Groupoid Models for C*-Algebras: Realizations and Obstructions

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Noncommutative Geometry and Graph Algebras Workshop
Western University - Fields Institute, London, Canada
July 2026

Outline

Dynamical Background: Partial Actions

Definition (Exel, 1994)

A **partial action** of a discrete group Γ on a topological space X consists of:

- Open subsets $\{D_g\}_{g\in\Gamma}$ of X, and
- Homeomorphisms $\theta_g : D_{g^{-1}} \to D_g$

such that $\theta_e = \mathrm{id}_X$ and $\theta_g \circ \theta_h \subseteq \theta_{gh}$ for all $g, h \in \Gamma$.

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- Partial actions generalize global group actions to local symmetries.
- They naturally describe systems where transformations are defined only on local domains.

Transformation Groupoids and Partial Crossed Products

A partial action θ induces a canonical **transformation groupoid** $X \rtimes_\theta \Gamma$:

- **Space:** $X \rtimes_\theta \Gamma = \{(x, g) \in X \times \Gamma : x \in D_{g^{-1}}\}$
- **Operations:** $s(x, g) = x, \quad r(x, g) = \theta_g(x)$
- $(x, g) \cdot (y, h) = (y, gh)$ if $x = \theta_h(y)$
- $(x, g)^{-1} = (\theta_g(x), g^{-1})$

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Associated C^*-Algebra

The reduced C^*-algebra of $X \rtimes_\theta \Gamma$ is the reduced **partial crossed product**:

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Example (Prototypical Example)

All AF-algebras can be realized as partial crossed products by $\mathbb{Z}$:

$$A \cong C_0(X) \rtimes_r \mathbb{Z}$$

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Why Partial Actions?

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- They allow us to import topological dynamics into operator algebra classification.
- They serve as the most natural model for C^*-algebras generated by partial isometries.

Testing Ground: Directed Graph Algebras

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Theorem (Classical Fact)

For any directed graph E, the graph algebra $C^*(E)$ is isomorphic to a partial crossed product:

$$C^*(E) \cong C_0(E^\infty) \rtimes_r \mathbb{F}_{|E^1|}$$

where E^∞ is the boundary path space and $\mathbb{F}_{|E^1|}$ is the free group on the edges.

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Key Takeaway

Every graph algebra admits a natural partial action model!

Higher-Rank Graphs (k-graphs)

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Theorem (B.–Kranz)

Path groupoids G_Λ associated to a broad class of k-graphs admit partial action models:

$$G_\Lambda \cong \Lambda^\infty \rtimes \Gamma \implies C^*(\Lambda) \cong C_0(\Lambda^\infty) \rtimes_r \Gamma$$

In particular, this holds whenever Λ embeds into its fundamental groupoid.

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Significance

Partial action models are completely universal for the entire UCT Kirchberg classification program!

Sketch of Construction for Kirchberg Algebras

How to construct the partial action model $C(X) \rtimes_r \Gamma$ for a UCT Kirchberg algebra A:

1. **Global Stable Models (Victor Wu):** Realize $A \otimes \mathcal{K}$ via global group actions on non-compact spaces:

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2. **Restriction to Partial Actions:** Restrict the global action to a compact open subset $X \subset Y$ to yield an induced partial action on X.

3. K-**Theory Control:** Use pure infiniteness and dynamical comparison to map K-theory generators directly into projections in $C(X)$.

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Our Goal

Identify structural and topological obstructions that prevent a groupoid G (and its C^*-algebra) from arising from a partial action.

Obstruction 1: Topological Connectivity

Definition (Deaconu–Renault Groupoids)

For a local homeomorphism $T : X \to X$, the Deaconu–Renault groupoid is:

$$G(X, T) := \{(x, m - n, y) \in X \times \mathbb{Z} \times X : T^m(x) = T^n(y)\}$$

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Remark

Holds even when $C^*_r(G(X, T))$ is a UCT Kirchberg algebra (e.g., $z \mapsto z^2$ on S^1). Topological connectivity creates an explicit obstruction to partial action models!

Obstruction 2: Asymptotic Spread (HLS & AFS)

Definition (HLS and AFS Groupoids)

Formed by approximations of a residually finite group Γ by its finite quotients Γ/Γ_n (Higson–Lafforgue–Skandalis / Alekseev–Finn-Sell).

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If Γ is not locally finite (e.g., $\Gamma = \mathbb{Z}^n$ or $\mathbb{F}_n$), the HLS and AFS groupoids fail fiber-bound properties required by transformation groupoids:

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Negative Answer to Exel's Question

There exist many explicit étale groupoids that **cannot** be written as $X \rtimes \Gamma$.

Broadening the Model: Twisted Groupoids

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Twisted Étale Groupoids (G, Σ)

- G is a locally compact Hausdorff étale groupoid.
- Σ is a central $\mathbb{T}$-groupoid extension (twist) over G:

$$\mathbb{T} \times G^{(0)} \to \Sigma \to G$$

- Generates the reduced twisted groupoid C^*-algebra $C_r^*(G, \Sigma)$.

Cartan Subalgebras and Renault's Duality

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Theorem (Renault, 2008)

For a C^*-algebra A, there is a 1-to-1 correspondence:

$$\text{Pairs } (A, B) \text{ where } B \subset A \text{ is a Cartan subalgebra}$$
$$\Updownarrow$$
$$\text{Effective twisted étale groupoids } (G, \Sigma)$$

where $A \cong C_r^*(G, \Sigma)$ and $B \cong C_0(G^{(0)})$.

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Moral

Every C^*-algebra with a Cartan subalgebra is a twisted étale groupoid algebra!

Fundamental Question 2

Can **every** C^*-algebra be realized as a reduced twisted étale groupoid algebra $C^*_r(G, \Sigma)$?

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- Twists allow non-trivial K-theory and cohomology classes.
- In [Buss–Sims, 2021], untwisted obstructions were found using opposite algebras ($A \not\cong A^{\mathrm{op}}$), but twists bypass this!
- What about classical self-opposite algebras like $B(\mathcal{H})$?

Main Obstruction Theorem for $B(\mathcal{H})$

Theorem (B.–Garcia-Pacheco, 2026, Adv. Math.)

Let $\mathcal{H}$ be an infinite-dimensional Hilbert space. There is **no** *LC Hausdorff étale groupoid G and* **no** *twist Σ over G such that*

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- First explicit example of a C^*-algebra that cannot be realized by *any* reduced twisted étale groupoid.
- Note: If $\dim(\mathcal{H}) = n < \infty$, $M_n(\mathbb{C})$ is the groupoid algebra of the pair groupoid on n points.

Proof Strategy for $B(\mathcal{H})$: Roadmap

Suppose $C_r^*(G,\Sigma) \cong B(\mathcal{H})$. Consider the canonical diagonal $A := C_0(G^{(0)}) \subseteq C_r^*(G,\Sigma)$ and the faithful conditional expectation $E : C_r^*(G,\Sigma) \to C_0(G^{(0)})$.

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The proof breaks into three main steps:

Step 1 Show that A is a von Neumann subalgebra of $B(\mathcal{H})$.

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Step 3 Exclude both possibilities for X:

- X finite $\implies$ forces a tracial state on $B(\mathcal{H})$.
- X infinite $\implies$ forces block sparsity incompatible with $B(\mathcal{H})$.

Step 1: A is a von Neumann Subalgebra

Proposition (B.–Garcia-Pacheco)

Let M be a von Neumann algebra and $A \subseteq M$ a commutative unital C^*-subalgebra. If there exists a faithful conditional expectation $E : M \to A$, then A is a von Neumann subalgebra of M.

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- Faithfulness gives $E(E(a) - a) = 0 \implies E(a) = a \in A$.
- Thus A is monotone complete. By Kadison's Theorem, A is a von Neumann algebra.

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Corollary

$A \cong \ell^\infty(X)$ for some set $X \cong G^{(0)}$.

Step 3A: Excluding Finite $G^{(0)}$ via Tracial States

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Construction of a Trace

Define $\tau : C_c(G, \Sigma) \to \mathbb{C}$ by

$$\tau(f) := \frac{1}{|G^{(0)}|} \sum_{u \in G^{(0)}} f(u) = \mu \circ E(f)$$

where μ is normalized counting measure on $G^{(0)}$.

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Proposition

If $G^{(0)}$ is finite, τ extends to a **tracial state** *on $C_r^*(G, \Sigma)$.*

Step 3A: Excluding Finite $G^{(0)}$ via Tracial States

Suppose $A \cong \ell^\infty(X)$ with $X \cong G^{(0)}$ finite.

Construction of a Trace

Define $\tau : C_c(G, \Sigma) \to \mathbb{C}$ by

$$\tau(f) := \frac{1}{|G^{(0)}|} \sum_{u \in G^{(0)}} f(u) = \mu \circ E(f)$$

where μ is normalized counting measure on $G^{(0)}$.

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Contradiction for Finite X

$B(\mathcal{H})$ admits **no tracial state** when $\mathcal{H}$ is infinite-dimensional. Thus $G^{(0)}$ cannot be finite!

Step 3B: Infinite $G^{(0)}$ and Block Sparsity

Now suppose X is infinite. Minimal projections $p_x = 1_{\{x\}}$ yield orthogonal decomposition $\mathcal{H} = \bigoplus_{x \in X} \mathcal{H}_x$.

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- Operators in $C_r^*(G, \Sigma)$ lie in the norm closure $\overline{\bigcup_{k \geq 1} \mathcal{S}_k}$ of k-sparse block operators.

Step 3C: The Spreading Operator Contradiction

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Final Contradiction

$V \notin \overline{\bigcup_{k\geq 1} \mathcal{S}_k} \implies C_r^*(G, \Sigma) \neq B(\mathcal{H})$. Thus no such (G, Σ) exists! `

Recent Developments (I): von Neumann & Embeddability

In joint work with Luiz F. Garcia and Tomás Pacheco (in progress), we completely classified which von Neumann algebras can be groupoid C^*-algebras:

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Embeddability & Calkin Algebra

- **Non-embeddability:** Neither $B(\mathcal{H})$ nor the Calkin algebra $\mathcal{C}(\mathcal{H}) = B(\mathcal{H})/\mathcal{K}(\mathcal{H})$ can embed into any $C_r^*(G, \Sigma)$.
- **Full C^*-Algebras:** $C^*(G, \Sigma)$ also cannot be isomorphic to $B(\mathcal{H})$ or $\mathcal{C}(\mathcal{H})$.

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Classification of von Neumann Algebras

The *only* von Neumann algebras realizable as reduced twisted groupoid C^*-algebras $C^*_r(G, \Sigma)$ are the **subhomogeneous** ones:

$$\bigoplus_{i=1}^{k} M_{n_i}(A_i)$$

where each A_i is a commutative von Neumann algebra.

Recent Developments (II): Non-Étale & Simple Separable Case

Non-Étale Groupoids (B.–Garcia–Pacheco, 2026)

- If $C^*_r(G,\Sigma)$ is unital for a LC groupoid G (with Haar system), then G is **necessarily étale**.
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The Simple Separable Case (Fernández Quero, Ioana & Tan, 2026)

- There exist **simple, separable**, non-nuclear, unital C^*-algebras A that **cannot** be realized as $C_r^*(G, \Sigma)$ for any étale groupoid twist Σ.
- These algebras are stably finite, with a unique tracial state, stable rank one, and real rank zero.

Summary of Realizations & Limits

Algebra / Groupoid	Target Model	Realizability
1-graphs	Partial Action $X \rtimes \Gamma$	Yes (Always)
k-graphs	Partial Action $X \rtimes \Gamma$	Yes (Broad class)
UCT Kirchberg	Partial Action $X \rtimes \Gamma$	Yes (Always)
Deaconu-Renault (Conn.)	Partial Action $X \rtimes \Gamma$	No (Topological)
HLS / AFS Groupoids	Partial Action $X \rtimes \Gamma$	No (Fibers)
Coarse Groupoids	Partial Action $X \rtimes \Gamma$	Coarse Embeddings
$B(\mathcal{H})$	Reduced Twisted Étale	**No** (Adv. Math.)
Calkin $\mathcal{C}(\mathcal{H})$	Reduced Twisted Étale	**No** (Adv. Math.)
Non-subhomogeneous vN	Reduced Twisted Étale	**No** (Adv. Math.)

Thank You & References

Thank you for your attention!

Happy 90th Birthday, Paul Baum!

References:

- A. Buss, L. F. Garcia, T. Pacheco. *$B(\mathcal{H})$ is not a twisted groupoid C^*-algebra*. Advances in Mathematics (2026). arXiv:2603.21946
- A. Buss, J. Kranz. *On groupoids beyond partial actions and models for Kirchberg algebras*. (2026). arXiv:2604.17921
- R. Exel. *Partial Dynamical Systems, Fell Bundles and Applications*. AMS (2017).
- J. Renault. *Cartan subalgebras in C^*-algebras*. Irish Math. Soc. Bull. (2008).