Laplace operators on quantum graphs

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[A. B., D. Jasiński, and P. Kasprzak, arXiv:2607.18473]

Why Laplacians?

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- Connectivity properties, Graph partitioning and clustering

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- Connectivity properties, Graph partitioning and clustering
- Random walks, Diffusion processes on graphs, Expansion properties
- ...

Why Quantum Graphs?

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- Confusability graphs for quantum communication channels [Duan-Severini-Winter, 2013], Quantum communication capacities

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- Confusability graphs for quantum communication channels [Duan-Severini-Winter, 2013], Quantum communication capacities
- Theory of quantum games and non-local correlations [Mancinska-Roberson, 2016], Entanglement-assisted protocols

Why Laplacians on Quantum Graphs?

Spectral framework to study quantum graphs

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Spectral framework to study quantum graphs

Link between quantum information theory and noncommutative geometry

Classical graph theory - a brief reminder

A finite graph is a pair $G = (V, E)$, where

- the finite set V is referred to as a set of vertices,
- $E \subseteq V \times V$ will be referred to as a set of edges.

The source and target maps are denoted by $s, t : E \to V$ respectively, where $s(i,j) = i$ and $t(i,j) = j$. We write $i \sim j$ to denote $(i,j) \in E$.

A graph G is called:

- directed if $i \sim j$ implies $j \nsim i$ for all $i, j \in V$,
- undirected if $i \sim j$ implies $j \sim i$ for all $i, j \in V$.

Every undirected graph can be upgraded to a directed one by an arbitrary choice of orientation for every edge.

If $(i,i) \in E$, then we say that G has a loop at the vertex i.

Classical graph theory - a brief reminder

Adjacency matrix $A \in \mathrm{Mat}_{|V|}(\mathbb{C})$ of a graph $G = (V, E)$

$$A_{ij} = \begin{cases} 1 & \text{if } (j, i) \in E \text{ (i.e., there is an edge from } j \text{ to } i\text{),} \\ 0 & \text{otherwise.} \end{cases}$$

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Incidence matrix $K \in \mathrm{Mat}_{|E| \times |V|}(\mathbb{C})$ for a directed graph

$$K_{e,v} = \begin{cases} 1 & \text{if } v = t(e), \\ -1 & \text{if } v = s(e), \\ 0 & \text{otherwise.} \end{cases}$$

The incidence matrix of an undirected graph is not uniquely defined: we can fix one by choosing an arbitrary orientation of edges.

Algebraic viewpoint

Incidence map

$\widetilde{K}(x) = [L_x, A], \; x \in \mathrm{C}(V),$ where $L_x : \mathrm{C}(V) \to \mathrm{C}(V)$ is given by $L_x(y) = xy$.

Algebraic viewpoint

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Proposition

- Let $G = (V, E)$ be a directed graph. Then $\widetilde{K}(\mathrm{C}(V)) \subseteq S_A := \mathrm{span}\{E_{ij} : (i,j) \in E\}$, and its coordinate matrix in the basis $\{E_{i,j}\}_{(i,j)\in E}$ is equal to K.
- Let $G = (V, E)$ be an undirected graph with a chosen orientation $E^+ \subset E$. Then $\widetilde{K}(\mathrm{C}(V)) \subseteq S_A^a := \mathrm{span}\{E_{ij} - E_{ji} : (j,i) \in E^+\}$, and its coordinate matrix in the basis $\{E_{ij} - E_{ji}\}_{(j,i)\in E^+}$ is K (under the same orientation).

Edge spaces

Proposition

The number of edges $|E|$ of directed (undirected) graph $G = (V, E)$ is given by any of the following formulas

- $|E| = \dim(S_A)$ $(|E| = \dim(S_A^a))$
- $|E| = \langle \Omega, A\Omega \rangle$ $(|E| = \frac{1}{2}\langle \Omega, A\Omega \rangle)$, where $\langle \cdot, \cdot \rangle$ is canonical scalar product (for the unnormalized trace) and $\Omega = [1, \ldots, 1]^T$.
- $|E| = \mathrm{Tr}(D)$ $(|E| = \frac{1}{2}\mathrm{Tr}(D))$, where D is the degree matrix of G.

Laplace operator

Definition

The Laplacian of a graph is defined by $\Delta := K^{\dagger}K$, where the adjoint † is taken with respect to the Hilbert-Schmidt scalar product on S_A.

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Proposition

The Laplace operator reads

$$\Delta = L_{(A+A^T)\Omega} - (A + A^T).$$

For directed graphs it reduces to $\Delta = D - (A + A^T)$, while for undirected ones we get $\Delta = 2D - 2A$.

Quantum spaces - preliminaries

Gelfand–Naimark theorem

The category of compact Hausdorff spaces $\mathcal{CS}$ is contravariantly equivalent to the category of commutative unital C^*-algebras $\mathcal{CC}^*$.

Quantum spaces

The category of quantum spaces $\mathcal{QS}$ is the opposite category to the category of C^*-algebras, i.e., $\mathcal{QS} := \mathcal{C}^{*\mathrm{op}}$.

GNS representation

Let $\mathcal{M}$ be a finite-dimensional C^*-algebra equipped with a faithful and positive functional $\Psi : \mathcal{M} \to \mathbb{C}$. The vector space $\mathcal{M}$, endowed with the scalar product $\langle m_1, m_2 \rangle_\Psi := \Psi(m_1^* m_2)$, is a finite-dimensional Hilbert space, called the GNS space of $(\mathcal{M}, \Psi)$, and denoted by $L^2(\mathcal{M}, \Psi)$.

Tomita–Takesaki for finite-dimensional algebras

Let $\mathcal{M}$ be a finite-dimensional C^*-algebra equipped with a faithful positive functional Ψ. There exists a unique one-parameter group of automorphisms $\sigma_z \in \mathrm{Aut}(\mathcal{M})$, $z \in \mathbb{C}$, called the modular group of $\mathcal{M}$ with respect to Ψ. This group is uniquely characterized by the KMS condition: $\Psi(ab) = \Psi(b\sigma_{-i}(a))$, for all $a, b \in \mathcal{M}$.

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Multiplication lifted to $L^2(\mathcal{M}, \Psi)$

Given a $(\mathcal{M}, \Psi)$, where $\mathcal{M}$ is a finite-dimensional C^*-algebra and Ψ is a faithful positive functional on $\mathcal{M}$, we define a linear map $m : L^2(\mathcal{M}, \Psi) \otimes L^2(\mathcal{M}, \Psi) \to L^2(\mathcal{M}, \Psi)$ by $m(a \otimes b) := ab$, $\forall a, b \in \mathcal{M} \simeq L^2(\mathcal{M}, \Psi)$, and denote by $m^\dagger$ its adjoint, i.e., the unique operator satisfying

$$\langle v, m(w \otimes u)\rangle_\Psi = \langle m^\dagger(v), w \otimes u\rangle_{\Psi\otimes\Psi} \quad \forall u, v, w \in L^2(\mathcal{M}, \Psi).$$

δ-forms

A positive linear functional $\Psi : \mathcal{M} \to \mathbb{C}$ on a C^*-algebra $\mathcal{M}$ is called a δ-form if $mm^\dagger = \delta^2 \mathrm{id}$.

Quantum system

The pair $(\mathcal{M}, \Psi)$ is called a quantum system if $\mathcal{M}$ is a C^*-algebra and Ψ is a faithful δ-form. If additionally $\delta = 1$, we call the pair $(\mathcal{M}, \Psi)$ a unit quantum system.

Quantum graphs

Definition

A quantum graph is a triple $G = (\mathcal{M}, \Psi, A)$, where:

- $(\mathcal{M}, \Psi)$ is a quantum system with some parameter δ,
- $A \in B(L^2(\mathcal{M}, \Psi))$ is an operator (called adjacency operator) satisfying:

$$(i) \quad m(A \otimes A)m^\dagger = \delta^2 A,$$

$$(ii) \quad A(x)^* = A(x^*) \quad \forall x \in L^2(\mathcal{M}, \Psi),$$

$$(iii) \quad A^\dagger(x)^* = A^\dagger(x^*) \quad \forall x \in L^2(\mathcal{M}, \Psi).$$

If $(\mathcal{M}, \Psi)$ is a unit quantum system, then G is called a unit quantum graph.

If A satisfies condition (ii), we call it $*$-preserving. If it satisfies both (ii) and (iii), it is called real.

Quantum graphs

A quantum graph $G = (\mathcal{M}, \Psi, A)$ is said to be

- undirected, if
$$(\mathrm{id} \otimes \eta^\dagger m)(\mathrm{id} \otimes A \otimes \mathrm{id})(m^\dagger \eta \otimes \mathrm{id}) = A,$$
  where $\eta : \mathbb{C} \ni \lambda \mapsto \lambda \mathbb{1} \in \mathcal{M}$,
- reflexive, if
$$\delta^{-2} m(A \otimes \mathrm{id}) m^\dagger = \mathrm{id},$$
- irreflexive, if
$$\delta^{-2} m(A \otimes \mathrm{id}) m^\dagger = 0.$$

Quantum spaces of edges

Left and right quantum space of edges

$$S_L(G) = \operatorname{span}\left\{\delta^{-2} m(A \otimes X)m^\dagger \mid X \in B(L^2(\mathcal{M}, \Psi))\right\} \subset B(L^2(\mathcal{M}, \Psi))$$
$$S_R(G) = \operatorname{span}\left\{\delta^{-2} m(X \otimes A)m^\dagger \mid X \in B(L^2(\mathcal{M}, \Psi))\right\} \subset B(L^2(\mathcal{M}, \Psi))$$

The following maps are projections:

$$P_A : B(\mathcal{M}) \ni X \mapsto \delta^{-2} m(A \otimes X)m^\dagger \in S_L(G)$$

$$P'_A : B(\mathcal{M}) \ni X \mapsto \delta^{-2} m(X \otimes A)m^\dagger \in S_R(G)$$

However, they are not orthogonal with respect to the Hilbert–Schmidt inner product.

Examples

Example 1

Let $G = (\mathrm{C}([n]), \Psi, A)$ be a classical graph, where:

- $\Psi(x) = \frac{1}{n}\sum_{i\in[n]} x_i$,
- $A \in \mathrm{Mat}_n(\mathbb{C})$ is an adjacency matrix.

We have $m^\dagger(e_i) = n\, e_i \otimes e_i$, where e_i denotes the canonical basis of $\mathrm{C}([n])$, and Ψ is a $\sqrt{n}$-form. Furthermore, condition (i) is equivalent to $A_{ij} \in \{0,1\}$ for all $i,j \in [n]$; condition for being *undirected* ensures symmetry $A = A^T$; the (un)reflexivity conditions agree with the classical notions.

Example 2: Complete quantum graph

For any quantum system $(\mathcal{M}, \Psi)$, the map $A_c := \delta^2\Psi(\cdot)\mathbb{1}$ yields a quantum graph. For a quantum graph $(\mathcal{M}, \Psi, A)$, its complement $(\mathcal{M}, \Psi, A_c - A)$ is again a quantum graph.

Schur product

Definition

$$M_1 \cdot_S M_2 := m(M_1 \otimes M_2)m^\dagger$$

Properties

- Let $a_j, b_j, c_s, d_s \in \mathcal{M}$ for $j \in I$, $s \in S$. Then:
$$m\Big(\sum_j |a_j\rangle\langle b_j| \otimes \sum_s |c_s\rangle\langle d_s|\Big) m^\dagger = \sum_{j,s} |a_j c_s\rangle\langle b_j d_s|.$$
- The Schur product is associative
- $(M_1 \cdot_S M_2)^\dagger = M_1^\dagger \cdot_S M_2^\dagger$

The opposite algebra and its representation

Let $(\mathcal{M}, \Psi)$ be a quantum system with some δ. We define two $*$-homomorphisms

$$\pi_\Psi : \mathcal{M} \to B(L^2(\mathcal{M}, \Psi)), \qquad \pi_\Psi^{\mathrm{op}} : \mathcal{M}^{\mathrm{op}} \to B(L^2(\mathcal{M}, \Psi))$$

by

$$\pi_\Psi(a)b = ab, \qquad \pi_\Psi^{\mathrm{op}}(a)b = b\sigma_{-i/2}(a).$$

The linear functional Ψ^{op} on $\mathcal{M}^{\mathrm{op}}$ defined by $\Psi^{\mathrm{op}}(x^{\mathrm{op}}) = \Psi(x)$ is a faithful δ-form.

Right and left transpositions

Definition

$$M^{TR} := (\mathrm{id} \otimes \eta^\dagger m)(\mathrm{id} \otimes M \otimes \mathrm{id})(m^\dagger \eta \otimes \mathrm{id}),$$
$$M^{TL} := (\eta^\dagger m \otimes \mathrm{id})(\mathrm{id} \otimes M \otimes \mathrm{id})(\mathrm{id} \otimes m^\dagger \eta).$$

Lemma

Let $(\mathcal{M}, \Psi)$ be a quantum system. Then for all families $(a_s), (b_s) \subseteq \mathcal{M}$ indexed by $s \in S$, the following identities hold:

i) $(\mathrm{id} \otimes \eta^\dagger m)(\mathrm{id} \otimes \sum_{s\in S} |a_s\rangle\langle b_s| \otimes \mathrm{id})(m^\dagger \eta \otimes \mathrm{id}) = \sum_{s\in S} |\sigma_{-i}(b_s^*)\rangle\langle a_s^*|$,

ii) $(\eta^\dagger m \otimes \mathrm{id})(\mathrm{id} \otimes \sum_{s\in S} |a_s\rangle\langle b_s| \otimes \mathrm{id})(\mathrm{id} \otimes m^\dagger \eta) = \sum_{s\in S} |b_s^*\rangle\langle \sigma_{-i}(a_s^*)|$.

Right and left transpositions

Properties

Let $(\mathcal{M}, \Psi)$ be a quantum system and $M, M_1, M_2 \in B(L^2(\mathcal{M}, \Psi))$. Then:

i) $(M^{TL})^{TR} = (M^{TR})^{TL} = M$,

ii) $(M_1 \cdot_S M_2)^{TR} = M_2^{TR} \cdot_S M_1^{TR}, \quad (M_1 \cdot_S M_2)^{TL} = M_2^{TL} \cdot_S M_1^{TL}$,

iii) $(M_1 M_2)^{TR} = M_2^{TR} M_1^{TR}, \quad (M_1 M_2)^{TL} = M_2^{TL} M_1^{TL}$,

iv) $\mathrm{Tr}(M) = \mathrm{Tr}(M^{TR}) = \mathrm{Tr}(M^{TL})$,

v) $(M^{TL})^\dagger = (M^\dagger)^{TR}, \quad (M^{TR})^\dagger = (M^\dagger)^{TL}$.

Right and left transpositions

Theorem

Let $(\mathcal{M}, \Psi)$ be a quantum system. Then for any $M \in B(L^2(\mathcal{M}, \Psi))$, any two of the following conditions imply the third:

i) M is right (and left) symmetric: $M = M^{TR}$ (and $M = M^{TL}$),

iI) M is $*$-preserving: $(Ma)^* = M(a^*)$ for all $a \in L^2(\mathcal{M}, \Psi)$,

iii) M is self-adjoint: $M = M^\dagger$.

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iii) M is self-adjoint: $M = M^\dagger$.

As a corollary we get: *If M is real, then $M^\dagger = M^{TR} = M^{TL}$*, and

Corollary

For any quantum graph $(\mathcal{M}, \Psi, A)$, we have $A^\dagger = A^{TL} = A^{TR}$.

Interplay with modular group

Lemma

Let $M \in B(L^2(\mathcal{M}, \Psi))$ be real. Then $\sigma_t \circ M = M \circ \sigma_t$ for all $t \in \mathbb{R}$. In particular, for any quantum graph $(\mathcal{M}, \Psi, A)$, the operators A and $A^\dagger$ commute with σ_t for all $t \in \mathbb{C}$.

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Theorem

Any $*$-preserving $M \in B(L^2(\mathcal{M}, \Psi))$ is real if and only if $[M, \sigma_t] = 0$ for all $t \in \mathbb{R}$.

Edge space projections

Let $(\mathcal{M}, \Psi, A)$ be a quantum graph with $P_A, P'_A : B(L^2(\mathcal{M}, \Psi)) \to B(L^2(\mathcal{M}, \Psi))$, the projections onto the edge spaces.

Theorem

Then $P_A^\dagger = P_{A \circ \sigma_i}$ and $(P'_A)^\dagger = P'_{\sigma_{-i} \circ A}$, where the adjoint is taken with respect to the Hilbert–Schmidt scalar product.

Idea of the proof

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Idea of the proof

It follows from the observation that for a finite-dimensional quantum system $(\mathcal{M}, \Psi)$ and $M, N, Z \in B(L^2(\mathcal{M}, \Psi))$ we have

$$\langle Z, M \cdot_S N\rangle_{HS} = \langle ((M^\dagger)^{TL} \circ \sigma_i) \cdot_S Z, N\rangle_{HS},$$

$$\langle Z, N \cdot_S M\rangle_{HS} = \langle Z \cdot_S (\sigma_{-i} \circ (M^\dagger)^{TL}), N\rangle_{HS}.$$

Edge space projections

Definition

We define the left and right scalar products on $B(L^2(\mathcal{M},\Psi))$ by:

$$\langle X, Y\rangle_L = \mathrm{Tr}((\sigma_i \circ X)^\dagger Y), \quad \langle X, Y\rangle_R = \mathrm{Tr}((X \circ \sigma_{-i})^\dagger Y),$$

for all $X, Y \in B(L^2(\mathcal{M},\Psi))$, with the corresponding adjoints ^{†_L} and ^{†_R}.

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We have $P_A^{\dagger_L} = P_A$ and $(P'_A)^{\dagger_R} = P'_A$.

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Theorem

We have $P_A^{\dagger_L} = P_A$ and $(P'_A)^{\dagger_R} = P'_A$.

Proposition

- π^{op} is an isometry into $(B(L^2(\mathcal{M}, \Psi)), \langle\cdot,\cdot\rangle_L)$,
- π is an isometry into $(B(L^2(\mathcal{M}, \Psi)), \langle\cdot,\cdot\rangle_R)$.

Edge space projections

Using $\pi_\Psi(\mathcal{M})' = \pi_\Psi^{\mathrm{op}}(\mathcal{M}^{\mathrm{op}})$, we recover Weaver's notion of a quantum graph:

Proposition

The left (resp. right) edge space is a closed $\pi_\Psi(\mathcal{M})'$–$\pi_\Psi(\mathcal{M})'$ (resp. $\pi_\Psi(\mathcal{M})$–$\pi_\Psi(\mathcal{M})$) bimodule with respect to the left (resp. right) scalar product.

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Proposition

$$\overline{S_L} := (S_L^\dagger)^{TL} = S_R \text{ and } \overline{S_R} = S_L.$$

As a consequence, any $\pi(\mathcal{M})$–$\pi(\mathcal{M})$ bimodule $S_R \subseteq B(L^2(\mathcal{M}, \Psi))$ defines a $\pi^{\mathrm{op}}(\mathcal{M}^{\mathrm{op}})$–$\pi^{\mathrm{op}}(\mathcal{M}^{\mathrm{op}})$ bimodule S_L (and vice versa).

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Theorem

Let $(\mathcal{M}, \Psi)$ be a quantum system with some δ, and let $S_R \in B(L^2(\mathcal{M}, \Psi))$ be any $\pi(\mathcal{M})$–$\pi(\mathcal{M})$ bimodule. Then there exists a unique $*$-preserving operator $A \in B(L^2(\mathcal{M}, \Psi))$ such that $A \cdot_S A = \delta^2 A$ and $S_R = \pi(\mathcal{M}) A \pi(\mathcal{M})$.

Quantum incidence map

Definition

The left and right incidence operators $\mathcal{K}_L : \mathcal{M}^{\mathrm{op}} \to S_L(G)$ and $\mathcal{K}_R : \mathcal{M} \to S_R(G)$ are defined by

$$\mathcal{K}_L(a) := \delta^{-2}(\pi^{\mathrm{op}}(a)A - A\pi^{\mathrm{op}}(a)),$$

$$\mathcal{K}_R(a) := \delta^{-2}(\pi(a)A - A\pi(a)).$$

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Proposition

$$\mathcal{K}_L(a) = \delta^{-2}A \cdot_S \left(|\sigma_{-i/2}(a)\rangle\langle\mathbb{1}| - |\mathbb{1}\rangle\langle\sigma_{-i/2}(a^*)|\right),$$
$$\mathcal{K}_R(a) = \delta^{-2}\left(|a\rangle\langle\mathbb{1}| - |\mathbb{1}\rangle\langle a^*|\right) \cdot_S A,$$
$$(\mathcal{K}_L)^{\dagger_L}(X) = (\sigma_{i/2} \circ X - \sigma_{-i/2} \circ X^{TL})\mathbb{1},$$
$$(\mathcal{K}_R)^{\dagger_R}(X) = (X - \sigma_i \circ X^{TR})\mathbb{1}.$$

Quantum incidence map

Theorem

For any quantum graph $(\mathcal{M}, \Psi, A)$, the incidence operators satisfy the Leibniz rule:

$$\mathcal{K}_L(a \cdot_{\mathrm{op}} b) = \pi_\Psi^{\mathrm{op}}(a)\mathcal{K}_L(b) + \mathcal{K}_L(a)\pi_\Psi^{\mathrm{op}}(b),$$

$$\mathcal{K}_R(ab) = \pi_\Psi(a)\mathcal{K}_R(b) + \mathcal{K}_R(a)\pi_\Psi(b).$$

Laplace operators

Definition

The left Laplacian $\Delta_L : \mathcal{M}^{\mathrm{op}} \to \mathcal{M}^{\mathrm{op}}$ and the right Laplacian $\Delta_R : \mathcal{M} \to \mathcal{M}$ of a graph $G = (\mathcal{M}, \Psi, A)$ are defined by

$$\Delta_L := \frac{\mathcal{K}_L^{\dagger_L}\mathcal{K}_L}{2}, \qquad \Delta_R := \frac{(\mathcal{K}_R)^{\dagger_R}\mathcal{K}_R}{2}.$$

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$$\Delta_L := \frac{\mathcal{K}_L^{\dagger_L} \mathcal{K}_L}{2}, \qquad \Delta_R := \frac{(\mathcal{K}_R)^{\dagger_R} \mathcal{K}_R}{2}.$$

Theorem

$$2\Delta_L(v) = \delta^{-2} \left((A^\dagger \mathbb{1}) \cdot_{\mathrm{op}} v + v \cdot_{\mathrm{op}} (A\mathbb{1}) - Av - A^\dagger v \right),$$
$$2\Delta_R(v) = \delta^{-2} \left((A^\dagger \mathbb{1}) \cdot v + v \cdot (A\mathbb{1}) - Av - A^\dagger v \right).$$

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$$2\Delta_R(v) = \delta^{-2} \left((A^\dagger \mathbb{1}) \cdot v + v \cdot (A\mathbb{1}) - Av - A^\dagger v \right).$$

Theorem

We have $\Delta_L = J \Delta_R J$ with the antiunitary operator $J(v) = \sigma_{-i/2}(v^*)$. As a consequence, $\mathrm{spec}(\Delta_L) = \mathrm{spec}(\Delta_R)$.

Definition [Matsuda '24]

Let $(\mathcal{M}, \Psi, A)$ be a quantum graph. The operator $\nabla_A : \mathcal{M} \to \mathcal{M} \otimes \mathcal{M}$ is defined by

$$\nabla_A(x) := \delta^{-2}(A^\dagger \otimes \mathrm{id} - \mathrm{id} \otimes A)m^\dagger(x).$$

Comparison to existing notions

Definition [Matsuda '24]

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$$\nabla_A(x) := \delta^{-2}(A^\dagger \otimes \mathrm{id} - \mathrm{id} \otimes A)m^\dagger(x).$$

- We define the linear map $\zeta : B(L^2(\mathcal{M}, \Psi)) \to \mathcal{M} \otimes \mathcal{M}$ by $\zeta(T) := (\mathrm{id} \otimes T)m^\dagger(\mathbb{1})$ for all $T \in B(L^2(\mathcal{M}, \Psi))$.

Comparison to existing notions

Definition [Matsuda '24]

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Proposition

We have $\mathcal{K}_L = \zeta^{-1} \circ \nabla_A \circ \sigma_{-i/2}$. As a result, $\Delta_L = \nabla_A^\dagger \nabla_A$.

Tracial graphs on Mat_2

- Let $\kappa : \mathrm{Mat}_2 \ni E_{ij} \mapsto e_{(i-1)2+j} \in \mathbb{C}^4$ be the vectorization map, where $\{e_k\}$ is the canonical basis of $\mathbb{C}^4$.

Define the set $J^{(1B)} := \{(\alpha, \beta) : (*) \text{ is fulfilled}\}$, where

$$(*) \qquad \begin{array}{lcr} \beta = 1 & \Longrightarrow & \alpha \in [0, \infty) \\ \beta \in [0, 1) & \Longrightarrow & \arg(\alpha) \in [0, \pi) \end{array}$$

Theorem [Kiefer-Schäfer, 26]

Any unit tracial quantum graph on Mat_2 with one quantum edge is isomorphic to either $\left(\mathrm{Mat}_2, 2\mathrm{Tr}, A^{(1A)}\right)$ or exactly one quantum graph of the form $\left(\mathrm{Mat}_2, 2\mathrm{Tr}, A^{(1B)}_{\alpha,\beta}\right)$, where $(\alpha, \beta) \in J^{(1B)}$,

$$\kappa A^{(1A)} \kappa^{-1} := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

Tracial graphs on Mat_2

$$\kappa A^{(1B)}_{\alpha,\beta}\kappa^{-1} := c^{-1}\begin{bmatrix} |\alpha|^2 & \alpha\beta_+ & \overline{\alpha}\beta_+ & \beta_+^2 \\ \alpha\beta_- & |\alpha|^2 & \beta_-\beta_+ & \overline{\alpha}\beta_+ \\ \overline{\alpha}\beta_- & \beta_-\beta_+ & |\alpha|^2 & \alpha\beta_+ \\ \beta_-^2 & \overline{\alpha}\beta_- & \alpha\beta_- & |\alpha|^2 \end{bmatrix},$$

$\beta_\pm = 1 \pm \beta$, and $c = |\alpha|^2 + 1 + \beta^2$.

Tracial graphs on Mat_2

$$\kappa A^{(1B)}_{\alpha,\beta}\kappa^{-1} := c^{-1}\begin{bmatrix} |\alpha|^2 & \alpha\beta_+ & \overline{\alpha}\beta_+ & \beta_+^2 \\ \alpha\beta_- & |\alpha|^2 & \beta_-\beta_+ & \overline{\alpha}\beta_+ \\ \overline{\alpha}\beta_- & \beta_-\beta_+ & |\alpha|^2 & \alpha\beta_+ \\ \beta_-^2 & \overline{\alpha}\beta_- & \alpha\beta_- & |\alpha|^2 \end{bmatrix},$$

$\beta_\pm = 1 \pm \beta$, and $c = |\alpha|^2 + 1 + \beta^2$. Moreover, the graph $\left(\mathrm{Mat}_2, 2\mathrm{Tr}, A^{(1B)}_{\alpha,\beta}\right)$ is:

- undirected if and only if $\alpha \in \mathbb{R}$ and $\beta = 0$,
- never reflexive,
- loop-free if and only if $\alpha = 0$.

Hence, the only loop-free one-edge quantum graph corresponds to the adjacency matrix $A^{(1B)}_{0,0}$.

Tracial graphs on Mat_2

Proposition

We have $\Delta_{\kappa A^{(\mathbf{1}A)}\kappa^{-\mathbf{1}}} = 0$ and

$$\Delta_{\kappa A^{(\mathbf{1}B)}_{\alpha,\beta}\kappa^{-\mathbf{1}}} = c^{-1} \begin{bmatrix} 1+\beta^2 & 0 & 0 & -(1+\beta^2) \\ 0 & (1+\beta)^2 & -(1-\beta^2) & 0 \\ 0 & -(1-\beta^2) & (1-\beta)^2 & 0 \\ -(1+\beta^2) & 0 & 0 & 1+\beta^2 \end{bmatrix}.$$

Therefore, $\operatorname{spec}\left(\Delta_{\kappa A^{(\mathbf{1}B)}_{\alpha,\beta}\kappa^{-\mathbf{1}}}\right) = \left\{0, 0, \frac{2(1+\beta^2)}{c}, \frac{2(1+\beta^2)}{c}\right\}$, and, since 0 is an eigenvalue with multiplicity greater than 1, the graph is not connected.

Tracial graphs on Mat_2

- There is only one disconnected two-edge unit tracial graph.
- All three-edge unit tracial graphs are connected.

Conclusions and outlook

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- We would like to relate our findings to the theory of Dirac operators and spectral noncommutative geometry on quantum graphs.
- One could use methods of spectral geometry to characterize certain aspects of quantum channels.

Noncommutative geometry for quantum information theory?

Thank you for your attention!