Graph algebras related to discrete boolean-antimonotone creators

Anna Wysoczańska-Kula
University of Wrocław

joint work with:
Jack Spielberg, Janusz Wysoczański

Noncommutative Geometry and Graph Algebras, London ON 2026

Idea

- In noncommutative probability there are many notions of independence.
- Often, models of independent random variables are creation operators on appropriate Fock-type spaces.
- C^*-algebras generated by all creators can have interesting properties.
- In some cases (free, monotone) they have been shown to be related to graph algebras.
- Does this apply to boolean-antimonotone realm?

Noncommutative probability (in 4 slides)

- noncommutative probability space : *-algebra $\mathcal{A}$ with a state φ
- random variable : $a = a^* \in \mathcal{A}$
- moments of a : sequence $(\varphi(a^n))_{n=0}^{\infty}$
- distribution of a : measure μ on $\mathbb{R}$ s.t. $\varphi(a^n) = \int_{\mathbb{R}} t^n d\mu(t)$.

Examples

1. $\mathcal{A} = L^1(\Omega, \Sigma, \mu)$, $\varphi(X) = \mathbb{E}[X]$;
2. $\mathcal{A} = M_n(\mathbb{C})$, $\varphi(M) = \mathrm{Tr}(M)$;
3. $\mathcal{A} = \mathbb{C}G$ where G is a group, $\varphi(\sum_{g\in G} \alpha_g g) = \alpha_e$;
4. $\mathcal{A} = B(H)$, $v \in H$ of norm 1, $\varphi(T) = \langle v, Tv \rangle$.

Noncommutative independences

A noncommutative independence is a rule that allows to compute mixed moments of independent elements out of their marginal moments.

Examples (take $a, b \in \mathcal{A}$ nc-independent)

- tensor (classical): random variables commute and mixed moments of alternating variables factorize, e.g. $\varphi(abab) = \varphi(a^2 b^2) = \varphi(a^2)\varphi(b^2)$

Noncommutative independences

A noncommutative independence is a rule that allows to compute mixed moments of independent elements out of their marginal moments.

Examples (take $a, b \in \mathcal{A}$ nc-independent)

- tensor (classical): random variables commute and mixed moments of alternating variables factorize, e.g. $\varphi(abab) = \varphi(a^2b^2) = \varphi(a^2)\varphi(b^2)$
- free: mixed moments of alternating, centered variables factorize to zero, e.g. $\varphi(abab) = \varphi(a)^2\varphi(b^2) + \varphi(a^2)\varphi(b)^2 - \varphi(a^2)\varphi(b^2)$

Noncommutative independences

A noncommutative independence is a rule that allows to compute mixed moments of independent elements out of their marginal moments.

Examples (take $a, b \in \mathcal{A}$ nc-independent)

- tensor (classical): random variables commute and mixed moments of alternating variables factorize, e.g. $\varphi(abab) = \varphi(a^2b^2) = \varphi(a^2)\varphi(b^2)$
- free: mixed moments of alternating, centered variables factorize to zero, e.g. $\varphi(abab) = \varphi(a)^2\varphi(b^2) + \varphi(a^2)\varphi(b)^2 - \varphi(a^2)\varphi(b^2)$
- boolean: mixed moments of alternating variables factorize completely, e.g. $\varphi(abab) = \varphi(a)^2\varphi(b)^2$

Noncommutative independences

A noncommutative independence is a rule that allows to compute mixed moments of independent elements out of their marginal moments.

Examples (take $a, b \in \mathcal{A}$ nc-independent)

- tensor (classical): random variables commute and mixed moments of alternating variables factorize, e.g. $\varphi(abab) = \varphi(a^2b^2) = \varphi(a^2)\varphi(b^2)$
- free: mixed moments of alternating, centered variables factorize to zero, e.g. $\varphi(abab) = \varphi(a)^2\varphi(b^2) + \varphi(a^2)\varphi(b)^2 - \varphi(a^2)\varphi(b^2)$
- boolean: mixed moments of alternating variables factorize completely, e.g. $\varphi(abab) = \varphi(a)^2\varphi(b)^2$
- monotone: index set is ordered and the mixed moments factor out the middle element if its index is a local maximum, e.g. if $a \in A_1$ and $b \in A_2$ then $\varphi(abab) = \varphi(b)\varphi(a^2b) = \varphi(a^2)\varphi(b)^2$
- antimonotone: index set is ordered and the mixed moments factor out the middle element if its index is a local *minimum*, e.g. if $a \in A_1$ and $b \in A_2$ then $\varphi(abab) = \varphi(a)\varphi(ab^2) = \varphi(a)^2\varphi(b^2)$

BM-independence

Definition (Wysoczański 2007)

Let $(\mathcal{A}, \varphi)$ be a ncps and let $(\mathtt{I}, \preceq)$ be a partially ordered set. A family $(\mathcal{A}_i)_{i\in\mathtt{I}}$ of $*$-subalgebras of $\mathcal{A}$ are called **boolean-monotone independent** if

BM1) $b_i b_j b_k = \varphi(b_j) b_i b_k$ if j is a local (weak) maximum ($i \prec j \succ k$ or $i \not\sim j \succ k$ or $i \prec j \not\sim k$)

BM2) $\varphi(b_{i_1} \dots b_{i_n}) = \prod_{j=1}^{n} \varphi(b_{i_j})$ if $i_1 \succ \dots \succ i_m \not\sim i_{m+1} \dots i_{k-1} \not\sim i_k \prec \dots \prec i_n$.

1. If $\mathtt{I}$ is linearily ordered, we recover the monotone indepedence.
2. If $\mathtt{I}$ is totally disordered, A_i's are boolean independence.

BA-independence

Definition

Let $(\mathcal{A}, \varphi)$ be a ncps and let $(\mathtt{I}, \preceq)$ be a partially ordered set. A family $(\mathcal{A}_i)_{i \in \mathtt{I}}$ of $*$-subalgebras of $\mathcal{A}$ are called **boolean-antimonotone independent** if

(BA1) $b_i b_j b_k = \varphi(b_j) b_i b_k$ if j is a local (weak) minimum ($i \succ j \prec k$ or $i \not\sim j \prec k$ or $i \succ j \not\sim k$)

(BA2) $\varphi(b_{i_1} \ldots b_{i_n}) = \prod_{j=1}^{n} \varphi(b_{i_j})$ if $i_1 \prec \ldots \prec i_m \not\sim i_{m+1} \ldots i_{k-1} \not\sim i_k \succ \ldots \succ i_n$.

1. If $\mathtt{I}$ is linearily ordered, we recover the antimonotone indepedence.
2. If $\mathtt{I}$ is totally disordered, A_i's are boolean independence.

Fock spaces and creators

The free Fock space

Question: Do free/monotone/bm-independent random variables exist?

The free Fock space

Question: Do free/monotone/bm-independent random variables exist?
Let H be a (separable) Hilbert space. Define the **full (free) Fock space** as

$$\mathcal{F}(H) := \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} H^{\otimes n}$$

with the inner product

$$\langle h_1 \otimes \ldots \otimes h_n, k_1 \otimes \ldots \otimes k_m \rangle = \delta_{n,m} \langle h_1, k_1 \rangle \ldots \langle h_n, k_n \rangle .$$

Given $h \in H$ of norm 1 we define the creation operator as

$$l(h)(\Omega) = h, \quad l(h)(h_1 \otimes \ldots \otimes h_n m) = h \otimes h_1 \otimes \ldots \otimes h_n .$$

Then $l(h) \in B(\mathcal{F}(H))$.

The free Fock space

Question: Do free/monotone/bm-independent random variables exist?
Let H be a (separable) Hilbert space. Define the **full (free) Fock space** as

$$\mathcal{F}(H) := \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} H^{\otimes n}$$

with the inner product

$$\langle h_1 \otimes \ldots \otimes h_n, k_1 \otimes \ldots \otimes k_m \rangle = \delta_{n,m} \langle h_1, k_1 \rangle \ldots \langle h_n, k_n \rangle.$$

Given $h \in H$ of norm 1 we define the creation operator as

$$l(h)(\Omega) = h, \quad l(h)(h_1 \otimes \ldots \otimes h_n m) = h \otimes h_1 \otimes \ldots \otimes h_n.$$

Then $l(h) \in B(\mathcal{F}(H))$.

Theorem (Voiculescu 1986)

If $h \perp k$, then $l(h)$ and $l(k)$ are freely independent.

The algebra of free creators

Let H be a Hilbert space of dimension $n \in \{2, \dots, \infty\}$.
Define **the algebra of free creators**, $\mathcal{A}_{free}(n)$, to be the sub-C*-algebra in $B(\mathcal{F}(H))$ generated by all free creation operators.

The algebra of free creators

Let H be a Hilbert space of dimension $n \in \{2, \dots, \infty\}$.
Define **the algebra of free creators**, $\mathcal{A}_{free}(n)$, to be the sub-C*-algebra in $B(\mathcal{F}(H))$ generated by all free creation operators.

Theorem (Evans 1980)

If $n \in \mathbb{N}$, then $\mathcal{A}_{free}(n) \cong \mathcal{O}_n \otimes \mathcal{K}(\mathcal{F}(H))$.
If $n = \infty$, then $\mathcal{A}_{free}(\infty) \cong \mathcal{O}_\infty$.

Where $\mathcal{O}_n$ denotes the Cuntz alegbra, that is, the universal C*-algebra generated by a family of isometries $\{S_i\}_{i=1}^n$ satisfying $\sum_{i=1}^n S_i S_i^* \leq 1$ with the equality when n is finite.
Note that $\mathcal{O}_n$ is separable, simple, purely infinite.

Let

- H : a separable Hilbert space with the ON basis $\{e_i\}_{i\in\mathtt{I}}$, $\mathtt{I} = \{1, \ldots, d\}$ or $\mathtt{I} = \mathbb{N}$. Denote $H_i = \mathbb{C}e_i$.
- $\mathrm{dec}(n) := \{\mathrm{i} = (i_1, \ldots, i_n) \in \mathtt{I}^n : \quad i_1 \geq i_2 \geq \cdots \geq i_n\}$: the set of nonincreasing sequences of indices in $\mathtt{I}$ of length n

We define the *weakly monotone Fock space*

$$\mathcal{F}_{wm} = \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} \bigoplus_{\mathrm{i}\in\mathrm{dec}(n)} H_{i_1} \otimes \ldots \otimes H_{i_n}.$$

being the closed subspace of $\mathcal{F}(H)$.

The algebra of monotone creators

Let

- H : a separable Hilbert space with the ON basis $\{e_i\}_{i\in\mathtt{I}}$, $\mathtt{I}=\{1,\ldots,d\}$ or $\mathtt{I}=\mathbb{N}$. Denote $H_i=\mathbb{C}e_i$.
- $\mathrm{dec}(n):=\{\mathrm{i}=(i_1,\ldots,i_n)\in\mathtt{I}^n:\quad i_1\geq i_2\geq\cdots\geq i_n\}$: the set of nonincreasing sequences of indices in $\mathtt{I}$ of length n

We define the *weakly monotone Fock space*

$$\mathcal{F}_{wm}=\mathbb{C}\Omega\oplus\bigoplus_{n=1}^{\infty}\bigoplus_{\mathrm{i}\in\mathrm{dec}(n)}H_{i_1}\otimes\ldots\otimes H_{i_n}.$$

being the closed subspace of $\mathcal{F}(H)$.
For $i\in I$ we define the monotone creation operator as

$$A_i^\dagger(\Omega)=e_i,\quad A_i^\dagger(e_{i_1}\otimes\ldots\otimes e_{i_n})=\begin{cases}e_i\otimes e_{i_1}\otimes\ldots\otimes e_{i_n} & \text{if } i\geq i_1,\\ 0 & \text{otherwise.}\end{cases}.$$

Then $A_i^\dagger\in B(\mathcal{F}_{wm}(H))$.

The algebra of monotone creators

Let $E = (E^0, E^1, r, s)$ be a finite directed graph. The graph algebra $C^*(E)$ is the universal C*-algebra generated by a family of mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$ with orthogonal ranges and satisfying:

$$s_e^* s_e = p_{r(e)} \quad \text{and} \quad p_v = \sum_{e : s(e) = v} s_e s_e^* \ \text{ for } |s^{-1}(v)| > 0.$$

Theorem (Griseta, Wysoczański 2025)

For $d \in \mathbb{N}$

$$\mathcal{A}_m(d) \cong C^*(E)/\mathcal{J},$$

where:

- $E^0 = \mathrm{I} \cup \{0\}$ and $E^1 = \{(i, j) : i \geq j\}$,
- $\mathcal{J} = \langle s_0 - s_0^* s_0 \rangle$.

The algebra of monotone creators

Let $E = (E^0, E^1, r, s)$ be a finite directed graph. The graph algebra $C^*(E)$ is the universal C*-algebra generated by a family of mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$ with orthogonal ranges and satisfying:

$$s_e^* s_e = p_{r(e)} \quad \text{and} \quad p_v = \sum_{e: s(e)=v} s_e s_e^* \ \text{ for } |s^{-1}(v)| > 0.$$

Theorem (Griseta, Wysoczański 2025)

For $d \in \mathbb{N}$

$$\mathcal{A}_m(d) \cong C^*(E)/\mathcal{J},$$

where:

- $E^0 = \mathtt{I} \cup \{0\}$ and $E^1 = \{(i,j) : i \geq j\}$,
- $\mathcal{J} = \langle s_0 - s_0^* s_0 \rangle$.

What happen when $\mathtt{I}$ is infinite?

The algebra of monotone creators

Assume $\mathtt{I}$ infinite (countable).

$\mathcal{A}_m(\mathtt{I})$: the C*-subalgebra of $B(\mathcal{F}_{wm})$ generated by all creators $\{A_i^\dagger\}_{i\in\mathtt{I}}$.

Observation (Crismale, Del Veccio, Rossi, Wysoczański 2025)

In $B(\mathcal{F}_{wm})$ the following relations hold:

$$P_i = A_i^\dagger A_i, \qquad Q_i = A_i A_i^\dagger,$$

where

- P_i : the projection onto the subspace of $\mathcal{F}_a$ consisting of vectors that starts in H_i
- Q_i : the projection onto the subspace of $\mathcal{F}_a$ spanned by Ω and vectors that starts in H_j with $j \geq i$.

Consequently, for any $i \in \mathtt{I}$, $A_i^\dagger$ is a partial isometry with the initial projection Q_i and the range projection P_i.

The algebra of monotone creators

Assume $\mathtt{I}$ infinite (countable).

$\mathcal{A}_m(\mathtt{I})$: the C*-subalgebra of $B(\mathcal{F}_{wm})$ generated by all creators $\{A_i^\dagger\}_{i\in\mathtt{I}}$.

Observation (Crismale, Del Veccio, Rossi, Wysoczański 2025)

In $B(\mathcal{F}_{wm})$ the following relations hold:

$$P_i = A_i^\dagger A_i, \qquad Q_i = A_i A_i^\dagger,$$

where

- P_i : the projection onto the subspace of $\mathcal{F}_a$ consisting of vectors that starts in H_i
- Q_i : the projection onto the subspace of $\mathcal{F}_a$ spanned by Ω and vectors that starts in H_j with $j \geq i$.

Consequently, for any $i \in \mathtt{I}$, $A_i^\dagger$ is a partial isometry with the initial projection Q_i and the range projection P_i.

Is $\mathcal{A}_m(\mathtt{I})$ a graph algebra?

In search of a generalization of (Cuntz-Krieger) graph algebras to infinite graphs two definitions appeared:

Definition (Fowler-Laca-Raeburn 2000)

Let $E = (E^0, E^1, r, s)$ be a directed graph. The (Cuntz-Krieger) graph C*-algebra of E, denoted by $C^*(E)$, is the universal C*-algebra generated by a family of mutually orthogonal projections $\{p_v \colon v \in E^0\}$ and partial isometries $\{s_e \colon e \in E^1\}$ with orthogonal ranges and satisfying:

(CK1) $s_e^* s_e = p_{r(e)}$,

(CK2) $s_e s_e^* \leq p_{s(e)}$,

(CK3) $p_v = \sum_{e:s(e)=v} s_e s_e^*$ for $0 < |s^{-1}(v)| < \infty$.

Exel-Laca algebras (1990)

Let $\mathtt{I}$ be an arbitrary (discrete) set, $M = (m(i,j))_{i,j\in\mathtt{I}}$ be a 0-1 matrix with no identically zero rows. The Exel-Laca algebra, $\mathcal{O}_M$, is the universal C*-algebra generated by a family of partial isometries $\{s_i\}_{i\in\mathtt{I}}$, with the initial projections $q_i := s_i^* s_i$ and the range projections $p_i := s_i s_i^*$, st :

(EL1) $q_i q_j = q_j q_i$,

(EL2) p_i and p_j are orthogonal for $i \neq j$,

(EL3) $q_i p_j = m(i,j) p_j$

(EL4) for each pair of finite subsets X, Y of $\mathtt{I}$ such that the set

$$J_{X,Y} := \{j \in \mathtt{I} : \prod_{x\in X} m(x,j) \prod_{y\in Y} [1 - m(y,j)] = 1\}$$

is finite the following relation holds

$$\prod_{x\in X} q_x \prod_{y\in Y} (1 - q_y) = \sum_{j\in J_{X,Y}} p_j .$$

The algebra of monotone creators

Theorem (Crismale, Del Veccio, Rossi and Wysoczański 2025)

Set $M = (m(i,j))_{i,j\in\mathcal{I}}$, with $\mathtt{I} = \mathbb{N}$ or $\mathbb{Z}$,

$$m(i,j) = \begin{cases} 1 & \text{if } i \geq j, \\ 0 & \text{if } i < j. \end{cases}$$

Then

1. The algebra of monotone creators indexed by $\mathbb{Z}$ is the Exel-Laca algebra associated to M: $\mathcal{A}_w(\mathbb{Z}) \cong \mathcal{O}_M$;
2. The algebra of monotone creators indexed by $\mathbb{N}$ is a quotient of the Exel-Laca algebra associated to M: $\mathcal{A}_w(\mathbb{N}) \cong \mathcal{O}_M/\mathcal{J}$;

[CDRW] also started studying the algebra of antimonotone creators indexed by $\mathbb{N}$.

Boolean-antimonotone creators

BA-Fock space

Let

- $(\mathtt{I}, \preceq)$: a partially ordered, countable set
- $\{\mathcal{H}_i : \ i \in \mathtt{I}\}$: a family of one-dimensional Hilbert spaces with a basic vector $e_i \in \mathcal{H}_i$ of norm 1
- $\mathrm{Inc}(n) := \{\bar{i} = (i_1, \dots, i_n) \in \mathtt{I}^n : \quad i_1 \preceq i_2 \preceq \dots \preceq i_n\}$: the set of (comparable and) nondecreasing sequences of indices in $\mathtt{I}$ of length n

We define the *discrete boolean-antimonotone Fock space*

$$\mathcal{F}_{ba} = \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} \bigoplus_{\bar{i} \in \mathrm{Inc}(n)} H_{i_1} \otimes \ldots \otimes H_{i_n}.$$

The creation operator $A_i^{\dagger}$ (by e_i) on $\mathcal{F}$:

$$A_i^{\dagger} h = \begin{cases} e_i & \text{if } h = \Omega \\ e_i \otimes h_{i_1} \otimes \dots \otimes h_{i_n} & \text{if } h = h_{i_1} \otimes \dots \otimes h_{i_n}, i \preceq i_1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

BA-algebra

$\mathcal{A}_{ba}(\mathtt{I})$: the C*-subalgebra of $B(\mathcal{F}_{ba})$ generated by ba-creators $\{A_i^\dagger\}_{i\in\mathtt{I}}$.

Observation

In $B(\mathcal{F}_{ba})$ the following relations hold:

$$P_i = A_i^\dagger A_i, \qquad Q_i = A_i A_i^\dagger,$$

where

- P_i : the projection onto the subspace of $\mathcal{F}_{ba}$ consisting of vectors that starts in H_i
- Q_i : the projection onto the subspace of $\mathcal{F}_{ba}$ spanned by Ω and vectors that starts in H_j with $j \succeq i$.

BA-algebra

$\mathcal{A}_{ba}(\mathtt{I})$: the C*-subalgebra of $B(\mathcal{F}_{ba})$ generated by ba-creators $\{A_i^\dagger\}_{i\in\mathtt{I}}$.

Observation

In $B(\mathcal{F}_{ba})$ the following relations hold:

$$P_i = A_i^\dagger A_i, \qquad Q_i = A_i A_i^\dagger,$$

where

- P_i : the projection onto the subspace of $\mathcal{F}_{ba}$ consisting of vectors that starts in H_i
- Q_i : the projection onto the subspace of $\mathcal{F}_{ba}$ spanned by Ω and vectors that starts in H_j with $j \succeq i$.

Moreover, let $m(i,j) = 1$ if $i \preceq j$ and $m(i,j) = 0$ otherwise. Then for any $i, j \in \mathtt{I}$, we have

$$\begin{aligned}
&(\mathrm{EL1})\ P_iP_j = \delta_{ij}\\
&(\mathrm{EL2})\ Q_iQ_j = Q_jQ_i \quad (= Q_{i\vee j} \text{ if } i\vee j \text{ exists})\\
&(\mathrm{EL3})\ Q_iP_j = m(i,j)P_j.
\end{aligned}$$

We are left with

(EL4) for each pair of finite subsets X, Y of $\mathtt{I}$ such that the set

$$J_{X,Y} := \{j \in \mathtt{I} \colon \prod_{x \in X} m(x,j) \prod_{y \in Y} [1 - m(y,j)] = 1\}$$
$$= \{j \in \mathtt{I} \colon \forall_{x \in X}\ (x \preceq j) \wedge \forall_{y \in Y}\ (j \prec y \text{ or } j \not\sim y)\}$$

is finite the following relation holds

$$\prod_{x \in X} q_x \prod_{y \in Y} (1 - q_y) = \sum_{j \in J_{X,Y}} p_j.$$

Counterexample

Let $\mathtt{I} = \{a, b, c\}$ with the ordering $a \prec b$, $a \prec c$ and $b \not\sim c$.

$$X = \{b, c\}, Y = \emptyset \ \Rightarrow J_{X,Y} = \emptyset \ \overset{(EL4)}{\Rightarrow} \ q_b q_b = 0.$$

But $Q_b Q_c = P_\Omega$, since Q_i is projection onto $\mathbb{C}\Omega$ and $H_j \otimes \ldots$ with $j \succeq i$

For the sequel of this talk we assume $\mathtt{I}$ is a countable, locally finite, leafless, rooted tree:

- **tree**: an undirected graph in which any two vertices are connected by exactly one path
- **rooted**: one of its vertex (the *root*) is singled out.
- **locally finite**: for every vertex, the number of edges that are incident to this vertex is finite.
- **leafless**: every vertex has at least one child.

Assumptions

For the sequel of this talk we assume $\mathtt{I}$ is a countable, locally finite, leafless, rooted tree:

- **tree**: an undirected graph in which any two vertices are connected by exactly one path
- **rooted**: one of its vertex (the *root*) is singled out.
- **locally finite**: for every vertex, the number of edges that are incident to this vertex is finite.
- **leafless**: every vertex has at least one child.

We consider the partial order on $\mathtt{I}$ given by the *out-orientation*, that is $i \prec j$ if i is closer to the root than j.

BA-algebra for rooted trees

Let $\mathtt{I}$ is a countable, locally finite, leafless, rooted tree with the partial order given by the out-orientation.

Lemma

(1) If $\mathtt{I}$ is linearly ordered (i.e. $\mathtt{I} = \mathbb{N}$), then the ba-creators satisfy (EL4).
(2) If $\mathtt{I}$ is not linearly ordered, it has at least two incomparable elements, say $i \not\sim k$, then

- $P_\Omega = Q_i Q_k \in \mathcal{A}_{ba}(\mathtt{I})$,
- (EL4) holds up to $P_\Omega = 0$.
- the closed two-sided $*$-ideal in $\mathcal{A}_{ba}(\mathtt{I})$ generated by P_Ω equals to the compact operators in $B(\mathcal{F}_{ba})$.

BA-algebra for rooted trees

Theorem (AK, Spielberg, Wysoczański)

Let $(\mathtt{I}, \preceq)$ be countable, locally finite, leafless, rooted tree with the out-order. Let $\mathcal{O}_M$ denotes the Excel-Laca algebra associated to the matrix $M = (m(i,j))_{i,j\in\mathtt{I}}$ with

$$m(i,j) = \begin{cases} 1 & \text{if } i \preceq j, \\ 0 & \text{if } i \succ j \text{ or } i \not\sim j. \end{cases}$$

Then the C*-algebra $\mathcal{A}_{ba}(\mathtt{I})$, generated by all creators $\{A_i^+\}_{i\in\mathtt{I}}$ on the ba-Fock space $\mathcal{F}_{ba}$, is $*$-isomorphic to:

1. $\mathcal{O}_M$ if $\mathtt{I}$ is linearly ordered;
2. the essential unital extension of $\mathcal{O}_M$ by $\mathcal{K}$ if $\mathtt{I}$ is not linearly ordered.

Relation to graph algebras

Graph C*-algebras and Exel-Laca algebras

Observation (Fowler, Laca, Raeburn 2000)

For a 0-1-matrix M indexed by I define $\mathrm{Gr}(M)$ to be the graph with the vertex set I and the adjacency matrix M. If M is a row-finite, then also $\mathrm{Gr}(M)$ is row-finite and $\mathcal{O}_M \simeq C^*(\mathrm{Gr}(M))$.

Observation (Fowler, Laca, Raeburn 2000)

For a 0-1-matrix M indexed by I define $\mathrm{Gr}(M)$ to be the graph with the vertex set I and the adjacency matrix M. If M is a row-finite, then also $\mathrm{Gr}(M)$ is row-finite and $\mathcal{O}_M \simeq C^*(\mathrm{Gr}(M))$.

Let A be a C*-algebra and let p be a projection.

- *corner* of A is the set $pAp = \{pap \colon a \in A\}$ (a C*-subalgebra of A)
- The corner is *full* if ApA spans a dense subspace of A.

Theorem (Katsura, Muhly, Sims, Tomforde 2010)

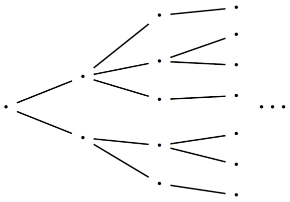
For any Exel-Laca algebra $\mathcal{O}_A$ there exists a directed graph E such that the $\mathcal{O}_A$ is $*$-isomorphic to a full corner of the graph algebra $C^*(E)$.

The construction can be quite complicated.

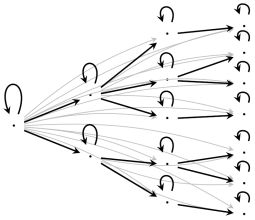
Let $\mathtt{I}$ be countable, locally finite, leafless, rooted tree with a out-order, and let M be given by

$$m(i,j) = \begin{cases} 1 & \text{if } i \preceq j, \\ 0 & \text{if } i \succ j \text{ or } i \not\sim j. \end{cases}$$

Consider the graph $\mathrm{Gr}(\mathtt{I})$ to have the vertex set $\mathtt{I}$ and the adjacency matrix M:



Example of a tree $\mathtt{I}$



$\mathrm{Gr}(\mathtt{I})$

We see that M is not row finite (and so is $\mathrm{Gr}(\mathtt{I})$), so we cannot expect $\mathcal{O}_M \cong C^*(\mathrm{Gr}(\mathtt{I}))$.

Definition

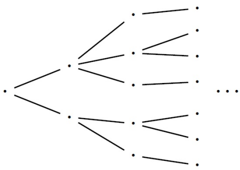
Fix $\mathtt{I}$ be as assumed. We define the graph $E = (E^0, E^1, s, r)$ as

$$E^0 = \{z_i \colon i \in \mathtt{I}\}, \quad E^1 = \{\alpha_i, \beta_i \colon i \in \mathtt{I}\}$$

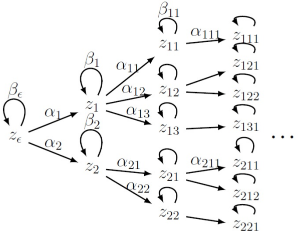
and

$$s(\alpha_i) = z_{p(i)}, \quad r(\alpha_i) = z_i, \qquad s(\beta_i) = r(\beta_i) = z_i, \qquad i \in \mathtt{I}.$$

Above $p(i)$ denotes the (unique) parent of i.



Example of a tree $\mathtt{I}$



The associated graph E

Note that E is row-finite and has no terminal circuits.

Exel-Laca vs. graph algebras

Given $\mathtt{I}$, $C^*(E)$ is generated by the mutually orthogonal projections $\{z_i \colon i \in \mathtt{I}\}$ and the partial isometries $\{\alpha_i, \beta_i \colon i \in \mathtt{I}\}$ which, for any $i \in \mathtt{I}$, satisfy

$$\alpha_j^* \alpha_j = z_j, \quad \beta_i^* \beta_i = z_i, \quad \beta_i \beta_i^* + \sum_{j \in \mathrm{CH}_i} \alpha_j \alpha_j^* = z_i.$$

(A warning: the names of the vertices and edges of E also denote the corresponding projections and partial isometries in $C^*(E)$.)
For $i, j \in \mathtt{I}$ with $i \prec j$ there exists the unique sequence $(i_1, i_2, \ldots, i_{n-1}, i_n)$ such that $i_1 = i$, $i_n = j$ and $p(i_{k-1}) = i_k$ for $k = 2, \ldots, n$.

$$\alpha_{i \to j} := \alpha_{i, i_2} \alpha_{i_2, i_3} \ldots \alpha_{i_{n-1}, j} \in C^*(E)$$

The algebra $\mathcal{A}_{ba}(\mathtt{I})$ vs. graph algebra

Theorem (AK, Spielberg, Wysoczański)

Let $\mathtt{I}$ be a countable, locally finite, leafless, rooted tree with the order given by out-orientation. Let $M = (m(i,j))_{i,j\in\mathtt{I}}$ be given by $m(i,j) = 1$ if $i \preceq j$ and 0 otherwise. Let E be the graph described on the previous page and let $C^*(E)$ be the associated graph algebra.
In $C^*(E)$ define

$$S_i = \alpha_{\epsilon\to i}\beta_i(\alpha_{\epsilon\to i})^*$$

Then:

$$\mathcal{O}_M \ni s_i \mapsto S_i \in z_\epsilon C^*(E) z_\epsilon$$

is a $*$-isomorphism.

Corollary

Let $\mathtt{I}$ be a countable, locally finite, leafless, rooted tree with the order given by out-orientation. Then $\mathcal{A}_{ba}(\mathtt{I})$ is either the full corner of the graph algebra $C^*(E)$ or its essential extension by compacts.

Properties of $\mathcal{A}_{ba}(\mathtt{I})$

Theorem (AK, Spielberg, Wysoczański)

For any countable, locally finite, leafless, rooted tree $\mathtt{I}$ with the out-orientation, the algebra $\mathcal{A}_{ba}(\mathtt{I})$ is:

- is nonsimple, not purely infinite and not of type I;
- it has the following K-groups:

$$K_0(\mathcal{A}_{ba}(\mathtt{I})) = \begin{cases} \mathbb{Z}, & \mathtt{I} \text{ linearly ordered} \\ \mathbb{Z}^p, & \mathtt{I} \text{ not linearly ordered}, \ p = 2 + \sum_{i\in\mathtt{I}}(|\mathrm{CH}_i| - 1) \end{cases}$$

$$K_1(\mathcal{A}_{ba}(\mathtt{I})) = \{0\}.$$

Compare with:

- $\mathcal{A}_{free}(\mathbb{N})$ is simple, purely infinite, not type I;
- $\mathcal{A}_{wm}(\mathbb{N})$ is type I;
- $\mathcal{A}_{am}(\mathbb{N})$ is not type I.

Final remarks and references

1. We show that the algebra of boolean-antimonotonic creators on countable, locally finite, leafless, rooted tree is closely related to graph algebra and conclude many of its properties.
2. For trees with leaves we have similarily that $\mathcal{A}_{ba}(\mathtt{I}) \cong O_M \otimes \mathcal{K}$.

Final remarks and references

1. We show that the algebra of boolean-antimonotonic creators on countable, locally finite, leafless, rooted tree is closely related to graph algebra and conclude many of its properties.
2. For trees with leaves we have similarily that $\mathcal{A}_{ba}(\mathtt{I}) \cong O_M \otimes \mathcal{K}$.

Thank you!