Asymmetric solutions of Dirac ensembles

Workshop on Noncommutative Geometry and Graph Algebras

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Joint work with B. Bukor, M. Khalkhali, S. Kováčik, K. Magdolenová, and J. Tekel.

Talk structure

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Random noncommutative geometries

Motivation: quantum gravity

- When constructing a theory of quantum gravity one tries to make sense of a partition function that "sums or integrates over manifolds" and a path integral over some matter field(s):

$$Z = \sum \int e^{-S(g,X)} dg\, dX.$$

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- We would like to consider such integrals over spectral triples instead of manifolds.

Motivation: Noncommutative Geometry

- In NCG, Spectral Triples $(\mathcal{A}, \mathcal{H}, D)$ mimic the data given by a smooth Riemannian manifold with spin structure.

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- $\mathcal{A}$ is a involutive complex algebra acting by bounded operators on a Hilbert space $\mathcal{H}$, and D is a self-adjoint (in general unbounded) operator acting on $\mathcal{H}$. This data is required to satisfy some regularity conditions.[1]

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- In particular Riemmanian spin^c manifolds can be reconstructed from their spectral triples via Conne's Reconstruction Theorem.[2]

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Fuzzy spectral triples

- In 2015 John Barrett[3] proposed constructing a Euclidean theory of quantum gravity by integrating over some moduli space of all possible Dirac operators for a fixed algebra and Hilbert space:

$$\int_{g} e^{-S(g)} dg \to \int_{\mathcal{D}} e^{-\operatorname{Tr} S(D)} dD.$$

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- To make sense of these integrals he considered certain natural real finite spectral triples where the algebra and Hilbert space are replaced by an algebra of matrices. Motivated by fuzzy noncommutative spaces, e.g. the fuzzy sphere and fuzzy torus[4].

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- Such integrals are well-defined for finite matrix size N. By taking the large N limit one hopes to recover a continuum theory.

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Fuzzy spectral triple

- A *fuzzy spectral triple* (aka fuzzy geometry) of signature (p, q) is a real finite spectral triple of the form

$$(M_N(\mathbb{C}), V \otimes M_N(\mathbb{C}), D; J, \Gamma)$$

where V is an irreducible Clifford module for the Clifford algebra $Cl_{p,q}$ with real structure J_V and grading Γ_V[5].

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- A *Dirac ensemble*[6] is a probability distribution over a set of fuzzy spectral triples with all data fixed except for the Dirac operator:

$$\frac{1}{Z} e^{-S(D)} dD.$$

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Fuzzy Spectral Triples

- The axioms of real spectral triples are solved to give formulas for the Dirac operator on fuzzy geometries[7]:

$$D = \sum_j \alpha_i \otimes [L_j, \cdot] + \sum_k \beta_k \otimes \{H_k, \cdot\} + \sum_\ell \alpha_\ell \otimes \{L_\ell, \cdot\} + \sum_r \beta'_\ell \otimes [H_r, \cdot],$$

  where the alpha's and beta's are certain products of gamma matrices that depend on the space of spinors. H's are Hermitian and L are traceless skew-Hermitian $N \times N$ matrices.

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- For example,
  - type $(1, 0)$, $D = \{H, \cdot\}$
  - type $(2, 0)$, $D = \gamma^1 \otimes \{H_1, \cdot\} + \gamma^2 \otimes \{H_2, \cdot\}$

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A type $(1,0)$ ensemble

- For example, let our spectral triple be $(M_N(\mathbb{C}), V \otimes M_N(\mathbb{C}), \{H, \cdot\})$ with
$$Z = \int_{\mathcal{D}} e^{-t_2 \operatorname{Tr} D^2 - t_4 \operatorname{Tr} D^4} dD,$$
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- The measure becomes the Lebesgue measure on the space of $N \times N$ Hermitian matrices:
$$dD = dH = \prod_{i=1}^{N} dH_{ii} \prod_{1 \leq i < j \leq N} d(\operatorname{Re}(H_{ij}))\, d(\operatorname{Im}(H_{ij})).$$

A type $(1,0)$ ensemble

- The integral

$$Z = \int_{\mathcal{D}} e^{-t_2 \,\mathrm{Tr}\, D^2 - t_4 \,\mathrm{Tr}\, D^4} dD$$

then becomes a bi-tracial matrix integral

$$= \int_{\mathcal{H}_N} \exp(-(2Nt_2 \mathrm{Tr} H^2 + 2t_2 (\mathrm{Tr} H)^2 + 2Nt_4 \mathrm{Tr}(H^4)$$
$$+ 8t_4 \mathrm{Tr} H \mathrm{Tr} H^3 + 6t_4 (\mathrm{Tr} H^2)^2)) dH.$$

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- In general, Dirac ensembles with polynomial potentials are bi-tracial multi-matrix ensembles.

Barrett and Glaser's conjecture

- Barrett and Glaser, conjectured that Dirac ensembles exhibit spectral phase transitions[8], where the distribution of the eigenvalues of D behaves like the spectrum of a Dirac operator on a manifold.

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- Work has been done on finding such phase transitions[9] as well as extracting geometric information[10]
- The goal of this talk is to explain the newly studied critical behaviour of several quartic Dirac ensembles.

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Spectral phase transitions

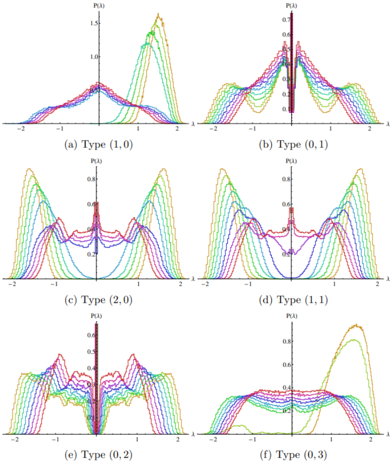

Figure: MC simulations of Barrett and Glaser. The eigenvalues of $S = \mathrm{Tr}\left(D^4 + g_2 D^2\right)$ for $n = 10$ and $g_2 = -1$, $-1.5, -2, -2.5, -3, -3.5, -4, -4.5, -5$. The lines are coloured from red ($g_2 = -1$) through to yellow ($g_2 = -5$).

The Riemann-Hilbert approach

The models

- In this talk we are interested in fuzzy geometries with

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- The quartic spectral action is

$$\begin{aligned} S(D) &= t_2 \operatorname{Tr} D^2 + t_4 \operatorname{Tr} D^4 \\ &= -2t_2(N \operatorname{Tr} H^2 + \varepsilon(\operatorname{Tr} H)^2) \\ &\quad - 2t_4(N \operatorname{Tr} H^4 + 4\varepsilon \operatorname{Tr} H \operatorname{Tr} H^3 + 3(\operatorname{Tr} H^2)^2) \end{aligned}$$

  where the sign parameter is

$$\varepsilon = 1 \quad \text{for } (1,0), \qquad \varepsilon = -1 \quad \text{for } (0,1).$$

Large N moments

Theorem (Khalkhali and N.)

The eigenvalue counting measure of H, and by extension D, converges weakly to a compactly supported measure with a continuous density function ρ i.e.

$$\frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i(H)}(x) \xrightarrow[\text{weakly}]{} \rho(x)dx,$$

and is the measure that minimizes the energy function

$$I[\mu] = \int\int \big(U(x,y) - \ln|x-y|\big)\, d\mu(x)d\mu(y),$$

where

$$U(x,y) = t_4(2(x^4+y^4)+12x^2y^2+8\epsilon(x^3y+xy^3)) + t_2(2(x^2+y^2)+4\epsilon xy).$$

- For these models, the support is either one or two compact intervals, “1-cut" or “2-cut" solutions.

Energy minimization

Theorem (Khalkhali and N.)

The large N density function of the distribution of eigenvalues of H, ρ, is solution of the saddle point equation:

$$\begin{aligned} P.V. \int_{\Sigma} \frac{\rho(y)}{y-x} dy &= 8t_4 x^3 + 24 t_4 m_2 x + 24\epsilon t_4 m_1 x^2 \\ &+ 8\epsilon t_4 m_3 x + 4t_2 x + 4\epsilon t_2 m_1 \\ &=: Q'(x), \end{aligned}$$

where $m_i := \int x^i \rho(x) dx$.

- Symmetric solutions were found in various works and there is a phase transition between the 1-cut and 2-cut solutions[11], rigorously proving Barrett and Glaser's conjecture for these models.

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Asymmetric solutions

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- However, the solutions of the minimizing solutions of the saddle point equation are not always the symmetric ones. This leads to “symmetry breaking" solutions.
- The same such asymmetric solutions appear in other approaches such as the Schwinger-Dyson equations and Monte Carlo simulations (MC) and are interesting from a physical perspective.

Resolvent formulation

- Define the Cauchy transform

$$W(z) = \int \frac{\rho(y)}{z - y} \, \mathrm{d}y$$

for $z \in \mathbb{C} \setminus \mathrm{supp}(\rho)$.

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$$W_+(x) + W_-(x) = Q'(x), \qquad x \in \mathrm{supp}(\rho).$$

- The density is recovered from the jump

$$\rho(x) = \frac{1}{2\pi i}\big(W_-(x) - W_+(x)\big).$$

Endpoint polynomial

- For k cuts, introduce

$$q(z) = \prod_{i=1}^{k} (z - a_i)(z - b_i).$$

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- The square root captures the endpoint singularities of W.

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- The jump of G is explicit in terms of Q'.
- The scalar boundary value problem is
$$G_+(z) - G_-(z) = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_{\mathrm{supp}(\rho)} \dfrac{\frac{i}{\pi} Q'(s)}{(s-z)(\sqrt{q(s)})_+}\,\mathrm{d}s, & z \in \mathrm{supp}(\rho), \\ 0, & z \in \mathbb{R} \setminus \mathrm{supp}(\rho). \end{cases}$$

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- The unknowns are now endpoints and moments.

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- The formula gives the resolvent once endpoints and moments are fixed and the discontinuity gives ρ. We can derive algebraic constraints from the asymptotic expansion of the resolvent.

Constraints from infinity

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- Here

$$H[\rho](x) = \frac{1}{\pi} \, \mathrm{P.V.} \int \frac{\rho(y)}{x - y} \, \mathrm{d}y.$$

Moment closure

- The Riemann–Hilbert formula produces ρ in terms of endpoint and moment parameters. Additional constraints are given by

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- For these quartic Dirac ensembles, the relevant moments are m_1, m_2, m_3 and m_4 (for the free energy later). Symmetric solutions set $m_1 = m_3 = 0$ while asymmetric solutions require solving the full nonlinear system.

One-cut density

Theorem

The spectral density function of the 1-cut $[a, b]$ solution of the type $(1, 0)$ and $(0, 1)$ quartic Dirac ensemble is

$$\rho(x) = \frac{1}{2\pi}\left(\frac{1}{\gamma^2} - \frac{24\epsilon t_4(x-\alpha)\left(\alpha + 24\alpha\gamma^4 t_4\right)}{24\gamma^4\epsilon t_4 - 1} + 8t_4\left(-2\alpha^2 - \gamma^2 + x^2 + \alpha x\right)\right)\sqrt{(x-a)(b-x)}_{[a,b]},$$

where $\alpha = \frac{a+b}{2}$ and $\gamma = \frac{a-b}{4}$ and are the solutions of two algebraic equations.

Symmetric 1-cut

- The 1-cut solution is of the form:

$$\rho(x) = \frac{1}{2\pi}\left(8t_4(x^2-\gamma^2)+\frac{1}{\gamma^2}\right)\sqrt{4\gamma^2-x^2}_{[-2\gamma,2\gamma]},$$

where the support $[-2\gamma, 2\gamma]$ is the real positive root of

$$192t_4^2\gamma^8 + 48t_4\gamma^4 + 4t_2\gamma^2 = 1,$$

which can be written as

$$\gamma^2 = \frac{1}{4\sqrt{3}}\left(\sigma\sqrt{-\frac{2}{t_4}+R}+\sqrt{-\frac{4}{t_4}-\sigma\frac{\sqrt{3}t_2}{t_4^2\sqrt{R}}-R}\right)$$

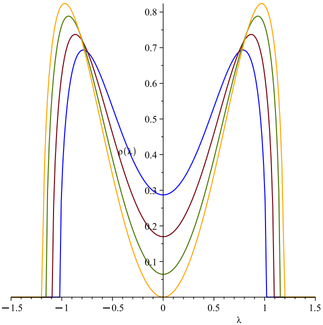
where

$$R := \sqrt[3]{\frac{3t_2^2+32t_4}{4t_4^4}}$$

and $\sigma = 1$ for $t_2 \leq 0$ and $\sigma = -1$ for $t_2 > 0$.

Transition to symmetric 2-cut

- In the quartic model there are two phases with a transition curve of $t_2 = -4\sqrt{2}t_4$.



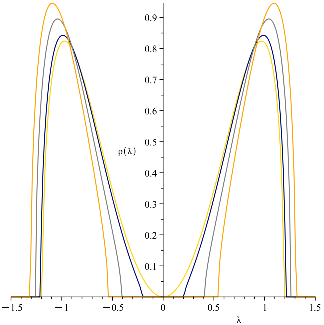


Figure: The quartic model plots of ρ for $t_4 = 1$ and various values of t_2.

Type $(0,1)$ from the RH equations

- For $\varepsilon = -1$, the asymmetric one-cut equations simplify strongly. The algebraic equations rule out a stable asymmetric phase. The preferred phases are **only** symmetric one-cut and symmetric two-cut and the transitions is a second order transition. However, this is not true in the type $(1,0)$ ensemble.

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- In general, the asymmetric solutions are extremely complicated. In the $(1,0)$ model there is a phase transition between the symmetric single cut phase and an asymmetric two cut phase. This was first discovered in[12].

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- Constraints are derived in the same manner but are even more complicated.

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- Candidate densities are compared using

$$I[\rho] = \iint \big(U(x,y) - \log|x-y|\big)\rho(x)\rho(y)\,\mathrm{d}x\,\mathrm{d}y.$$

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- The preferred transition is

$$\text{symmetric one-cut} \quad \longrightarrow \quad \text{asymmetric two-cut.}$$

  and the symmetric two-cut solution is bypassed completely.

Spectral density plots

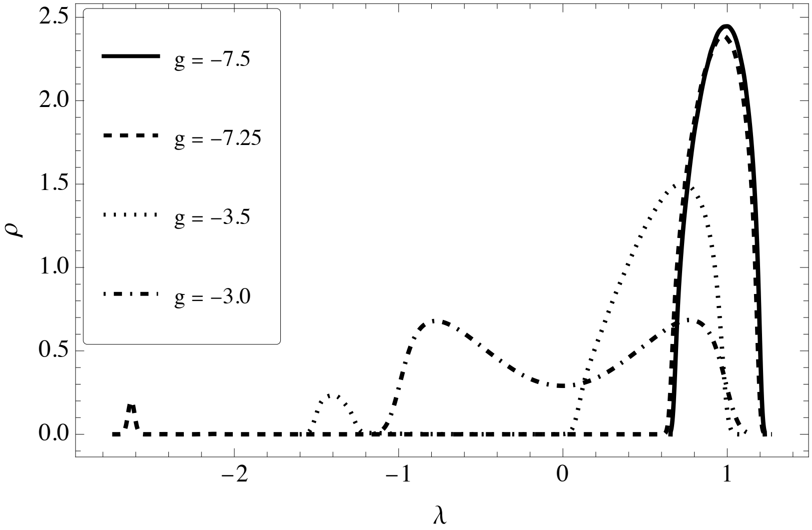

Figure: Comparison of the eigenvalue density function for various solutions of the $(1,0)$ model for $t_4 = 1$.

Bootstrapping with positivity

Bootstrapping with positivty

- Bootstrapping with posivity was first introduced to matrix models by Lin[13] and has been used to study the symmetric solutions of Dirac ensembles[14].

[13] Henry W Lin (2020). "Bootstraps to strings: solving random matrix models with positivity". In: *Journal of High Energy Physics* 2020.6, pp. 1–28.

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- This technique takes a truncation of Schwinger–Dyson equations and constraints derived from the positivity of the measure, and then uses them to bound the possible values of moments in terms of the coupling constants.

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Schwinger–Dyson equations

- One can derive the SDE as a consequence of the total derivative vanishing
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- For $\ell>0$, the equations of these quartic models are

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- The left-hand side comes from the Vandermonde/eigenvalue repulsion and the right-hand side comes from differentiating the multi-trace action.

Moment positivity

- The Hamburger moment problem can be formulated as follows: given a sequence $(m_0, m_1, m_2, \ldots)$ of real numbers, one asks if there is a positive Borel measure μ on the real line so that m_k is the k th moment of μ, that is

$$m_k = \int_{\mathbb{R}} x^k d\mu(x), \quad k = 0, 1, 2, \ldots$$

It is known that a necessary and sufficient condition for the existence of μ is that the Hankel matrix of the moments

$$\mathcal{M} = \begin{bmatrix} m_0 & m_1 & m_2 & m_3 & \cdots \\ m_1 & m_2 & m_3 & m_4 & \cdots \\ m_2 & m_3 & m_4 & m_5 & \cdots \\ m_3 & m_4 & m_5 & m_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

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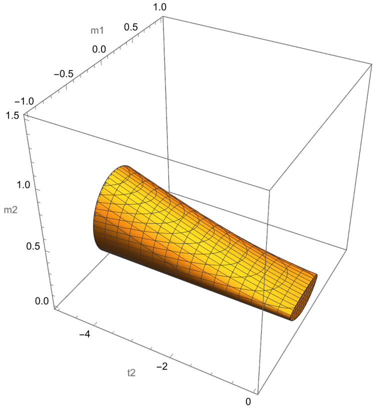
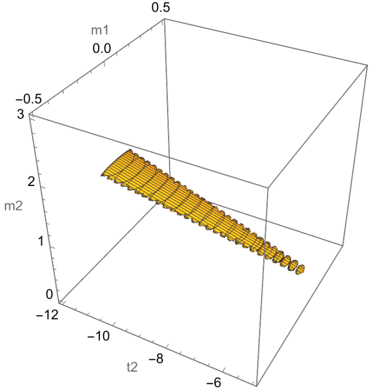
- Progressively larger submatrices and more SDE give sharper constraints. Bootstrap estimates can be used to reconstruct the eigenvalue distribution and free energy.

Bootstrap region for type $(0, 1)$





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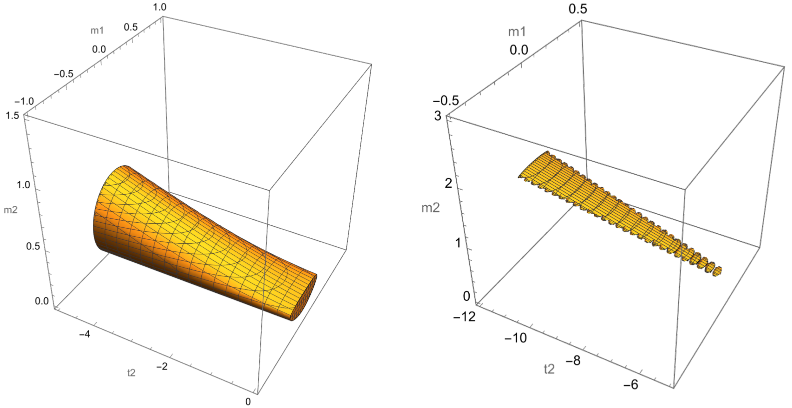

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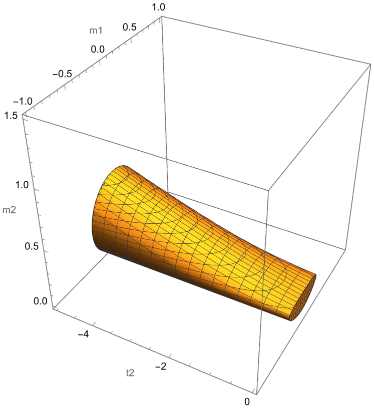

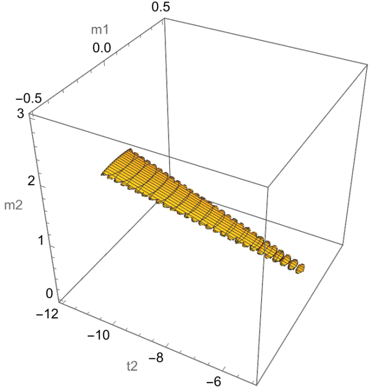

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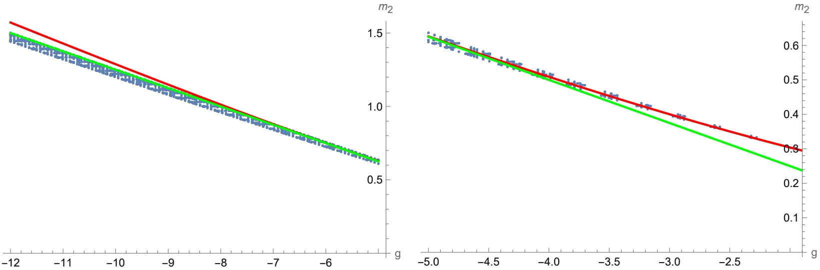

- The bootstrapped m_2 values follow the analytic symmetric solutions.

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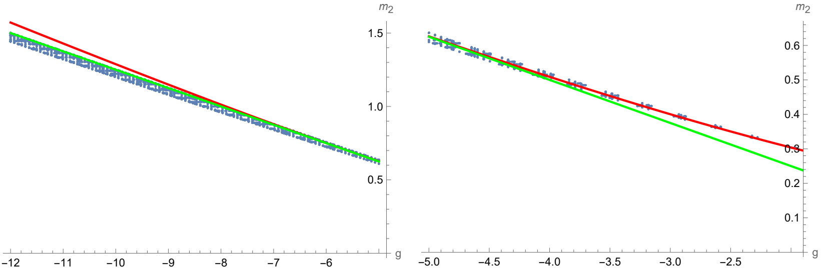

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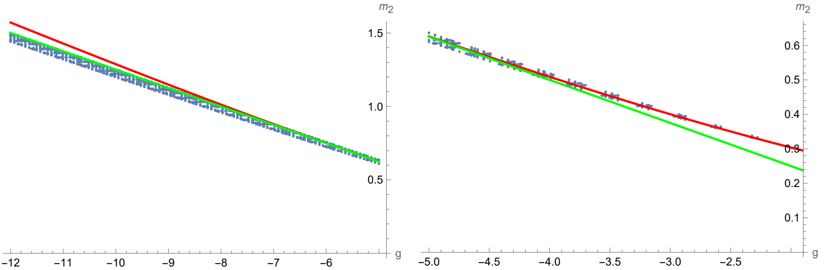

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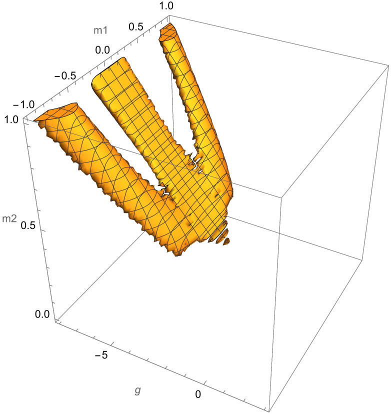
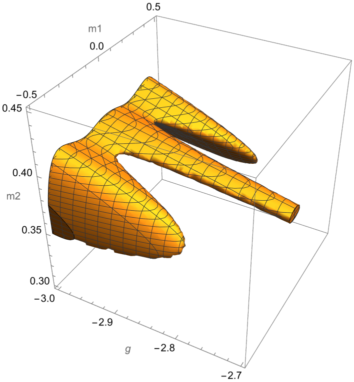
Bootstrap fork for type $(1, 0)$





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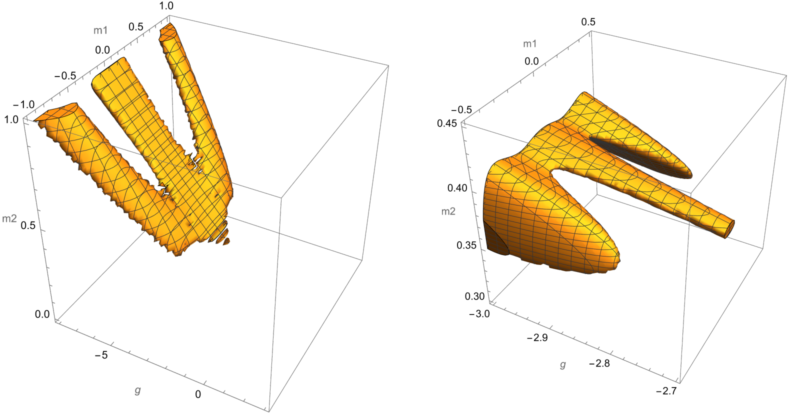

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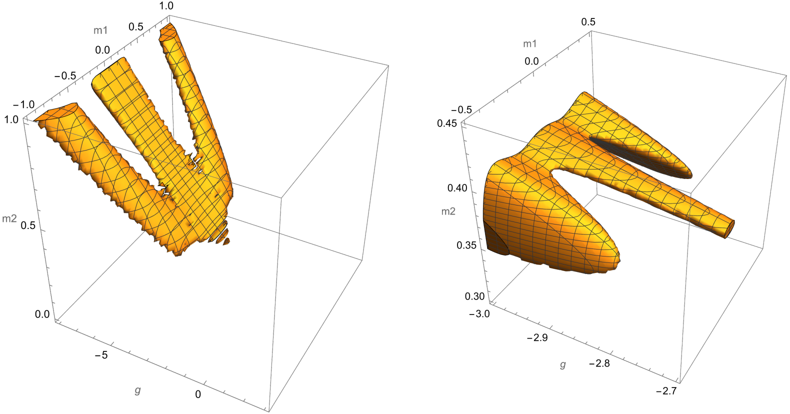

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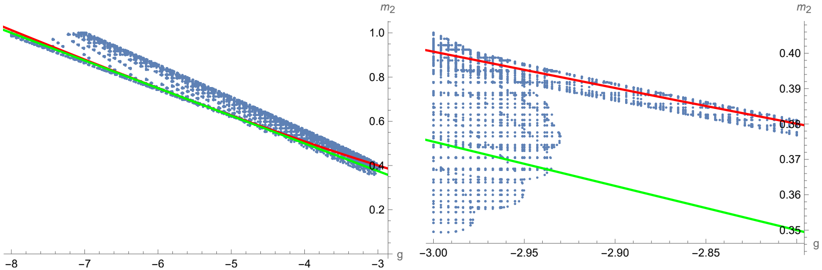

- The refined search separates symmetric and asymmetric solutions.

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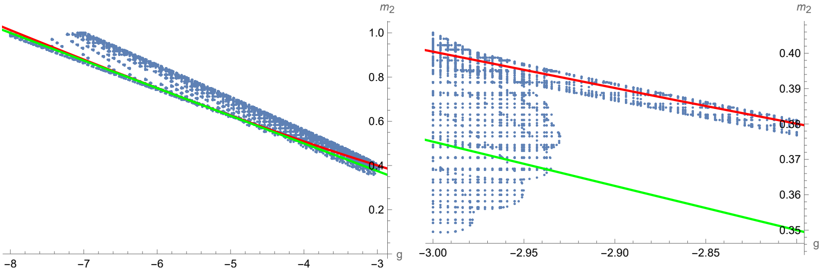

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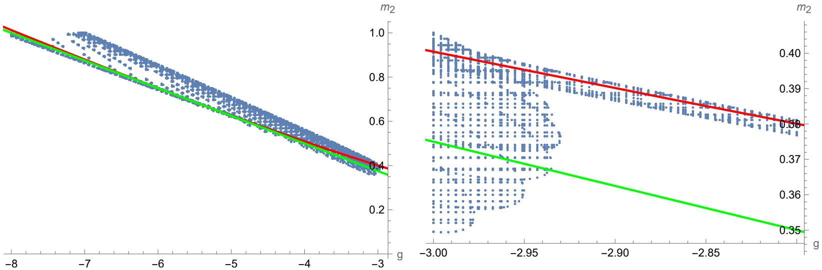

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Reconstructed asymmetric densities

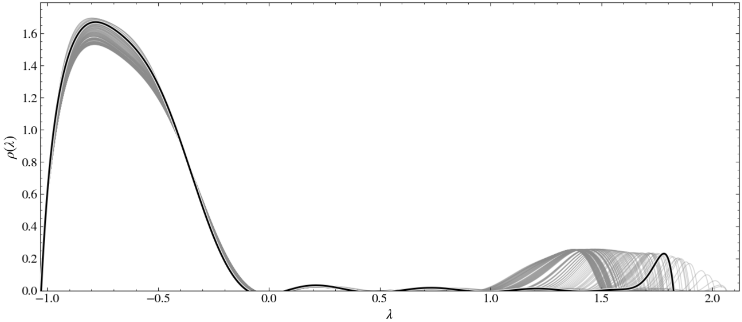
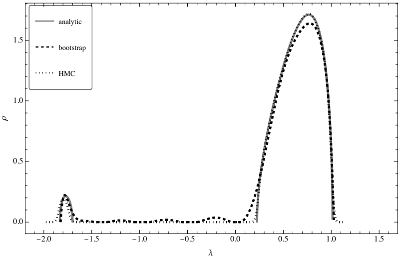
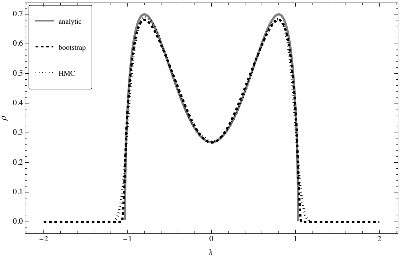


Figure: Two-cut asymmetric eigenvalue distributions reconstructed from the bootstrap moments for $g = -4$. By reconstructing the distributions across different values of g, we determine the energetically preferred solutions and compare the symmetric and asymmetric phases. The preferred solution for this value of g is highlighted

Analytic, bootstrap, and HMC comparison





- The analytic Riemann–Hilbert density, bootstrap reconstruction, and HMC histogram agree well.

Summary

- Quartic Dirac ensembles lead to multi-trace one-matrix models.
- The large N saddle point equation is a scalar Riemann–Hilbert problem.
- The Riemann–Hilbert solution gives explicit one-cut and two-cut density formulae as well as formulae for the free energy.
- $(0, 1)$ remains symmetric: one-cut to symmetric two-cut.
- $(1, 0)$ has a stable asymmetric two-cut phase.
- Bootstrap and HMC independently recover the same picture.

Thank you. Questions?