THE MAXIMAL DIMENSIONS OF PATH AND GRAPH ALGEBRAS

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29.07.2026

Notation

Definition: Directed Graph

A directed graph is a quadruple $E := (E^0, E^1, s, t)$ consisting of

1. the set of vertices E^0 and the set of edges E^1,
2. the source and target maps $s, t\colon E^1 \to E^0$ assigning to each edge its source and target vertex, respectively.

Definition: Sources and Sinks

A vertex v in a directed graph E is called

- a source if $t^{-1}(\{v\}) = \varnothing$,
- a sink, if $s^{-1}(\{v\}) = \varnothing$.

Notation

Definition: Path

A finite path in a directed graph E is a single vertex or a finite tuple $p_n := (e_1, \ldots, e_n)$ of edges satisfying

$$t(e_1) = s(e_2), \quad t(e_2) = s(e_3), \quad \ldots, \quad t(e_{n-1}) = s(e_n).$$

- The beginning $s(p_n)$ of p_n is $s(e_1)$,
- The end $t(p_n)$ of p_n is $t(e_n)$,
- The length of a path is 0 when it's a vertex or the size of the tuple otherwise,
- Path p_n is called a cycle if $s(p_n) = t(p_n)$,
- A directed graph without cycles is called acyclic,

Notation

Definition: Connectedness

The directed graph (E^0, E^1, s, t) is connected if it is connected as an undirected graph.

Definition: Forests, Trees and Trunks

Let (E^0, E^1, s, t) be an acyclic graph with $N \geqslant 1$ edges, then

- if it has no isolated vertexes we call it a *N-forest graph*,
- if it's a connected N-forest we call it a *N-tree graph*,
- if its set of verteces can indexed by $\{1, \ldots, n\}$ and satisfy:

$$\forall_{e \in E^1}\ \exists_{1 \leqslant i \leqslant n-1}\ s(e) = v_i \text{ and } t(e) = v_{i+1}$$

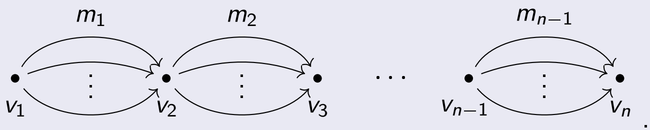
  we call it a *N-trunk graph*.

We denote the set of all forest, tree and trunk graphs with $N \geqslant 1$ edges by $\mathscr{E}^N$, $\mathscr{F}^N$, and $\mathscr{T}^N$, respectively.

Notation

Presentation of trunks

Given a trunk E with verteces $E^0 = \{v_1, \ldots, v_n\}$ we will depict it in the following way:



The trunk graph as in the above picture may be described by its associated sequence $(m_1, \ldots, m_{n-1})$.

Induced graphs: Trunks

If E is a trunk graph with $E^0 = \{v_1, \ldots, v_n\}$ then for $1 \leqslant i \leqslant j \leqslant n$ let $E_{i,j}$ denote a full subgraph of E induced by verteces $\{v_i, \ldots, v_j\}$.

Calculating the number of paths

A. Chirvasitu, P.M. Hajac 2019

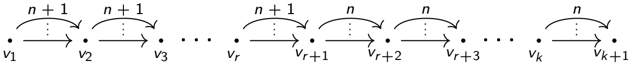
Let E be an acyclic directed graph with $N \geqslant 1$ edges, and let

$$1 \leqslant k \leqslant N =: nk + r, \quad 0 \leqslant r \leqslant k - 1.$$

Then there are at most

$$\boxed{P_k^N := (n+1)^r n^{k-r}}$$

different paths of length k, and the bound is optimal.



Definition: Path algebra

Definition: Path algebra of a graph

Let $E = (E^0, E^1, s, t)$ be a directed graph, $\mathbb{K}$ a field.
Let's consider the free algebra $\mathbb{K}\langle E^0 \sqcup E^1 \rangle$ together with relations:

- $(V) \quad \forall v, u \in E^0 \quad vu = \delta_{v,u} v,$
- $(E) \quad \forall e \in E^1 \quad s(e)e = e = e\, t(e),$

then the algebra $\mathbb{K}\langle E^0 \sqcup E^1 | V, E \rangle$ is called the *path algebra* of E over $\mathbb{K}$ and will be denoted by $\mathbb{K}E$.

Notation

Number of paths

Let E be a forest graph, then

- $P(v, w)$ - number of paths from v to w,
- $P_s(v)$ - amount of paths begining in v,
- $P_t(v)$ - number of paths ending in v,
- $P(E)$ - number of paths in E.

Moreover if E has unique source/sink then

- $P_s(E)$ - number of paths beginning at the source of E,
- $P_t(E)$ - number of paths ending at the sink of E.

Fact:

For a directed graph E the following equality holds:

$$dim\ \mathbb{K}E = P(E).$$

Calculating the number of paths

P.M. Hajac, O.M. Stachowiak

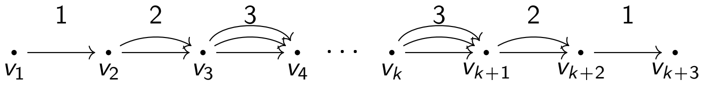
For $E \in \mathcal{E}^N$ the following inequality holds:

$$P(E) \leqslant N \left(e^{\frac{1}{e}}\right)^N .$$

Three Sisters: Path Algebras

Theorem: Dimension of the Path Algebra

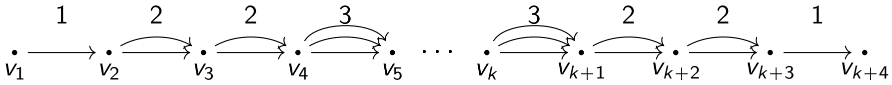
If $N \geqslant 16$, $N = 3k$ and F maximizes the dimension of $\mathbb{K}E$ among $\mathcal{E}^N$, then $P(F) = \frac{1}{4}\left(121 \cdot 3^{k-2} - 2k + 7\right)$



Three Sisters: Path Algebras

Theorem: Dimension of the Path Algebra

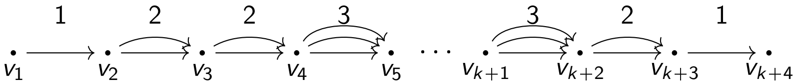
If $N \geqslant 16$, $N = 3k+1$ and F maximizes the dimension of $\mathbb{K}E$ among $\mathcal{E}^N$, then $P(F) = \frac{1}{4}\left(529 \cdot 3^{k-3} - 2k + 25\right)$



Three Sisters: Path Algebras

Theorem: Dimension of the Path Algebra

If $N \geqslant 16$, $N = 3k + 2$ and F maximizes the dimension of $\mathbb{K}E$ among $\mathcal{E}^N$, then $P(F) = \frac{1}{4}\left(253 \cdot 3^{k-2} - 2k + 15\right)$



Definition: Levitt Path Algebra

Definition:Doubled Graph

For a directed graph $E = (E^0, E^1, s, t)$ we define the doubled graph $\tilde{E} = \left(E^0, E^1 \sqcup \left(E^1\right)^*, \tilde{s}, \tilde{t}\right)$ via formulas:

- $\left(E^1\right)^* = \{e^* : e \in E^1\}$,
- $\forall e \in \left(E^1\right)^* \quad \tilde{s}(e^*) = t(e)$ and $\forall e \in E^1 \quad \tilde{s}(e) = s(e)$,
- $\forall e \in \left(E^1\right)^* \quad \tilde{t}(e^*) = s(e)$ and $\forall e \in E^1 \quad \tilde{t}(e) = t(e)$.

Leavitt Path Algebra

Definition: Leavitt path algebra of a graph

Let $E = (E^0, E^1, s, t)$ be a directed graph, $\mathbb{K}$ a field.
Let's consider the path algebra of the doubled graph $\tilde{E}$:
$\mathbb{K}\langle E^0 \sqcup E^1 \sqcup \left(E^1\right)^* \,|V, E\rangle$ together with relations:

- $(CK1) \quad \forall e^* \in \left(E^1\right)^* \forall f \in E^1 \; e^* f = \delta_{e,f} t(e)$,
- $(CK2) \quad \forall v \in E^0 \quad 0 < \left|s^{-1}(\{v\})\right| < +\infty \Rightarrow v = \sum\limits_{e \in s^{-1}(\{v\})} ee^*$.

The algebra $\mathbb{K}\langle E^0 \sqcup E^1 \sqcup \left(E^1\right)^* \,|V, E, CK1, CK2\rangle$ is called the *Leavitt path algebra* of E over $\mathbb{K}$ and is denoted by $L_{\mathbb{K}}(E)$.

Leavitt Path Algebras properties

Theorem

Let E be an N-forest graph and $\{u_i\}_1^n$ be the set of all its sinks. Then the following equality holds

$$L_{\mathbb{K}}(E) = \bigoplus_{i=1}^{n} M_{P_t(u_i)}(\mathbb{K}) .$$

Lemma

Let $\mathcal{E}_s^N$ denote the set of all N-forests with a single sink. Then the following equality holds:

$$\sup_{E\in\mathcal{E}^N} dim\ L_{\mathbb{K}}(E) = \sup_{E\in\mathcal{E}_s^N} dim\ L_{\mathbb{K}}(E) = \sup_{E\in\mathcal{E}_s^N} (P_t(E))^2 .$$

Note that $\mathcal{T}^N \subset \mathcal{E}_s^N$.

Three Sisters: Leavitt Path Algebras

Theorem: Dimension of the Leavitt Path Algebra

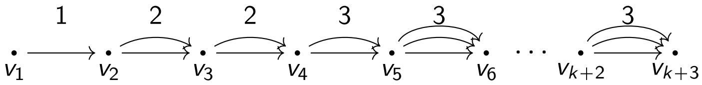
If $N \geqslant 8$, $N = 3k$ and F maximazes the dimension of $L_{\mathbb{K}}(E)$ among $\mathcal{E}^N$, then $P_t(F) = \frac{1}{2}\left(11 \cdot 3^{k-1} - 1\right)$,

$$\begin{tikzcd}
\underset{v_1}{\bullet} \arrow[r, "1"] & \underset{v_2}{\bullet} \arrow[r, "2"] \arrow[r, bend left] & \underset{v_3}{\bullet} \arrow[r, "3"] \arrow[r, bend left] \arrow[r, bend left=40] & \underset{v_4}{\bullet} & \cdots & \underset{v_{k+1}}{\bullet} \arrow[r, "3"] \arrow[r, bend left] \arrow[r, bend left=40] & \underset{v_{k+2}}{\bullet}
\end{tikzcd}$$

Three Sisters: Leavitt Path Algebras

Theorem: Dimension of the Leavitt Path Algebra

If $N \geqslant 8$, $N = 3k+1$ and F maximazes the dimension of $L_{\mathbb{K}}(E)$ among $\mathcal{E}^N$, then $P_t(F) = \frac{1}{2}\left(47 \cdot 3^{k-2} - 1\right)$,



Three Sisters: Leavitt Path Algebras

Theorem: Dimension of the Leavitt Path Algebra

If $N \geqslant 8$, $N = 3k + 2$ and F maximazes the dimension of $L_{\mathbb{K}}(E)$ among $\mathcal{E}^N$, then $P_t(F) = \frac{1}{2}\left(23 \cdot 3^{k-1} - 1\right)$,

$$
\begin{tikzcd}
\underset{v_1}{\bullet} \arrow[r, "1"] & \underset{v_2}{\bullet} \arrow[r, shift left, "2"] \arrow[r, shift right] & \underset{v_3}{\bullet} \arrow[r, shift left, "2"] \arrow[r, shift right] & \underset{v_4}{\bullet} \arrow[r, shift left=2, "3"] \arrow[r] \arrow[r, shift right=2] & \underset{v_5}{\bullet} & \cdots & \underset{v_{k+2}}{\bullet} \arrow[r, shift left=2, "3"] \arrow[r] \arrow[r, shift right=2] & \underset{v_{k+3}}{\bullet}
\end{tikzcd}
$$

Path Algebra

Path Algebra Case - Methods

From Trees to Trunks

Lemma:

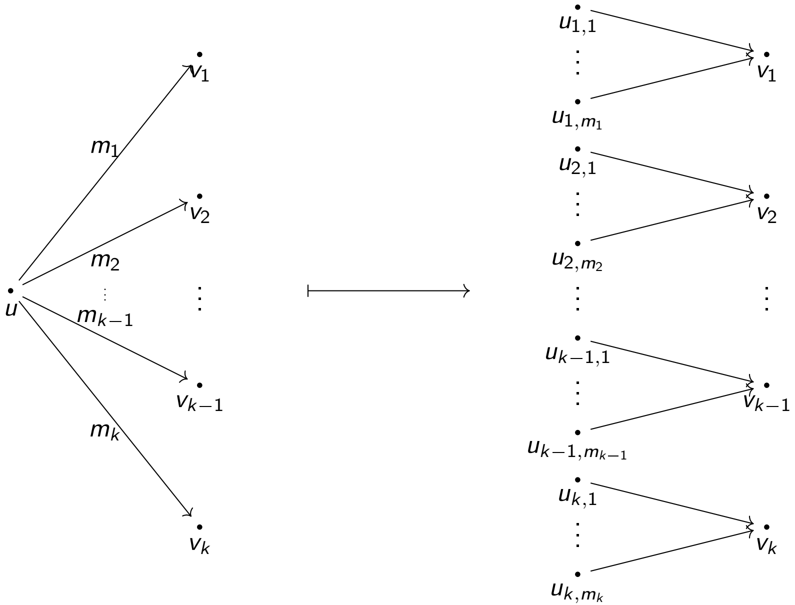
The following equality holds:

$$\sup_{E \in \mathcal{E}^N} P(E) = \sup_{E \in \mathcal{T}^N} P(E).$$

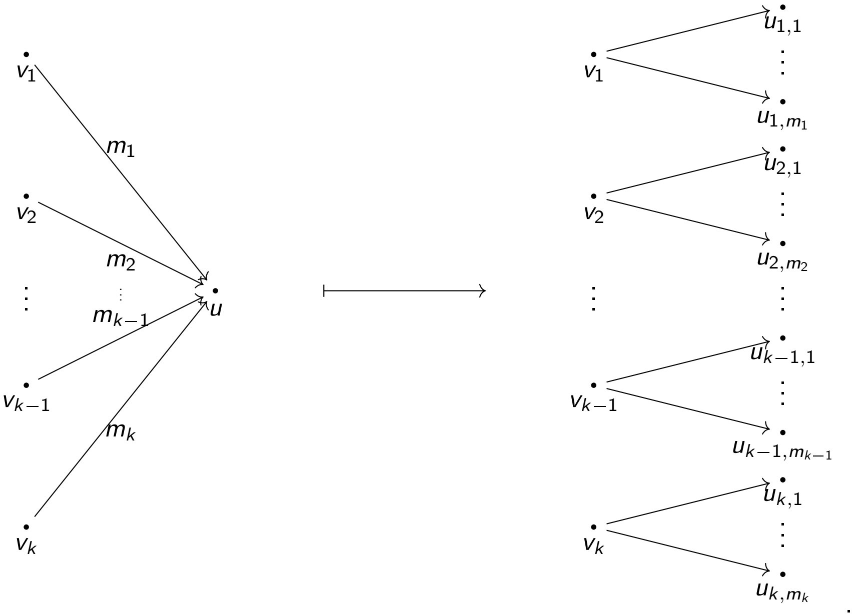
Moves

- Spliting sinks and sources,
- Resolving alternative paths,
- Combining stray paths,
- Fusion of trunks.

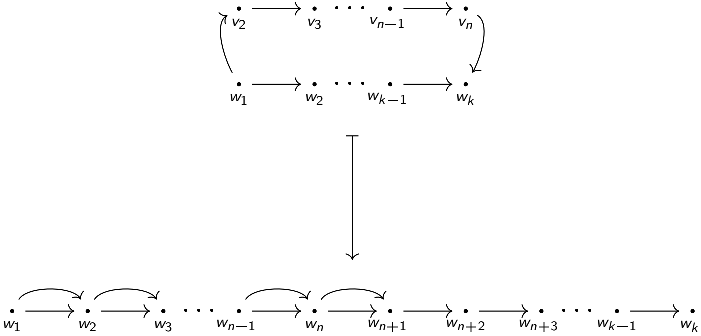
Moves - Spliting Sources



Moves - Spliting Sinks



Moves - Resolving alternative paths



Moves - Combing stray paths from sources

$$
\begin{tikzcd}
\underset{v_1}{\bullet} \arrow[r] & \underset{v_2}{\bullet} & \cdots & \underset{v_{n-1}}{\bullet} \arrow[r] & \bullet\, v_n \arrow[d] \\
\underset{w_1}{\bullet} \arrow[r] & \underset{w_2}{\bullet} & \cdots & \underset{w_{k-1}}{\bullet} \arrow[r] & \underset{w_k}{\bullet}
\end{tikzcd}
$$

$$\Big\downarrow\!\!\!\!\!\!\!\top \quad (\mapsto)$$

$$
\begin{tikzcd}
\underset{v_1}{\bullet} \arrow[r] & \underset{v_2}{\bullet} & \cdots & \underset{v_{n-1}}{\bullet} \arrow[r] & \bullet\, v_n \arrow[d] & & & \\
& & & & \underset{w_1}{\bullet} \arrow[r] & \underset{w_2}{\bullet} & \cdots & \underset{w_{k-1}}{\bullet} \arrow[r] & \underset{w_k}{\bullet}
\end{tikzcd}
$$

Moves - Combining stray paths to sinks

$$\begin{tikzcd}
v_1 \arrow[r] & v_2 \arrow[r, phantom, "\cdots"] & v_{n-1} \arrow[r] & v_n \\
w_1 \arrow[u] \arrow[r] & w_2 \arrow[r, phantom, "\cdots"] & w_{k-1} \arrow[r] & w_k
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[d, mapsto] \\
{}
\end{tikzcd}$$

$$\begin{tikzcd}
& & & v_1 \arrow[r] & v_2 \arrow[r, phantom, "\cdots"] & v_{n-1} \arrow[r] & v_n \\
w_1 \arrow[r] & w_2 \arrow[r, phantom, "\cdots"] & w_{k-1} \arrow[r] & w_k \arrow[u]
\end{tikzcd}$$

Moves - Fusion of trunks

$$v_1 \xrightarrow{m_1} v_2 \xrightarrow{m_2} v_3 \cdots v_{n-1} \xrightarrow{m_{n-1}} v_n \qquad w_1 \xrightarrow{l_1} w_2 \xrightarrow{l_2} w_3 \cdots w_{k-1} \xrightarrow{l_{k-1}} w_k$$

$$\big\downarrow$$

$$v_1 \xrightarrow{m_1} v_2 \xrightarrow{m_2} v_3 \cdots v_{n-1} \xrightarrow{m_{n-1}} v_n \xrightarrow{l_1} w_2 \xrightarrow{l_{k-1}} w_3 \cdots w_{k-1} \xrightarrow{l_{k-1}} w_k$$

Python

N edges	Maximal number of paths	Optimal trunks	Length of longest path
3	3	(1,1,1)	3
4	15	(1,1,1,1)	4
5	21	(1,1,1,1,1), (1,1,2,1),(1,2,1,1)	4 or 5
6	31	(1,2,2,1)	4
7	44	(1,1,2,2,1),(1,2,2,1,1)	5
8	66	(1,2,2,2,1)	5
9	91	(1,1,2,2,2,1),(1,2,2,2,1,1),(1,2,3,2,1)	5 or 6
10	137	(1,2,2,2,2,1)	6
11	192	(1,2,2,3,2,1),(1,2,3,2,2,1)	6
12	280	(1,2,2,2,2,2,1)	7
13	401	(1,2,2,3,2,2,1)	7
14	571	(1,2,2,3,3,2,1),(1,2,3,3,2,2,1)	7
15	820	(1,2,2,2,3,2,2,1),(1,2,2,3,2,2,2,1)	8
16	1194	(1,2,2,3,3,2,2,1)	8
17	1709	(1,2,2,3,3,3,2,1),(1,2,3,3,3,2,2,1)	8
18	2449	(1,2,3,3,3,3,2,1)	8
19	3574	(1,2,2,3,3,3,2,2,1)	9
20	5124	(1,2,2,3,3,3,3,2,1),(1,2,3,3,3,3,2,2,1)	9
21	7349	(1,2,3,3,3,3,3,2,1)	9
22	10715	(1,2,2,3,3,3,3,2,2,1)	10
23	15370	(1,2,2,3,3,3,3,3,2,1),(1,2,3,3,3,3,3,2,2,1)	10
24	22050	(1,2,3,3,3,3,3,3,2,1)	10
25	32139	(1,2,2,3,3,3,3,3,2,2,1)	11

From Trunks to bounded Trunks

Definition: Width, Weight, Length

Let E be a trunk graph with $N \geqslant 1$ edges and n vertices.

- The *width* of E is $w(E) := \max_{v \in E^0} |s_E^{-1}(v)| = \max_{1 \leqslant k \leqslant n-1} m_k$,
- The *weight* of E is N,
- The *length* of E is $n - 1$.

We denote the set of all trunk graphs with weight N and width w by $\mathcal{T}_w^N$.

Theorem:

The following equality holds:

$$\sup_{E \in \mathcal{T}^N} P(E) = \sup_{E \in \mathcal{T}_3^N} P(E).$$

Proof

Let us consider graph $E \in \mathscr{T}^N \backslash \mathscr{T}_3^N$ and let v_k be a vertex satisfying $m_k > 3$.

$$\begin{tikzcd}[column sep=large]
E_{1,k} \arrow[r, bend left=50, "m_k"] \arrow[r, bend left=15] \arrow[r, bend right=50] \arrow[r, phantom, "\vdots"] & E_{k+1,n}
\end{tikzcd}$$

and compare it with a graph G given by decomposition:

$$\begin{tikzcd}[column sep=large]
E_{1,k} \arrow[r, bend left=50, "\left\lfloor \frac{m_k}{2} \right\rfloor"] \arrow[r, bend left=15] \arrow[r, bend right=50] \arrow[r, phantom, "\vdots"] & \underset{w}{\bullet} \arrow[r, bend left=50, "\left\lceil \frac{m_k}{2} \right\rceil"] \arrow[r, bend left=15] \arrow[r, bend right=50] \arrow[r, phantom, "\vdots"] & E_{k+1,n}
\end{tikzcd}$$

$$\begin{aligned}
P(G) - P(E) = 1 &+ P_t(v_k) \cdot \left\lfloor \frac{m_k}{2} \right\rfloor + \left\lceil \frac{m_k}{2} \right\rceil \cdot P_s(v_{k+1}) \\
&+ P_t(v_k) \cdot \left\lfloor \frac{m_k}{2} \right\rfloor \cdot \left\lceil \frac{m_k}{2} \right\rceil \cdot P_s(v_{k+1}) - P_t(v_k) \cdot m_k \cdot P_s(v_{k+1}).
\end{aligned}$$

Properties

Lemma:

Let $F \in \mathcal{T}_3^N$ be the optimal graph then:

$$m_1 = m_{n-1} = 1.$$

$$\begin{tikzcd}
\underset{v_1}{\bullet} \arrow[r, bend left=50, "m_1"] \arrow[r, bend left=15] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=50] & F_{2,n}
\end{tikzcd}$$

and compare it with graph G:

$$\begin{tikzcd}
\underset{v_{1,1}}{\bullet} \arrow[r, "1"] & \underset{v_{1,2}}{\bullet} \arrow[r, "1"] & \underset{v_{1,3}}{\bullet} & \cdots & \underset{v_{1,m_1}}{\bullet} \arrow[r, "1"] & F_{2,n}
\end{tikzcd}$$

Properties

Lemma:

Let $F \in \mathscr{T}_3^N$ be the optimal graph then:

$$\forall_{1 \leqslant k \leqslant n-2} \; \left| m_k - m_{k+1} \right| \leqslant 1.$$

$$\begin{tikzcd}
F_{1,k} \arrow[r, "1"] & \underset{v_{k+1}}{\bullet} \arrow[r] \arrow[r, bend left=20] \arrow[r, bend left=40, "3"] & F_{k+2,n}
\end{tikzcd}$$

and compare

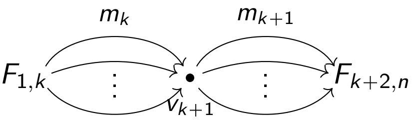
$$\begin{tikzcd}
F_{1,k} \arrow[r] \arrow[r, bend left=25, "2"] & \underset{v_{k+1}}{\bullet} \arrow[r] \arrow[r, bend left=25, "2"] & F_{k+2,n}
\end{tikzcd}$$

Properties

Lemma:

Let $F \in \mathcal{T}_3^N$ be the optimal graph and $k^* = \min\{k \colon P_t(v_k) \geqslant P_s(v_{k+2})\}$, then:

- $\forall_{1 \leqslant k < k*} \; m_k \leqslant m_{k+1}$,
- $\forall_{k* \leqslant k \leqslant n-2} \; m_{k+1} \leqslant m_k$.



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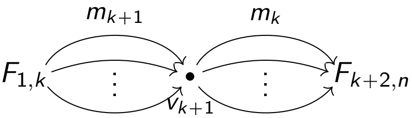
and compare it with a graph G given by decomposition:



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Properties

Lemma: Summary

Let $F \in \mathscr{T}_3^N$ be the optimal graph and $k^* = \min\{k\colon P_t(v_k) \geqslant P_s(v_{k+2})\}$, then:

- $m_1 = m_{n-1} = 1$,
- $\forall_{1 \leqslant k \leqslant n-2} \ |m_k - m_{k+1}| \leqslant 1$,
- $\forall_{1 \leqslant k < k*} \ m_k \leqslant m_{k+1}$,
- $\forall_{k* \leqslant k \leqslant n-2} \ m_{k+1} \leqslant m_k$.

Sequence

Definition: Associated Sequence

All lemmas that we have proven thus far yield a very particular form of the associated sequence of edges $I := (m_k)_{k=1}^{n-1}$ in an optimal graph.
Namely, we know that I is a sequence of the form:

$$I = (1, \ldots, 1, 2, \ldots, 2, 3, \ldots, 3, 2, \ldots, 2, 1, \ldots, 1).$$

Say that there are n_1 copies of 1's at the beginning, followed by n_2 2's, n_3 3's, n_4 2's and finally n_5 1's at the end.
We will use the following shorthand for the sequence I:

$$I =: (n_1, n_2, n_3, n_4, n_5).$$

Notation:

Note that when $n_3 = 0$ then n_2, n_4 are not well defined, similarly when $n_2 = n_4 = 0$ then n_1, n_5 are not well defined.

Trunks

Notation: "Critical Index"

For $j \in \{1,2,3,4,5\}$ let k_j denote the j-th index such that $m_j \neq m_{j+1}$.

For critical indices the following equality holds: $k_j = \sum_{i=1}^{j} n_i$.

$$\underset{v_1}{\cdot} \xrightarrow{1} \underset{v_2}{\cdot} \xrightarrow{1} \cdots \xrightarrow{1} \underset{v_{k_1+1}}{\cdot} \xrightarrow{2} \underset{v_{k_1+2}}{\cdot} \xrightarrow{2} \cdots \xrightarrow{2} \underset{v_{k_2+1}}{\cdot} \xrightarrow{3} \underset{v_{k_2+2}}{\cdot} \xrightarrow{3} \cdots \xrightarrow{3} \underset{v_{k_3+1}}{\cdot} \xrightarrow{2} \underset{v_{k_3+2}}{\cdot} \xrightarrow{2} \cdots \xrightarrow{2} \underset{v_{k_4+1}}{\cdot} \xrightarrow{1} \underset{v_{k_4+2}}{\cdot} \xrightarrow{1} \cdots \xrightarrow{1} \underset{v_{k_5+1}}{\cdot}$$

Formula for paths

Formula:

If E has associated sequence $(n_1, n_2, n_3, n_4, n_5)$ then

$$P_t(E) = \left(\left((n_1+2)\,2^{n_2} - \frac{1}{2}\right)3^{n_3} + \frac{1}{2}\right)2^{n_4} + n_5 - 1 \tag{1}$$

$$\begin{aligned} P(E) = {} & \left[(n_1+2)\,2^{n_2}\,(n_5+2)\,2^{n_4}\right]3^{n_3} - \frac{1}{2}\left[(n_1+2)\,2^{n_2} + (n_5+2)\,2^{n_4}\right]3^{n_3} \\ & + \frac{1}{4}3^{n_3} + \frac{1}{2}\left[(n_1+2)\,2^{n_2} + (n_5+2)\,2^{n_4}\right] \\ & + \frac{1}{2}\left[n_1(n_1-1) + n_5(n_5-1)\right] - \left[n_2+n_4\right] - \frac{1}{2}n_3 - \frac{13}{4} \end{aligned} \tag{2}$$

Sketch

Proof

Let E have associated sequence $(n_j)_1^5$ and $\{k_j\}_1^5$ be the critical indices, then

$$
\begin{aligned}
P_t(E) &= \sum_{k=1}^{n} P(v_k, v_n) \\
&= \sum_{k=k_5+1}^{n} P(v_k, v_n) + \sum_{k=k_4+1}^{k_5} P(v_k, v_n) + \sum_{k=k_3+1}^{k_4} P(v_k, v_n) \\
&+ \sum_{k=k_2+1}^{k_3} P(v_k, v_n) + \sum_{k=k_1+1}^{k_2} P(v_k, v_n) + \sum_{k=1}^{k_1} P(v_k, v_n) \\
&= 1 + \sum_{k=1}^{n_5} 1^k + \sum_{k=1}^{n_4} 2^k \\
&+ \sum_{k=1}^{n_3} 3^k \cdot 2^{n_4} + \sum_{k=1}^{n_2} 2^k \cdot 3^{n_3} \cdot 2^{n_4} + \sum_{k=1}^{n_1} 1^k \cdot 2^{n_2} \cdot 3^{n_3} \cdot 2^{n_4}
\end{aligned}
$$

Sketch

Proof

$$
\begin{aligned}
P(F) &= \sum_{k=1}^{n}\sum_{l=1}^{k} P(v_l, v_k) = \sum_{k=1}^{n} P_t\left(v_k\right) = \sum_{k=1}^{n} P_t\left(E_{1,k}\right) \\
&= P_t(E_{1,1}) + \sum_{k=2}^{k_1+1} P_t\left(E_{1,k}\right) + \sum_{k=k_1+2}^{k_2+1} P_t\left(E_{1,k}\right) \\
&+ \sum_{k=k_2+2}^{k_3+1} P_t\left(E_{1,k}\right) \\
&+ \sum_{k=k_3+2}^{k_4+1} P_t\left(E_{1,k}\right) \\
&+ \sum_{k=k_4+2}^{k_5+1} P_t\left(E_{1,k}\right)
\end{aligned}
$$

Sketch

Proof

Applying formula (1) we get:

$$\begin{aligned} P(E) = 1 &+ \sum_{k=1}^{n_1} [k+1] + \sum_{k=1}^{n_2} \left[(n_1+2)\, 2^k - 1 \right] \\ &+ \sum_{k=1}^{n_3} \left[\left((n_1+2)\, 2^{n_2} - \frac{1}{2} \right) 3^k - \frac{1}{2} \right] \\ &+ \sum_{k=1}^{n_4} \left[\left(\left((n_1+2)\, 2^{n_2} - \frac{1}{2} \right) 3^{n_3} + \frac{1}{2} \right) 2^k - 1 \right] \\ &+ \sum_{k=1}^{n_5} \left[\left(\left((n_1+2)\, 2^{n_2} - \frac{1}{2} \right) 3^{n_3} + \frac{1}{2} \right) 2^{n_4} + k - 1 \right] \end{aligned}$$

Sequences and their properties

Lemma:

If F is the optimal graph then its sequence $(n_1, n_2, n_3, n_4, n_5)$ satisfies:

- $n_2 \geqslant 1$ whenever $N \geqslant 5$,
- $n_1 = n_5 = 1$ whenever $N \geqslant 8$,
- $n_3 \geqslant 1$ whenever $N \geqslant 13$,
- $n_3 \geqslant 2$ whenever $N \geqslant 16$.

Moreover

- $n_1, n_5 \leqslant 3$ whenever $n_2 \geqslant 1$,
- $|n_2 - n_4| \leqslant 1$ whenever $n_3 \geqslant 1$,
- $\max\{n_2, n_4\} \leqslant 2$ whenever $n_3 \geqslant 2$.

Moves

List

1. $(n_1, n_2, n_3, n_4, n_5) \mapsto (n_1 - 2, n_2 + 1, n_3, n_4, n_5)$
2. $(n_1, n_2, n_3, n_4, n_5) \mapsto (n_1 - 1, n_2 - 1, n_3 + 1, n_4, n_5)$
3. $(n_1, n_2, n_3, n_4, n_5) \mapsto (n_1 - 1, n_2, n_3 + 1, n_4 - 1, n_5)$
4. $(n_1, n_2, n_3, n_4, n_5) \mapsto (n_1 - 1, n_2 + 1, n_3 - 1, n_4 + 1, n_5)$
5. $(n_1, n_2, n_3, n_4, n_5) \mapsto (n_1, n_2 - 2, n_3 + 2, n_4 - 1, n_5)$
6. $(n_1, n_2, n_3, n_4, n_5) \mapsto (n_1, n_2 - 1, n_3, n_4 + 1, n_5)$

Moves

Move 1

$(n_1, n_2, n_3, n_4, n_5) \mapsto (n_1 - 2, n_2 + 1, n_3, n_4, n_5)$

$$\begin{tikzcd} E_{1,n_1-1} \arrow[r, "1"] & \underset{v_{n_1}}{\bullet} \arrow[r, "1"] & E_{k_1+1,n} \end{tikzcd}$$

And let's compare it with graph G

$$\begin{tikzcd} E_{1,n_1-1} \arrow[r, bend left, "2"] \arrow[r] & E_{k_1+1,n} \end{tikzcd}$$

Moves

Move 2

$(n_1, n_2, n_3, n_4, n_5) \mapsto (n_1 - 1, n_2 - 1, n_3 + 1, n_4, n_5)$

$$
\begin{tikzcd}
\overset{\bullet}{v_1} \arrow[r, "1"] & E_{2,k_3} \arrow[r, bend left, "2"] \arrow[r] & E_{k_3+1,n}
\end{tikzcd}
$$

Compare it with

$$
\begin{tikzcd}
E_{2,k_3} \arrow[r, bend left=50, "3"] \arrow[r, bend left] \arrow[r] & E_{k_3+1,n}
\end{tikzcd}
$$

Moves

Move 3

$(n_1, n_2, n_3, n_4, n_5) \mapsto (n_1 - 1, n_2, n_3 + 1, n_4 - 1, n_5)$

$$\begin{tikzcd} \underset{v_1}{\bullet} \arrow[r, "1"] & E_{2,k_4+1} \arrow[r] \arrow[r, bend left, "2"] & E_{k_4+2,n} \end{tikzcd}$$

Compare it with

$$\begin{tikzcd} E_{2,k_4+1} \arrow[r] \arrow[r, bend left] \arrow[r, bend left=60, "3"] & E_{k_4+2,n} \end{tikzcd}$$

Moves

Move 4

$$(n_1, n_2, n_3, n_4, n_5) \mapsto (n_1 - 1, n_2 + 1, n_3 - 1, n_4 + 1, n_5)$$

$$\begin{tikzcd}
E_{1,n_1} \arrow[r, "1"] & E_{k_1+1,k_3} \arrow[r, bend left=40, "3"] \arrow[r, bend left=20] \arrow[r] & E_{k_3+1,n}
\end{tikzcd}$$

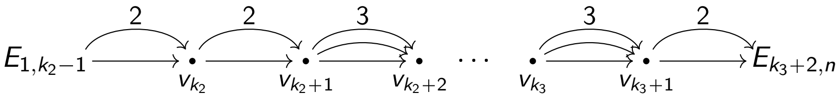
Compare it with

$$\begin{tikzcd}
E_{1,n_1} \arrow[r, bend left=20, "2"] \arrow[r] & E_{k_1+1,k_3} \arrow[r, bend left=20, "2"] \arrow[r] & E_{k_3+1,n}
\end{tikzcd}$$

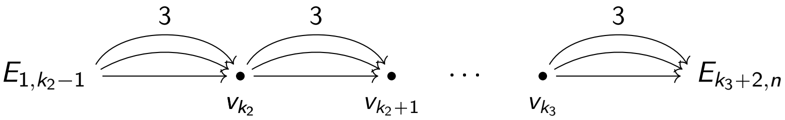
Moves

Move 5

$$(n_1, n_2, n_3, n_4, n_5) \mapsto (n_1, n_2 - 2, n_3 + 2, n_4 - 1, n_5)$$



Let us considet the graph G



Moves

Move 6

$(n_1, n_2, n_3, n_4, n_5) \mapsto (n_1, n_2 - 1, n_3, n_4 + 1, n_5)$

$$\begin{tikzcd} E_{1,k_2} \arrow[r, bend left, "2"] \arrow[r] & \underset{v_{k_2+1}}{\bullet} \arrow[r, bend left, "3"] \arrow[r, bend left=15] \arrow[r] & \underset{v_{k_2+2}}{\bullet} & \cdots & \underset{v_{k_2+n_3}}{\bullet} \arrow[r, bend left, "3"] \arrow[r, bend left=15] \arrow[r] & E_{k_3+1,n} \end{tikzcd}.$$

and compare it with graph G

$$\begin{tikzcd} E_{1,k_2} \arrow[r, bend left, "3"] \arrow[r, bend left=15] \arrow[r] & \underset{v_{k_2+1}}{\bullet} & \cdots & \underset{v_{k_2+n_3-1}}{\bullet} \arrow[r, bend left, "3"] \arrow[r, bend left=15] \arrow[r] & \underset{v_{k_2+n_3}}{\bullet} \arrow[r, bend left, "2"] \arrow[r] & E_{k_3+1,n} \end{tikzcd}.$$

Three Sisters

Theorem: Form of Optimal Graph

If $N \geqslant 16$ and F maximizes the dimension of $\mathbb{K}E$ among $\mathcal{E}^N$, then

- $F = \left(1, 1, \frac{N-6}{3}, 1, 1\right) = \left(1, 1, \left\lfloor \frac{N}{3} \right\rfloor - 2, 1, 1\right)$,
- $F = \left(1, 2, \frac{N-10}{3}, 2, 1\right) = \left(1, 2, \left\lfloor \frac{N}{3} \right\rfloor - 3, 2, 1\right)$,
- $F = \left(1, 2, \frac{N-8}{3}, 1, 1\right) = \left(1, 2, \left\lfloor \frac{N}{3} \right\rfloor - 2, 1, 1\right)$,

depending on the remainder of division of N by 3.

Theorem: Dimension of the Path Algebra

If $N \geqslant 16$ and F maximizes the dimension of $\mathbb{K}E$ among $\mathcal{E}^N$, then

- $P(F) = \frac{1}{4}\left(121 \cdot 3^{\left\lfloor \frac{N}{3} \right\rfloor - 2} - 2\left\lfloor \frac{N}{3} \right\rfloor + 7\right)$,
- $P(F) = \frac{1}{4}\left(529 \cdot 3^{\left\lfloor \frac{N}{3} \right\rfloor - 3} - 2\left\lfloor \frac{N}{3} \right\rfloor + 25\right)$,
- $P(F) = \frac{1}{4}\left(253 \cdot 3^{\left\lfloor \frac{N}{3} \right\rfloor - 2} - 2\left\lfloor \frac{N}{3} \right\rfloor + 15\right)$,

depending on the remainder of division of N by 3.

Leavitt Path Algebra

Leavitt Path Algebra Methods

From Trees to Trunks

Theorem

The following equality holds:

$$\sup_{E \in \mathscr{E}_s^N} P_t(E) = \sup_{E \in \mathscr{T}^N} P_t(E).$$

Moves

- Resolving alternative paths,
- Combing stray paths.

Moves - Resolving alternative paths

$$\begin{tikzcd}
\underset{v_2}{\bullet} \arrow[r] & \underset{v_3}{\bullet} & \cdots & \underset{v_{n-1}}{\bullet} \arrow[r] & \underset{v_n}{\bullet} \arrow[d, bend left] \\
\underset{w_1}{\bullet} \arrow[u, bend left] \arrow[r] & \underset{w_2}{\bullet} & \cdots & \underset{w_{k-1}}{\bullet} \arrow[r] & \underset{w_k}{\bullet}
\end{tikzcd}$$

$$\big\downarrow$$

$$\begin{tikzcd}
\underset{w_1}{\bullet} \arrow[r] \arrow[r, bend left] & \underset{w_2}{\bullet} \arrow[r] \arrow[r, bend left] & \underset{w_3}{\bullet} & \cdots & \underset{w_{n-1}}{\bullet} \arrow[r] \arrow[r, bend left] & \underset{w_n}{\bullet} \arrow[r] \arrow[r, bend left] & \underset{w_{n+1}}{\bullet} \arrow[r] & \underset{w_{n+2}}{\bullet} \arrow[r] & \underset{w_{n+3}}{\bullet} & \cdots & \underset{w_{k-1}}{\bullet} \arrow[r] & \underset{w_k}{\bullet}
\end{tikzcd}$$

Moves - Combing stray paths from sources

$$\begin{tikzcd}
\underset{v_1}{\bullet} \arrow[r] & \underset{v_2}{\bullet} & \cdots & \underset{v_{n-1}}{\bullet} \arrow[r] & \bullet\, v_n \arrow[d] \\
\underset{w_1}{\bullet} \arrow[r] & \underset{w_2}{\bullet} & \cdots & \underset{w_{k-1}}{\bullet} \arrow[r] & \underset{w_k}{\bullet}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[d, mapsto] \\
{}
\end{tikzcd}$$

$$\begin{tikzcd}
\underset{v_1}{\bullet} \arrow[r] & \underset{v_2}{\bullet} & \cdots & \underset{v_{n-1}}{\bullet} \arrow[r] & \bullet\, v_n \arrow[d] & & & & \\
& & & & \underset{w_1}{\bullet} \arrow[r] & \underset{w_2}{\bullet} & \cdots & \underset{w_{k-1}}{\bullet} \arrow[r] & \underset{w_k}{\bullet}
\end{tikzcd}$$

Python

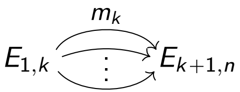
N edges	Maximal number of paths	Optimal trunks	Length of longest path
3	5	(1,2)	2
4	7	(1,1,2),(1,3),(2,2)	2 or 3
5	11	(1,2,2)	3
6	16	(1,2,3)	3
7	23	(1,2,2,2)	4
8	34	(1,2,2,3)	4
9	49	(1,2,3,3)	4
10	70	(1,2,2,2,3)	5
11	103	(1,2,2,3,3)	5
12	148	(1,2,3,3,3)	5
13	211	(1,2,2,2,3,3)	6
14	310	(1,2,2,3,3,3)	6
15	445	(1,2,3,3,3,3)	6
16	634	(1,2,2,2,3,3,3)	7
17	931	(1,2,2,3,3,3,3)	7
18	1336	(1,2,3,3,3,3,3)	7
19	1903	(1,2,2,2,3,3,3,3)	8
20	2794	(1,2,2,3,3,3,3,3)	8
21	4009	(1,2,3,3,3,3,3,3)	8
22	5710	(1,2,2,2,3,3,3,3,3)	9
23	8383	(1,2,2,3,3,3,3,3,3)	9
24	12028	(1,2,3,3,3,3,3,3,3)	9
25	17131	(1,2,2,2,3,3,3,3,3,3)	10

Ones and differences

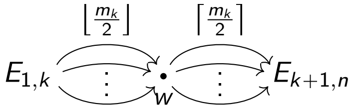
Lemma

The following equality holds:

$$\sup_{E \in \mathscr{T}^N} P_t(E) = \sup_{E \in \mathscr{T}^N_3} P_t(E).$$



and compare it with a graph G given by decomposition:



$$P_t(G) - P_t(E) = \left\lceil \frac{m_k}{2} \right\rceil \cdot P_s(v_{k+1}) + P_t(v_k) \cdot \left\lfloor \frac{m_k}{2} \right\rfloor \cdot \left\lceil \frac{m_k}{2} \right\rceil \cdot P_s(v_{k+1})$$
$$- P_t(v_k) \cdot m_k \cdot P_s(v_{k+1}).$$

Properties

Lemma:

Let $F \in \mathcal{T}_3^N$ be the optimal graph then:

$$m_1 = 1.$$

$$\overbrace{\begin{tikzcd} \underset{v_1}{\bullet} \arrow[r, bend left] \arrow[r, phantom, "\vdots"] \arrow[r, bend right] & F_{2,n} \end{tikzcd}}^{m_1}$$

and compare it with graph G:

$$\underset{v_{1,1}}{\bullet} \xrightarrow{\quad 1 \quad} \underset{v_{1,2}}{\bullet} \xrightarrow{\quad 1 \quad} \underset{v_{1,3}}{\bullet} \quad \cdots \quad \underset{v_{1,m_1}}{\bullet} \xrightarrow{\quad 1 \quad} F_{2,n}$$

Properties

Lemma:

Let $F \in \mathcal{T}_3^N$ be the optimal graph then:

$$\forall_{1 \leqslant k \leqslant n-2} \left| m_k - m_{k+1} \right| \leqslant 1.$$

$$\begin{tikzcd}
E_{1,k} \arrow[r, "1"] & \underset{v_{k+1}}{\bullet} \arrow[r] \arrow[r, bend left=20] \arrow[r, bend left=40, "3"] & E_{k+2,n}
\end{tikzcd}$$

and compare

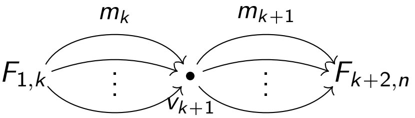
$$\begin{tikzcd}
E_{1,k} \arrow[r] \arrow[r, bend left=30, "2"] & \underset{v_{k+1}}{\bullet} \arrow[r] \arrow[r, bend left=30, "2"] & E_{k+2,n}
\end{tikzcd}$$

Properties

Lemma:

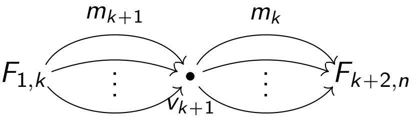
Let $F \in \mathcal{T}_3^N$ be the optimal graph then:

$$\forall_{1 \leqslant k \leqslant n-2}\ m_k \leqslant m_{k+1}.$$



,

and compare it with a graph G given by decomposition:



.

Properties

Lemma: Summary

Let $F \in \mathcal{T}_3^N$ be the optimal graph then:

- $m_1 = 1$,
- $\forall_{1 \leqslant k \leqslant n-2} \left| m_k - m_{k+1} \right| \leqslant 1$,
- $\forall_{1 \leqslant k \leqslant n-2} \; m_k \leqslant m_{k+1}$.

Paths to the sink

Formula

If E has associated sequence (n_1, n_2, n_3) then

$$P_t(E) = \left((n_1 + 2)\, 2^{n_2} - \frac{1}{2} \right) 3^{n_3} - \frac{1}{2}$$

Proof

We apply formula (1)

Sequences and their properties

Lemma:

If F is the optimal graph then its sequence (n_1, n_2, n_3) satisfies:

- $n_2 \geqslant 1$ whenever $N \geqslant 3$,
- $n_1 = 1$ whenever $N \geqslant 5$,
- $n_3 \geqslant 1$ whenever $N \geqslant 8$.

Moreover

- $n_2 \leqslant 3$ whenever $n_3 \geqslant 1$.

Moves

Moves

Let us consider a trunk with associated sequence (n_1, n_2, n_3).

1. $(n_1, n_2, n_3) \mapsto (n_1 - 2, n_2 + 1, n_3)$
2. $(n_1, n_2, n_3) \mapsto (n_1, n_2 - 3, n_3 + 2)$
3. $(n_1, n_2, n_3) \mapsto (n_1 - 1, n_2 - 1, n_3 + 1)$

Moves

Move 1

$(n_1, n_2, n_3) \mapsto (n_1 - 2, n_2 + 1, n_3)$

$$\begin{tikzcd} E_{1,n_1-1} \arrow[r, "1"] & \underset{v_{n_1}}{\bullet} \arrow[r, "1"] & E_{k_1+1,n} \end{tikzcd}$$

And let's compare it with graph G

$$\begin{tikzcd} E_{1,n_1-1} \arrow[r, bend left] \arrow[r] & E_{k_1+1,n} \end{tikzcd} \quad 2$$

Moves

Move 2

$(n_1, n_2, n_3) \mapsto (n_1, n_2 - 3, n_3 + 2)$

$$
\begin{tikzcd}
E_{1,k_2-2} \arrow[r] \arrow[r, bend left, "2"] & \underset{v_{k_2-1}}{\bullet} \arrow[r] \arrow[r, bend left, "2"] & \underset{v_{k_2}}{\bullet} \arrow[r] \arrow[r, bend left, "2"] & \underset{v_{k_2+1}}{\bullet} \arrow[r] \arrow[r, bend left=20] \arrow[r, bend left=40, "3"] & E_{k_2+2,n}
\end{tikzcd}
$$

Let us consider the graph G

$$
\begin{tikzcd}
E_{1,k_2-2} \arrow[r] \arrow[r, bend left=20] \arrow[r, bend left=40, "3"] & \underset{v_{k_2-1}}{\bullet} \arrow[r] \arrow[r, bend left=20] \arrow[r, bend left=40, "3"] & \underset{v_{k_2}}{\bullet} \arrow[r] \arrow[r, bend left=20] \arrow[r, bend left=40, "3"] & E_{k_2+2,n}
\end{tikzcd}
$$

Moves

Move 3

$(n_1, n_2, n_3) \mapsto (n_1 - 1, n_2 - 1, n_3 + 1)$

$$\begin{tikzcd} \underset{v_1}{\bullet} \arrow[r, "1"] & E_{2,k_2} \arrow[r, bend left, "2"] \arrow[r] & E_{k_2+1,n} \end{tikzcd}$$

Compare it with

$$\begin{tikzcd} E_{2,k_2} \arrow[r, bend left=40, "3"] \arrow[r, bend left] \arrow[r] & E_{k_2+1,n} \end{tikzcd}$$

Optimal graphs and Three sisters

Theorem:

If $N \geqslant 8$ and F maximizes the dimension of $L_{\mathbb{K}}(E)$ among $\mathcal{E}_s^N$ then

- $F = \left(1, 1, \frac{N-3}{3}\right) = \left(1, 1, \left\lfloor \frac{N}{3} \right\rfloor - 1\right)$, $P_t(F) = \frac{1}{2}\left(11 \cdot 3^{\left\lfloor \frac{N}{3} \right\rfloor - 1} - 1\right)$
- $F = \left(1, 3, \frac{N-7}{3}\right) = \left(1, 3, \left\lfloor \frac{N}{3} \right\rfloor - 2\right)$, $P_t(F) = \frac{1}{2}\left(47 \cdot 3^{\left\lfloor \frac{N}{3} \right\rfloor - 2} - 1\right)$
- $F = \left(1, 2, \frac{N-5}{3}\right) = \left(1, 2, \left\lfloor \frac{N}{3} \right\rfloor - 1\right)$, $P_t(F) = \frac{1}{2}\left(23 \cdot 3^{\left\lfloor \frac{N}{3} \right\rfloor - 1} - 1\right)$

depending on the value of $N \mod 3$.

Open Problem

A new problem appears!

Gap Problem

Necessary and Sufficient numbers

For any natural number $n \in \mathbb{N}$ let us consider numbers:

$$m(n) = \max\left\{N\colon \sup_{E \in \mathcal{E}_s^N} P_t(E) < n\right\} + 1$$

$$M(n) = \min\left\{N\colon M_n(\mathbb{K}) = L_{\mathbb{K}}(E),\ |E^1| = N\right\}$$

called necessary and sufficient numbers for construction respectively.

Gap problem

What is the behavior of $M(n)$ compared to $m(n)$?
In particular:

$$\sup_{n \in \mathbb{N}} \left(M(n) - m(n)\right) < +\infty.$$

The End

Thank you!!!