Categorical traces and Hopf-cyclic coefficients[1]

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[1]Work in progress, no C^*-algebras ... yet.

Context (with pictures)

- Considering Hopf-cyclic cohomology[2] for a Hopf algebra $H \in \mathcal{B}$ braided, we know to pass to the monoidal category $\mathcal{C} = H_{\mathcal{B}}$-mod. Then the coefficients form the category $HC(\mathcal{C})$, a twisted analogue of the center $\mathcal{Z}(\mathcal{C})$.
- I wanted to emulate the description of the coefficients via the twisted Drinfeld double, and was led to yet another 2-category of "coefficients": the module categories over what turns out to be a monoidal (unlike $HH(\mathcal{C})$ in general) category $HH(\mathcal{B})$.
- The product on $HH(\mathcal{B})$ is known as the annuli stacking product from factorization homology[3] (creates *TQFTs* from algebraic data like $\mathcal{B}$, which makes a $2d$ theory), and stacking annuli leads to the product via the factorization property.

[2] Hajac-Khalkhali-Rangipour-Sommerhäuser
[3] Ayala-Francis

Motivation

- So $HH(\mathcal{B})$ has a natural product given by the braiding on $\mathcal{B}$.
- If $\mathcal{B}$ and $\mathcal{C}$ are dual categories, like $(Rep(G), \otimes)$ and $(\mathrm{Vec}_G, *)$ for a finite group G, then
$$HH(\mathcal{B}) = HH(\mathcal{C}).$$
- The Question: What additional structure on $\mathcal{C}$ (it isn't the braiding, since $\mathcal{C}$ is not braided) remembers the braiding on $\mathcal{B}$ and enables the construction of a product on $HH(\mathcal{C})$?
- Categorical traces will explain everything.[4]

[4] I should have listened to Masoud

Monoidal categories (Why?)

A monoidal category $\mathcal{C}$ has a functor:

$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ with an associator $(X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z)$

and a unit $u : \mathrm{Vec} \to \mathcal{C}$. There are *many* conditions.

- Examples: Vec, $M_n(\mathrm{Vec})$, $Rep(G)$, Vec_G, Vec_G^{ω}[5], $A \curvearrowright pt$, A-mod[6], H-mod, H-comod[7], $Rep(\mathcal{G})$, $\mathrm{Vec}_{\mathcal{G}}$[8].
- Not examples: A-mod[9], $Rep^{\phi}(G)$ or more generally $Rep^{\phi}(\mathcal{G})$[10].

[5] G finite group, ω a 3-cocycle
[6] A commutative
[7] H bialgebra
[8] $\mathcal{G}$ finite groupoid
[9] A noncommutative, though *BimodA* would work
[10] ϕ a 2-cocycle

Duality

Morita equivalence for monoidal categories (instead of algebras).

- Non-trivial examples: $(\mathrm{Vec}, M_n(\mathrm{Vec}))$, $(Rep(G), \mathrm{Vec}_G, S_K(G))$, ($H$-mod, H^*-mod)[11], $(Rep(\mathcal{G}), \mathrm{Vec}_{\mathcal{G}})$.

Focus on (H-mod, H^*-mod):

- $Alg(H\text{-mod}) = Alg(H^*\text{-mod})$

$$\begin{aligned} A \in Alg(H\text{-mod}) &\mapsto A\tilde{\otimes}H \in Alg(H^*\text{-mod}) \\ &\mapsto (A\tilde{\otimes}H)\tilde{\otimes}H^* \in Alg(H\text{-mod}) \end{aligned}$$

- A is Morita equivalent to $(A\tilde{\otimes}H)\tilde{\otimes}H^*$ in H-mod and so in Vec (thanks to the fiber functor).

[11] H a finite dimensional Hopf algebra

Linear algebra review (with a picture)[12]

- $\mathcal{D}$ is a symmetric n-category, $X \in \mathcal{D}$ dualizable object, and $f \in End_{\mathcal{D}}(X)$; let $Tr(X) = Tr(Id_X)$ and $Tr(f) \in End_{\mathcal{D}}(1)$:

$$1 \stackrel{coev}{\to} X \otimes X^* \stackrel{f \otimes Id}{\to} X \otimes X^* \stackrel{\tau}{\to} X^* \otimes X \stackrel{ev}{\to} 1$$

  note that $End_{\mathcal{D}}(1)$ is a symmetric $n-1$-category.

- $Tr(f \otimes g) = Tr(f) \otimes Tr(g)$, $Tr(fg) = Tr(gf)$.
- Let $f \in Hom_{\mathcal{D}}(X, Y)$ if $\exists {}^*f$ $(\exists f^*)$ then we have: $f_* : Tr(X) \to Tr(Y)$ $(f^! : Tr(Y) \to Tr(X))$:

$$f_* : Tr(Id_X) \to Tr({}^*ff) = Tr(f^*f) \to Tr(Id_Y)$$

- Let $f, g \in Hom_{\mathcal{D}}(X, Y), \alpha : f \to g$, if α is suitably dualizable, have $\alpha_* : f_* \to g_*$ $(n \geq 3$ for this).

[12] Toën-Vezzosi, Hoyois-Scherotzke-Sibilla

Examples I

- $\mathcal{D} = \text{Vec}$, V finite dimensional vector space, $f \in End(V)$ then $Tr(f) = \sum_i e^i(f(e_i))$ is the usual trace.
- $\mathcal{D} = Alg(\text{Vec})$: 2-category of algebras, bimodules, bimodule morphisms. Let A be an algebra, M an A-bimodule.
  - A considered as an $(A \otimes A^{op}, 1)$-bimodule is co-evaluation
  - A considered as an $(1, A^{op} \otimes A)$-bimodule is evaluation

    $$Tr(M) = A \otimes_{A \otimes A^{op}} M \otimes A^{op} \otimes_{A \otimes A^{op}} A = M/[M, A] = HH_0(A, M).$$

  - If F is an (A, B)-bimodule, finitely generated and projective over A, then $F_* : HH_0(B) \to HH_0(A)$ is the usual map.

Examples II

- $\mathcal{D} = Alg(Alg(\mathrm{Vec}))$ (this can and will continue ...). Consider these as algebras with extra structure or monoidal categories.
- Let H be a Hopf algebra then it is an object in $\mathcal{D}$ via:
  - Multiplication: $H \otimes H$ as a $(H, H \otimes H)$-bimodule via the comultiplication on the left.
  - Unit: 1 as a $(H, 1)$-bimodule via the counit on the left.

  Or just note that H-mod is a monoidal category.
- Let A be a commutative algebra then it is an object in $\mathcal{D}$ via:
  - Multiplication: A as a $(A, A \otimes A)$-bimodule via the multiplication on the right.
  - Unit: A as a $(A, 1)$-bimodule via the unit on the right.

  Or just note that A-mod is a monoidal category.

Hopf-cyclic Theory (the example continues)

Let $\mathcal{C}$ be a monoidal category, then $Tr(\mathcal{C}) \in Alg(\text{Vec})$ so an algebra (or the category of its modules).

- When $\mathcal{C} = H$-mod we get $Tr(\mathcal{C}) = D^a(H)$, the twisted Drinfeld double (aYD modules).[13]
- Let $A \in Alg(\mathcal{C})$:
  - $mod_{\mathcal{C}} A$ as a $(\mathcal{C}, \text{Vec})$-bimodule category yields $A \in Hom_{\mathcal{D}}(1, \mathcal{C})$
  - $A_* \in Hom_{Alg(\text{Vec})}(1, Tr(\mathcal{C}))$, i.e., $A_* \in Tr(\mathcal{C})$
  - For $M \in Tr(\mathcal{C})$, we have $HC(A, M) = Hom_{Tr(\mathcal{C})}(A_*, M)$.
- Let $S \in A_{\mathcal{C}}$-mod:
  - $S : A \to 1$ in $Hom(1, \mathcal{C})$
  - $S_* : A_* \to 1_*$ so $S_* \in HC(A, 1_*)$.

[13] The stability part of the story is suppressed here.

Algebras in $\mathcal{D}$

Consider $A \in Alg(\mathcal{D})$, there are two ways to go:

- We can compute $Tr_{Alg(\mathcal{D})}(A) \in End_{Alg(\mathcal{D})}(1) = \mathcal{D}$. In fact $Tr_{Alg(\mathcal{D})}(A) = A \otimes_{A \otimes A^{op}} A$.
- We can compute $Tr_{\mathcal{D}}(A) \in End_{\mathcal{D}}(1)$.
  - If all the algebra structure maps of A in $\mathcal{D}$ are right dualizable then m_*, u_*, etc equip $Tr_{\mathcal{D}}(A)$ with an algebra structure.
  - If $\mathcal{D} = Alg(\mathcal{D}')$ then $Alg(End_{\mathcal{D}}(1)) = \mathcal{D}$, so that $Tr_{\mathcal{D}}(A) \in \mathcal{D}$ as well.

Question: $Tr_{Alg(\mathcal{D})}(A) = Tr_{\mathcal{D}}(A)$? No. But maybe sometimes?

The two traces compared

Let $\mathcal{D} = Alg(\text{Vec})$, let H be a Hopf algebra, then $H \in Alg(\mathcal{D})$.

- $Tr_{Alg(\mathcal{D})}(H) = D^a(H)$.
- $Tr_{\mathcal{D}}(H) = H/[H,H]$.
  - If H is only a bi-algebra, then the algebra structure is left dualizable, so that $H/[H,H]$ is a coalgebra (and it is just Δ induced).
  - If H is a finite dimensional Hopf algebra then the algebra structure is right dualizable and $H/[H,H]$ is a Frobenius algebra.
- Let $H = \mathcal{O}_G$ then $Tr_{Alg(\mathcal{D})}(H) = \mathcal{O}_G^{ad} \rtimes G$, $Tr_{\mathcal{D}}(H) = kG$.
- Let $H = kG$ then $Tr_{Alg(\mathcal{D})}(H) = \mathcal{O}_G^{ad} \rtimes G$, $Tr_{\mathcal{D}}(H) = \mathcal{O}_G^{adG}$.

Even more Alg, this is the last time, I promise

Let $\mathcal{D}' = Alg(Alg(\mathrm{Vec}))$ and $\mathcal{D} = Alg(\mathcal{D}')$, this is one more Alg than before. View the objects of $\mathcal{D}$ as monoidal categories (objects of $\mathcal{D}'$) with extra structure.

- Let $\mathcal{C} \in \mathcal{D}$ then $Tr_{\mathcal{D}'}(\mathcal{C})$ is a monoidal category.
- Let $\mathcal{B} \in \mathcal{D}'$ be a braided category, then $\mathcal{B} \in \mathcal{D}$:
  - Multiplication: $\mathcal{B}$ as a $(\mathcal{B}, \mathcal{B} \otimes \mathcal{B})$-bimodule category via the product on the right, and the braiding giving the monoidal structure on the action functor.
  - Unit: $\mathcal{B}$ as a $(\mathcal{B}, \mathrm{Vec})$-bimodule via the unit on the right.
  - Just like for A a commutative algebra.
  - So $HH(\mathcal{B}) = Tr_{\mathcal{D}'}(\mathcal{B})$ is *monoidal*.
- Braided categories arise when the Hopf algebra is quasi-triangular (special case: cocommutative).

Same amount of Alg continued

Recall: $\mathcal{D}' = Alg(Alg(\mathrm{Vec}))$ and $\mathcal{D} = Alg(\mathcal{D}')$.

- Let $\mathcal{H} \in \mathcal{D}'$ be a Hopf category, then $\mathcal{H} \in \mathcal{D}$:
  - Multiplication: $\mathcal{H} \otimes \mathcal{H}$ as a $(\mathcal{H}, \mathcal{H} \otimes \mathcal{H})$-bimodule via Δ on the left.
  - Unit: Vec as a $(\mathcal{H}, \mathrm{Vec})$-bimodule via the counit functor on the left.
  - Just like for H a Hopf algebra.
  - So $HH(\mathcal{H}) = Tr_{\mathcal{D}'}(\mathcal{H})$ is *monoidal*.
- Hopf categories arise when the Hopf algebra is commutative.[14]

[14] The co-quasi-triangular case is much more subtle.

Some answers

Question: What structure on a monoidal category $\mathcal{C}$ ensures that $HH(\mathcal{C})$ is monoidal?

Answer: $\mathcal{C}$ is an *algebra* in $Alg(Alg(\text{Vec}))$, i.e., $\mathcal{C} \in Alg(Alg(Alg(\text{Vec})))$.

Question: Why does a dual of a braided category have a product on its coefficients?

Answer: A braided category is an algebra in $Alg(Alg(\text{Vec}))$, its dual is isomorphic to it in $Alg(Alg(\text{Vec}))$, so it can be given an algebra structure too; and now they are isomorphic algebras. Note that this is stronger than being isomorphic in $Alg(Alg(Alg(\text{Vec})))$, i.e., isomorphic Vs Morita equivalent.

Dual Hopf algebras

Let H be a finite dimensional cocommutative Hopf algebra. Then its dual H^* is a commutative Hopf algebra.

- H and H^* are isomorphic in $Alg(Alg(\mathrm{Vec}))$, indeed Vec is the $(H\text{-mod}, H^*\text{-mod})$-bimodule category that gives the isomorphism.
- H and H^* are both algebras[15] in $Alg(Alg(\mathrm{Vec}))$ and the isomorphism above is that of algebras.
- Thus, their Hopf-cyclic coefficients are equivalent as *monoidal* categories.
- This covers the $Rep(G)$ Vs Vec_G question with $H = kG$ cocommutative.

[15] Since H is cocommutative (braided) and H^* is commutative (Hopf)

Comparing the two traces, again, but now with more *Alg*

Consider the Hopf category $\mathcal{C}$ which is $(\mathrm{Vec}_G, \otimes_G)$ as a monoidal category and with Hopf structure: $\delta_g \mapsto \oplus_{\alpha\beta=g}\delta_\alpha \otimes \delta_\beta$.

- $\mathcal{C}$ is not isomorphic, in $\mathcal{D}'$, to $\mathcal{E}$, the Hopf category $(\mathrm{Vec}_G, *)$ with $\delta_g \mapsto \delta_g \otimes \delta_g$, obtained as the dual of $\mathcal{B}$, the braided category $Rep(G)$; $\mathcal{B}$ and $\mathcal{E}$ are isomorphic in $\mathcal{D}'$, as algebras.
- $Tr_{\mathcal{D}'}(\mathcal{C}) = (\mathrm{Vec}_G, *)$, while $Tr_{\mathcal{D}}(\mathcal{C}) = (\mathrm{Vec}_{G \curvearrowright G^{ad}}, *)$: *not the same*.
- $\mathcal{C} \simeq \mathcal{E} \simeq \mathcal{B}$ in $\mathcal{D}$, so $Tr_{\mathcal{D}}(\mathcal{C}) \simeq Tr_{\mathcal{D}}(\mathcal{E}) \simeq Tr_{\mathcal{D}}(\mathcal{B})$.

$Tr_{\mathcal{D}'}(\mathcal{B}) \simeq (Rep(G \curvearrowright G^{ad}), \otimes) \simeq (\mathrm{Vec}_{G \curvearrowright G^{ad}}, *) = Tr_{\mathcal{D}}(\mathcal{C}) \simeq Tr_{\mathcal{D}}(\mathcal{B})$.

Same!?

New questions, probably known ...

If $\mathcal{B}$ is a rigid braided category then we have

$$Tr_{\mathcal{D}'}(\mathcal{B}) \to Tr_{\mathcal{D}}(\mathcal{B})$$

in fact *this* is what originally led me to consider the annuli stacking product (on the LHS) before I knew of its existence. I actually needed the RHS (before I knew what it was).

- For $\mathcal{B} = Rep(G)$ it is an equivalence, is it always?
- Why not for other algebras in $\mathcal{D}'$?
- Is there at least a map? There is in the examples that we considered, but there are more examples.

Why is this nice?

- Categorical traces and dualizable objects are a big deal, with well developed foundations, that I haven't even begun to understand, with applications to everything.
- There is still a lot more structure that could be explored and applied in this setting.
- I didn't even go beyond 3 Alg's!

Thank you!