C_r^*\Gamma$

Question. Is it possible for a torsion-free discrete group Γ to have no idempotent elements (other than 0 and 1) in $l^1\Gamma$ — and to have idempotent elements (other than 0 and 1) in $C_r^*\Gamma$?