Quantum Complex Grassmannian Gr(2,4)

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1. Grassmann Mannifold Gr(2,4)

- **Definition:** $\mathrm{Gr}(2,4) := \{V \subset \mathbb{C}^4 : \dim(V) = 2\}$
- **Compact homogenous space:** $\mathrm{Gr}(2,4) = \mathrm{SU}(4)/\mathrm{S}\big(\mathrm{U}(2)\times\mathrm{U}(2)\big)$
- **Complex manifold:** $\mathrm{Gr}(2,4) := \mathrm{SL}(4,\mathbb{C})/P$

$$P = \left\{ A \in \mathrm{SL}(4,\mathbb{C}) : A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix},\ A_{ij} \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) \right\}$$

- **Schubert Calculus:** $\mathrm{Gr}(2,4) := \{\ell \subset \mathbb{CP}^3 : \ell = V/\mathbb{C} \text{ is a line}\}$
- **Noncommutative Version:**

$$\mathcal{O}(\mathrm{Gr}_q(2,4)) := \mathcal{O}\Big(\mathrm{SU}_q(4)\,/\,\mathrm{S}\big(\mathrm{U}_q(2)\times\mathrm{U}_q(2)\big)\Big)$$

- **Question:** Do Schubert's counting methods survive q-deformations?

2. Plücker Embedding of Gr(2,4)

- **Definition:** $\mathbb{P}(\Lambda^2\mathbb{C}^4) \cong \mathbb{CP}^5 = \{[v] : v \in \mathbb{C}^6 \setminus \{0\}\}$

  $$\mathrm{Gr}(2,4) \ni V = \mathrm{span}\{u,v\} \longmapsto [u \wedge v] \in \mathbb{P}(\Lambda^2\mathbb{C}^4)$$

  $$\mathbb{CP}^3 \ni \ell = [V] \longmapsto [u \wedge v] \in \mathbb{P}(\Lambda^2\mathbb{C}^4)$$

$\Rightarrow$ independent of the basis $\{u,v\}$ $\Rightarrow$ injective

$\Rightarrow$ not surjective: $(u\wedge v) \wedge (u \wedge v) = 0$

- **Plücker Coordinates and Plücker relation:**

  $$\Lambda^2\mathbb{C}^4 = \mathrm{span}\{e_1\wedge e_2,\ e_1\wedge e_3,\ e_1\wedge e_4,\ e_2\wedge e_3,\ e_2\wedge e_4,\ e_3\wedge e_4\}$$

  $$\omega = x_{12}\, e_1\wedge e_2 + x_{13}\, e_1\wedge e_3 + x_{14}\, e_1\wedge e_4 + x_{23}\, e_2\wedge e_3 + x_{24}\, e_2\wedge e_4 + x_{34}\, e_3\wedge e_4$$

  $$\omega \wedge \omega = 0 \ \Rightarrow\ x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} = 0$$

  $\dim\big(\mathbb{P}(\Lambda^2\mathbb{C}^4)\big) - 1 = 4 = \dim(\mathrm{Gr}(2,4))$, irreducible

$\Rightarrow$ $\mathrm{Gr}(2,4) \cong \{[(z_1,z_2,z_3,z_4,z_5,z_6)] \in \mathbb{CP}^5 : z_1 z_6 - z_2 z_5 + z_3 z_4 = 0\}$

3. Cell Decomposition of Gr(2,4)

- **Standard Flag:** $\{0\} \subset \mathbb{C} \subset \mathbb{C}^2 \subset \mathbb{C}^3 \subset \mathbb{C}^4, \;\; 1 \leq \sigma_1 < \sigma_2 \leq 4$

$$b_1 = (0,0,0,1),\; b_2 = (0,0,1,0),\; b_3 = (0,1,0,0),\; b_4 = (1,0,0,0)$$

$$e(\sigma_1,\sigma_2) = \{V \in \mathrm{Gr}(2,4) : \dim(V \cap \mathbb{C}^{\sigma_1 - 1}) = 0,\; \dim(V \cap \mathbb{C}^{\sigma_1}) = 1,$$
$$\dim(V \cap \mathbb{C}^{\sigma_2 - 1}) = 1,\; \dim(V \cap \mathbb{C}^{\sigma_2}) = 2\}$$

- **Schubert Cells:** $z_1,\, z_2,\, z_3,\, z_4 \in \mathbb{C}$

$$\emptyset : \left[\begin{matrix}(0 & 1 & z_4 & z_3)\\ (1 & 0 & z_2 & z_1)\end{matrix}\right] = e(3,4), \qquad \square : \left[\begin{matrix}(0 & 0 & 1 & z_3)\\ (1 & z_2 & 0 & z_1)\end{matrix}\right] = e(2,4)$$

$$\ydiagram{1,1} : \left[\begin{matrix}(0 & 0 & 1 & z_2)\\ (0 & 1 & 0 & z_1)\end{matrix}\right] = e(2,3), \qquad \ydiagram{2} : \left[\begin{matrix}(0 & 0 & 0 & 1)\\ (1 & z_2 & z_1 & 0)\end{matrix}\right] = e(1,4)$$

$$\ydiagram{2,1} : \left[\begin{matrix}(0 & 0 & 0 & 1)\\ (0 & 1 & z_1 & 0)\end{matrix}\right] = e(1,3), \qquad \ydiagram{2,2} : \left[\begin{matrix}(0 & 0 & 0 & 1)\\ (0 & 0 & 1 & 0)\end{matrix}\right] = e(1,2)$$

$\Rightarrow$ 6 open cells. **Problem:** Gluing functions?

4. Plücker Coordinates

- $\mathrm{Gr}(2,4) = \left\{ \begin{bmatrix} (u_1 & u_2 & u_3 & u_4) = u \\ (v_1 & v_2 & v_3 & v_4) = v \end{bmatrix} : \dim(\mathrm{span}\{u,v\}) = 2 \right\}$
- $x_{ij} := \det \begin{pmatrix} u_i & u_j \\ v_i & v_j \end{pmatrix} \Rightarrow (u_1 b_i + u_j b_j) \wedge (v_1 b_i + v_j b_j) = x_{ij}\, b_i \wedge b_j, \ \ i < j$

$\Rightarrow \quad x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} = 0$

$$\begin{bmatrix} (0 & 1 & z_4 & z_3) \\ (1 & 0 & z_2 & z_1) \end{bmatrix} = e(3,4), \qquad x_{12} \neq 0$$

$$\begin{bmatrix} (0 & 0 & 1 & z_3) \\ (1 & z_2 & 0 & z_1) \end{bmatrix} = e(2,4) \qquad x_{12} = 0, \ \ x_{13} \neq 0$$

$$\begin{bmatrix} (0 & 0 & 1 & z_2) \\ (0 & 1 & 0 & z_1) \end{bmatrix} = e(2,3), \qquad x_{12} = x_{13} = x_{14} = 0, \ \ x_{23} \neq 0$$

$$\begin{bmatrix} (0 & 0 & 0 & 1) \\ (1 & z_2 & z_1 & 0) \end{bmatrix} = e(1,4) \qquad x_{12} = x_{13} = x_{23} = 0, \ \ x_{14} \neq 0$$

$$\begin{bmatrix} (0 & 0 & 0 & 1) \\ (0 & 1 & z_1 & 0) \end{bmatrix} = e(1,3), \qquad x_{12} = x_{13} = x_{14} = x_{23} = 0, \ \ x_{24} \neq 0$$

$$\begin{bmatrix} (0 & 0 & 0 & 1) \\ (0 & 0 & 1 & 0) \end{bmatrix} = e(1,2) \qquad x_{12} = \ldots = x_{24} = 0, \ \ x_{34} \neq 0$$

5. Orthogonal Plücker Coordinates

- $\mathrm{Gr}(2,4) = \left\{ \left[\begin{matrix} (u_{11} & u_{12} & u_{13} & u_{14}) = u_1 \\ (u_{21} & u_{22} & u_{23} & u_{24}) = u_2 \end{matrix} \right] : \langle u_l, u_k \rangle_{\mathbb{C}^4} = \delta_{kl},\ \ k,i = 1,2 \right\}$
- **Plücker Relation:** $x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} = 0$
- $\mathrm{Gr}(2,4) = \left\{ \left[\begin{matrix} (0 & \bar{x}_{34} & \bar{x}_{24} & \bar{x}_{23}) \\ (t & x_{12} & -x_{13} & x_{14}) \end{matrix} \right] : (x_{12} : x_{13} : x_{14} : x_{23} : x_{24} : x_{44}) \in \mathbb{CP}^5 \right\}$
- $|x_{34}|^2 + |x_{24}|^2 + |x_{24}|^2 = 1, \quad t^2 = 1 - (|x_{12}|^2 + |x_{13}|^2 + |x_{14}|^2), \ \ t \geq 0$

$\Rightarrow \quad |x_{ij}| \leq 1 \ \ \Rightarrow \ \ x_{ij} \in \bar{\mathbb{D}} := \{ z \in \mathbb{C} : |z| \leq 1 \}$

- **Circle Bundle:**

$$z_1 := x_{12},\ \ z_2 := x_{13},\ \ z_3 := x_{14},\ \ z_4 := t x_{23},\ \ z_5 := t x_{24},\ \ z_6 := t x_{34}$$

$\Rightarrow \ \ |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2 + |z_6|^2 = 1 - t^2 + t^2 = 1$

$\Rightarrow \ \ \mathbb{S}^{11} \ni (z_1, z_2, z_3, z_4, z_5, z_6) \mapsto (z_1 : z_2 : z_3 : z_4 : z_5 : z_6) \in \mathbb{CP}^5 = \mathbb{S}^{11}/U(1)$

6. CW-Decomposition in Disc Coordinates

$$\mathrm{Gr}(2,4) = \left\{ \begin{bmatrix} (0 & z_6 & \bar z_5 & \bar z_4) \\ (t & z_1 & -z_2 & z_3) \end{bmatrix} : \begin{array}{l} z_1 z_6 - z_2 z_5 + z_3 z_4 = 0, \quad t \in [0,1] \\ |z_4|^2 + |z_5|^2 + z_6^2 = 1, \\ t^2 + |z_1|^2 + |z_2|^2 + |z_3|^2 = 1 \end{array} \right\}$$

$$X^0 = e(1,2) = \begin{bmatrix} (0 & 0 & 0 & 1) \\ (0 & 0 & 1 & 0) \end{bmatrix} \cong \{\mathrm{pt}\}$$

$$X^2 = e(1,3) = \begin{bmatrix} (0 & 0 & 0 & 1) \\ (0 & y_1 & -\sqrt{1-|y_1|^2} & 0) \end{bmatrix} \cong \mathbb{CP}^1$$

$$\begin{tikzcd}
y_1 \mapsto (y_1{:}-\sqrt{1-|y_1|^2}) & & X^2 = \mathbb{CP}^1 & & 1 \mapsto (1{:}0) \\
& \bar{\mathbb{D}} \arrow[ur, "\Phi_2^L"] & & X^0 = \{1\} \arrow[ul, hook', "\Phi_2^R"'] & \\
e^{\mathrm{i}\theta} \mapsto e^{\mathrm{i}\theta} & & \mathbb{S}^1 \arrow[ul, hook, "\iota"] \arrow[ur, "\rho_2"'] & & e^{\mathrm{i}\theta} \mapsto 1
\end{tikzcd}$$

7. CW-Decomposition: e(1,4)

$$X^2 = e(1,3) = \left[\begin{matrix} (0 & 0 & 0 & 1) \\ (0 & y_1 & -\sqrt{1-|y_1|^2} & 0) \end{matrix} \right] \cong \mathbb{CP}^1 \cong \bar{\mathbb{D}}/\sim$$

$$e(1,4) = \left[\begin{matrix} (& 0 & 0 & 0 & 1) \\ (& \sqrt{(1-|y_1|^2)(1-|y_2|^2)} & y_1 & -\sqrt{1-|y_1|^2}\,y_2 & 0) \end{matrix} \right] \cong \mathbb{CP}^2$$

$$\begin{tikzcd}
 & \mathbb{CP}^2 & \\
\bar{\mathbb{D}} \times \bar{\mathbb{D}} \arrow[ur, "\varphi_4^L"] & & X^2 = \mathbb{CP}^1 = \bar{\mathbb{D}}/\sim \arrow[ul, hook, "\varphi_4^R"'] \\
 & \partial(\bar{\mathbb{D}} \times \bar{\mathbb{D}}) \arrow[ul, hook, "\iota"] \arrow[ur, "\rho_4^1 = \Phi_2^L \times \{1\}"'] &
\end{tikzcd}$$

$$\left(\sqrt{(1-|y_1|^2)(1-|y_2|^2)} : y_1 : -\sqrt{1-|y_1|^2}\,y_2\right)$$

$$(y_1 : -\sqrt{1-|y_1|^2}) \mapsto (0 : y_1 : -\sqrt{1-|y_1|^2})$$

$$(y_1, y_2) \mapsto (y_1, y_2)$$

$$(y_1, y_2) \mapsto (y_1 : -\sqrt{1-|y_1|^2})$$

$$\Rightarrow \quad \mathbb{CP}^2 = (\bar{\mathbb{D}} \times \bar{\mathbb{D}}) / \sim$$

8. CW-Decomposition: e(2,4)

$$X^2 = e(1,3) = \left[\begin{matrix} (0 & 0 & 0 & 1) \\ (0 & y_1 & -\sqrt{1-|y_1|^2} & 0) \end{matrix}\right] \cong \mathbb{CP}^1 \cong \bar{\mathbb{D}}/\sim$$

$$e(2,3) = \left[\begin{matrix} (0 & 0 & \sqrt{1-|y_3|^2} & \bar{y}_3 &) \\ (0 & y_1 & -\sqrt{1-|y_1|^2}y_3 & \sqrt{1-|y_1|^2}\sqrt{1-|y_3|^2} &) \end{matrix}\right] \cong \mathbb{CP}^2$$

$$\Rightarrow \sqrt{1-|y_1|^2}(-y_3, \sqrt{1-|y_3|^2}) \perp (\sqrt{1-|y_3|^2}, \bar{y}_3) \quad \text{(unique up to scalars)}$$

$$\begin{tikzcd}
& \mathbb{CP}^2 & \\
\bar{\mathbb{D}} \times \bar{\mathbb{D}} \arrow[ur, "\phi_4^L"] & & X^2 = \mathbb{CP}^1 = \bar{\mathbb{D}}/\sim \arrow[ul, hook', "\phi_4^R"'] \\
& \partial(\bar{\mathbb{D}} \times \bar{\mathbb{D}}) \arrow[ul, hook, "\iota"] \arrow[ur, "\rho_4^2 = \Phi_2^L \times \{1\}"] &
\end{tikzcd}$$

$\left(y_1 : -\sqrt{1-|y_1|^2}\,y_3 : \sqrt{(1-|y_1|^2)(1-|y_3|^2)}\right)$

$(y_1 : -\sqrt{1-|y_1|^2}) \mapsto (y_1 : -\sqrt{1-|y_1|^2} : 0)$

$(y_1, y_3) \mapsto (y_1, y_3)$

$(y_1, y_3) \mapsto (y_1 : -\sqrt{1-|y_1|^2})$

$$\Rightarrow \mathbb{CP}^2 = (\bar{\mathbb{D}} \times \bar{\mathbb{D}}) / \sim$$

$$X^4 = \left[\begin{matrix} (& 0 & 0 & 0 & 1) \\ (& \sqrt{(1-|y_1|^2)(1-|y_2|^2)} & y_1 & -\sqrt{1-|y_1|^2}\,y_2 & 0) \end{matrix}\right]$$

$$\coprod_{\mathbb{CP}^1} \left[\begin{matrix} (0 & 0 & \sqrt{1-|y_3|^2} & \bar{y}_3 &) \\ (0 & y_1 & -\sqrt{1-|y_1|^2}\,y_3 & \sqrt{1-|y_1|^2}\sqrt{1-|y_3|^2} &) \end{matrix}\right]$$

- $\lim(y_3 : \sqrt{1-|y_3|^2}) = (1:0) \Rightarrow \lim(e(1,4)) = e(1,3) \cong \mathbb{CP}^1$
- $\lim(y_2 : \sqrt{1-|y_2|^2}) = (1:0) \Rightarrow \lim(e(2,3)) = e(1,3) \cong \mathbb{CP}^1$

$$\begin{tikzcd}
 & X^4 = e(1,4) \coprod\limits_{\mathbb{CP}^1} e(2,3) & \\
\bar{\mathbb{D}}\times\bar{\mathbb{D}} \coprod \bar{\mathbb{D}}\times\bar{\mathbb{D}} \arrow[ur, "\varphi_4^L \sqcup \phi_4^L"'] & & X^2 = \mathbb{CP}^1 \arrow[ul, hook', "\Phi_4^R", "(y_1 : -\sqrt{1-|y_1|^2}) \mapsto (0 : y_1 : -\sqrt{1-|y_1|^2} : 0)"'] \\
 & \partial(\bar{\mathbb{D}}\times\bar{\mathbb{D}} \coprod \bar{\mathbb{D}}\times\bar{\mathbb{D}}) \arrow[ul, hook', "\iota \sqcup \iota"] \arrow[ur, "\rho_4 = \rho_4^1 \sqcup \rho_4^2"'] &
\end{tikzcd}$$

$$X^6 = e(2,4) = \left[\begin{matrix} (0 & 0 & \sqrt{1-|y_3|^2} & \bar{y}_3 &) \\ (t & y_1 & -\sqrt{1-|y_1|^2}y_2y_3 & \sqrt{1-|y_1|^2}\sqrt{1-|y_3|^2}\,y_2 &) \end{matrix} \right]$$

- $\lim(y_3 : \sqrt{1-|y_3|^2}) = (1:0) \Rightarrow \lim(e(2,4)) = e(1,4)$
- $\lim(y_2 : \sqrt{1-|y_2|^2}) = (1:0) \Rightarrow \lim(e(2,4)) = e(2,3)$

$$\begin{tikzcd}
& X^6 = e(2,4) & \\
\bar{\mathbb{D}} \times \bar{\mathbb{D}} \times \bar{\mathbb{D}} \arrow[ur, "\Phi_6^L"] & & X^4 = \mathbb{CP}^2 \coprod_{\mathbb{CP}^1} \mathbb{CP}^2 \arrow[ul, hook', "\Phi_6^R"'] \\
& \partial(\bar{\mathbb{D}} \times \bar{\mathbb{D}} \times \bar{\mathbb{D}}) \arrow[ul, hook, "\iota"] \arrow[ur, "\rho_6 = \varphi_4^L \underset{\mathbb{CP}^1}{\times} \phi_4^L"'] &
\end{tikzcd}$$

$(t : y_1 : -\sqrt{1-|y_1|^2}\,y_2 : 0) \ \coprod$

$(0 : y_1 : -\sqrt{1-|y_1|^2}\,y_3 : \sqrt{1-|y_1|^2}\sqrt{1-|y_3|^2})$

11. CW-Decomposition: X^8

$$X^6 = e(2,4) = \begin{bmatrix} (0 & 0 & \sqrt{1-|y_3|^2} & \bar{y}_3 &) \\ (t & y_1 & -\sqrt{1-|y_1|^2}\,y_2y_3 & \sqrt{1-|y_1|^2}\sqrt{1-|y_3|^2}\,y_2 &) \end{bmatrix}$$

$$X^8 = \begin{bmatrix} (0 & t_3t_4 & t_3\bar{y}_4 & \bar{y}_3 &) \\ (t_1t_2 & y_1y_4 - t_1t_4y_2y_3 & -t_4y_1 - t_1y_2y_3y_4 & t_1t_3y_2 &) \end{bmatrix}$$

$$t_1 := \sqrt{1-|y_1|^2},\ t_2 := \sqrt{1-|y_2|^2},\ t_3 := \sqrt{1-|y_3|^2},\ t_4 := \sqrt{1-|y_4|^2}$$

$$\begin{tikzcd}
 & X^8 = e(3,4) & \\
\bar{\mathbb{D}} \times \bar{\mathbb{D}} \times \bar{\mathbb{D}} \times \bar{\mathbb{D}} \arrow[ur, "\phi_8^L"'] & & X^6 = \bar{\mathbb{D}}\times\bar{\mathbb{D}}\times\bar{\mathbb{D}}/\sim \arrow[ul, hook', "\Phi_8^R", "(t\,:\,y_1\,:\,-\sqrt{1-|y_1|^2}\,y_2y_3\,:\,\sqrt{1-|y_1|^2}\sqrt{1-|y_3|^2}\,y_2)"'] \\
 & \partial(\bar{\mathbb{D}} \times \bar{\mathbb{D}} \times \bar{\mathbb{D}} \times \bar{\mathbb{D}}) \arrow[ul, hook', "\iota"'] \arrow[ur, "\rho_8 = \phi_6^L \times \{1\}"] &
\end{tikzcd}$$

12. Coordinate Functions

$$X^8 = \begin{bmatrix} (0 & t_3 t_4 & t_3 \bar{y}_4 & \bar{y}_3 \;) \\ (t_1 t_2 & y_1 y_4 - t_1 t_4 y_2 y_3 & -t_4 y_1 - t_1 y_2 y_3 y_4 & t_1 t_3 y_2) \end{bmatrix}$$

- $z_j : \bar{\mathbb{D}} \times \bar{\mathbb{D}} \times \bar{\mathbb{D}} \times \bar{\mathbb{D}} \longrightarrow \mathbb{C}, \quad j = 1, \ldots, 6,$

$$z_1(y_1, y_2, y_3, y_4) := y_1 y_4 - \sqrt{1 - |y_2|^2}\sqrt{1 - |y_4|^2}\, y_2 y_3$$
$$z_2(y_1, y_2, y_3, y_4) := \sqrt{1 - |y_4|^2}\, y_1 + \sqrt{1 - |y_1|^2}\, y_2 y_3 y_4$$
$$z_3(y_1, y_2, y_3, y_4) := \sqrt{1 - |y_1|^2}\sqrt{1 - |y_3|^2}\, y_2$$
$$z_4(y_1, y_2, y_3, y_4) := y_3$$
$$z_5(y_1, y_2, y_3, y_4) := \sqrt{1 - |y_3|^2}\, y_4$$
$$z_6(y_1, y_2, y_3, y_4) := \sqrt{1 - |y_3|^2}\sqrt{1 - |y_4|^2}$$

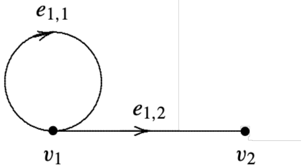
$\Rightarrow \quad z_1 z_6 - z_2 z_5 + z_4 z_3 = 0$

$$\mathrm{Gr}(2,4) = \left\{ \begin{bmatrix} (0 & \bar{z}_6 & \bar{z}_5 & \bar{z}_4) \\ (t & z_1 & -z_2 & z_3) \end{bmatrix} : (y_1, y_2, y_3, y_4) \in \bar{\mathbb{D}} \times \bar{\mathbb{D}} \times \bar{\mathbb{D}} \times \bar{\mathbb{D}} \right\}$$

$\Rightarrow \quad \mathrm{spec}\big(C^*(\{1, z_1, z_2, z_3, z_4, z_5, z_6\})\big) \cong \mathrm{Gr}(2,4)$

13. Noncommutative Version: Quantum Disk

Toepliz algebra $\mathcal{T}$:



Quantum Disc: $\mathcal{T}$ is the C*-algebra generated by y and y^* with

$$yy^* - q^2 y^* y = 1 - q^2, \quad q \in [0,1).$$

Representation: $y(e_n) = \sqrt{1-q^{2n}}\, e_{n-1}$ on $\ell_2(\mathbb{N}) = \overline{\mathrm{span}}\{e_i : i \in \mathbb{N}\}$

$\Rightarrow\ t(e_n) := \sqrt{1-|y|^2}(e_n) = q^n e_n \ \Rightarrow\ t \in \mathcal{K}(\ell_2(\mathbb{N}))$

C*-Algebra Extension:

$$\{0\} \longrightarrow \mathcal{K}(\ell_2(\mathbb{N})) \longrightarrow \mathcal{T} \longrightarrow C(\mathbb{S}^1) \xrightarrow{\sigma} \{0\}$$

Classically: $\{0\} \longrightarrow C_0(\mathbb{D}) \longrightarrow C(\bar{\mathbb{D}}) \longrightarrow C(\mathbb{S}^1) \longrightarrow \{0\}$

$$X^8 = \begin{bmatrix} (0 & t_3t_4 & t_3y_4^* & y_3^* \;) \\ (t_1t_2 & y_1y_4 - t_1t_4y_2y_3 & -t_4y_1 - t_1y_2y_3y_4 & t_1t_3y_2) \end{bmatrix}$$

$$= \begin{bmatrix} (0 & z_6^* & z_5^* & z_4^*) \\ (t_1t_2 & z_1 & -z_2 & z_3) \end{bmatrix}$$

Hilbert Space Representation: $\mathcal{H} = \ell(\mathbb{N}) \otimes \ell(\mathbb{N}) \otimes \ell(\mathbb{N}) \otimes \ell(\mathbb{N})$

- $y_1|n_1n_2n_3,n_4\rangle = \sqrt{1-q^{2n_1}}|n_1-1,n_2,n_3,n_4\rangle$

  $y_2|n_1n_2n_3,n_4\rangle = \sqrt{1-q^{2n_2}}|n_1,n_2-1,n_3,n_4\rangle$

  $y_3|n_1n_2n_3,n_4\rangle = \sqrt{1-q^{2n_3}}|n_1,n_2,n_3-1,n_4\rangle$

  $y_4|n_1n_2n_3,n_4\rangle = \sqrt{1-q^{2n_4}}|n_1,n_2,n_3,n_4-1\rangle$

$\Rightarrow \quad t_j|n_1n_2n_3,n_4\rangle = q^{n_j}|n_1,n_2,n_3,n_4-1\rangle, \quad j = 1,\dots,4$

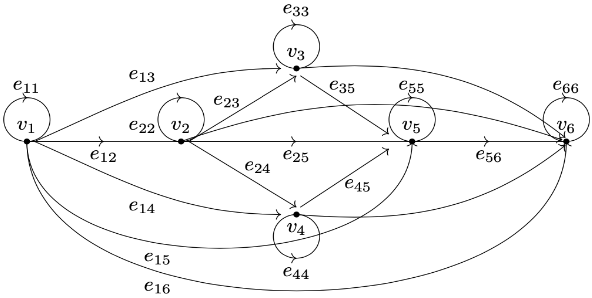
$\Rightarrow \quad q^2z_6z_1 - qz_5z_2 + z_4z_3 = 0$ q-Plücker relation!

- $\left\{ \begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ u_{21} & u_{22} & u_{23} & u_{24} \end{pmatrix} : u_{ij} \in \mathcal{O}(\mathrm{SU}_q(4)),\ \sum_{i=1}^{4} u_{ki}u_{li}^* = \delta_{kl} \right\}$
- $z_{jk} := u_{j1}u_{k2} - u_{k1}u_{j2}$, $(j,k) \in \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$

$\Rightarrow$ $z_{12}z_{12}^* + q^2 z_{13}z_{13}^* + q^4 z_{14}z_{14}^* + q^4 z_{23}z_{23}^* + q^6 z_{24}z_{24}^* + q^8 z_{34}z_{34}^* = 1$

$\Rightarrow$ $q^2 z_{34}z_{12} - q z_{24}z_{12} + z_{23}z_{13} = 0$

$\Rightarrow$ U(1)-bundle over $\mathrm{Gr}_q(2,4)$



- $\mathbb{CP}^1$

$$\begin{tikzcd}[column sep=small]
 & & X^2 = \mathbb{CP}^1 & & \\
y_1 \mapsto (y_1 : -\sqrt{1-|y_1|^2}) & \bar{\mathbb{D}} \arrow[ur, "\Phi_2^L"'] & & X^0 = \{1\} \arrow[ul, hook', "\Phi_2^R"] & 1 \mapsto (1:0) \\
e^{i\theta} \mapsto e^{i\theta} & & \mathbb{S}^1 \arrow[ul, hook', "\iota"'] \arrow[ur, "\rho_2"] & & e^{i\theta} \mapsto 1
\end{tikzcd}$$

- $C(\mathbb{CP}^1_q)$

$$\begin{tikzcd}[column sep=small]
 & & C(\mathbb{CP}^1_q) = \mathcal{T} \underset{(\sigma_2,\iota_2)}{\oplus} \mathbb{C} \arrow[dl, hook', "\mathrm{pr}_2^L"] \arrow[dr, "\mathrm{pr}_2^R"] & & \\
 & \mathcal{T} \arrow[dr, "\sigma_2"'] & & \mathbb{C}(X^0) = \mathbb{C}\{1\} \arrow[dl, hook', "\iota_2"] & \\
z \mapsto e^{i\theta} & & C(\mathbb{S}^1) = \mathcal{T}/\mathcal{K} & & 1 \mapsto 1
\end{tikzcd}$$

17. CW Decomposition $\mathbb{CP}^2_q$

$C(X^2_q) = C(\mathbb{CP}^1_q)$:

$$\begin{tikzcd}[column sep=small]
 & C(\mathbb{CP}^1_q) = \mathcal{T} \underset{(\sigma_2,\iota_2)}{\oplus} \mathbb{C} \arrow[dl, "\mathrm{pr}_2^L"'] \arrow[dr, "\mathrm{pr}_2^R"] & \\
\mathcal{T} \arrow[dr, "\sigma_2"'] & & \mathbb{C}(X^0) = \mathbb{C}\{1\} \arrow[dl, hook', "\iota_2"', "1 \mapsto 1"] \\
 & \mathcal{T}/\mathcal{K} &
\end{tikzcd}$$

- $C(\mathbb{CP}^2_q)$:

$$\begin{tikzcd}[column sep=small]
 & C(\mathbb{CP}^2_q) = \mathcal{T}\otimes\mathcal{T} \underset{(\sigma_4,\iota_4)}{\oplus} C(X^2_q) \arrow[dl, "\mathrm{pr}_4^L"'] \arrow[dr, "\mathrm{pr}_2^R"] & \\
\mathcal{T}\otimes\mathcal{T} \arrow[dr, "\sigma_2"'] & & \mathbb{C}(X^2) = C(\mathbb{CP}^1_q) \arrow[dl, hook', "\iota_2"', "\mathrm{pr}_2^L \otimes 1"] \\
 & \frac{\mathcal{T}\otimes\mathcal{T}}{\mathcal{K}\otimes\mathcal{K}} &
\end{tikzcd}$$

18. CW Decomposition of $\mathrm{Gr}_q(1,n)$

Induction: $C(\mathbb{CP}_q^{n-1}) \hookrightarrow \mathcal{T}\otimes\ldots\otimes\mathcal{T}$ $((n-1)$-times$)$

- $C(\mathbb{CP}_q^{n})$:

$$C(\mathbb{CP}_q^{n}) = \mathcal{T}\otimes\ldots\otimes\mathcal{T}\otimes\mathcal{T} \underset{(\sigma_n,\iota_n)}{\oplus} C(\mathbb{CP}_q^{n-1})$$

$$\begin{tikzcd}
 & C(\mathbb{CP}_q^{n}) \arrow[dl, hook, "\mathrm{pr}_n^L"'] \arrow[dr, "\mathrm{pr}_n^R"] & \\
\mathcal{T}\otimes\ldots\otimes\mathcal{T}\otimes\mathcal{T} \arrow[dr, "\sigma_n"] & & C(\mathbb{CP}_q^{n-1}) \arrow[dl, hook, "\iota_n"', "\mathrm{pr}_{n-1}^L\otimes 1"] \\
 & \dfrac{(\mathcal{T}\otimes\ldots\otimes\mathcal{T})\otimes\mathcal{T}}{\mathcal{K}\otimes\ldots\otimes\mathcal{K}\otimes\mathcal{K}} &
\end{tikzcd}$$

- $\mathrm{Gr}_q(2,4))$?

  Embed generators and consider the generated (sub)-C*-algebras ...

19. CW Decomposition of $\mathbf{Gr(2,4)}$

Schubert Cells: $1 \le \sigma_1 < \sigma_2 \le 4, \quad d := (\sigma_1 - 1) + (\sigma_2 - 2)$

$$\begin{tikzcd}
 & e(\sigma_1,\sigma_2) & \\
\underset{i=1}{\overset{d}{\times}} \bar{\mathbb{D}} \arrow[ur, "\phi_d^L"] & & X^{d-2} = \Big(\underset{i=1}{\overset{d-1}{\times}} \bar{\mathbb{D}}\Big)/\sim \arrow[ul, hook', "\phi_d^R"'] \\
 & \partial\Big(\underset{i=1}{\overset{d}{\times}} \bar{\mathbb{D}}\Big) \arrow[ul, hook', "\iota"'] \arrow[ur, two heads, "\rho_d"] &
\end{tikzcd}$$

Dualizing:

$$\begin{tikzcd}
 & C(e(\sigma_1,\sigma_2)) \arrow[dl, hook', "\mathrm{pr}_d^L"] \arrow[dr, two heads, "\mathrm{pr}_n^R"'] & \\
\underset{i=1}{\overset{d}{\bigotimes}} C(\bar{\mathbb{D}}) \arrow[dr, two heads, "\sigma_d"] & & C(X^{2d-2}) \subset \underset{i=1}{\overset{d-2}{\bigotimes}} C(\bar{\mathbb{D}}) \arrow[dl, hook', "\iota_d"', "\color{red}{\mathrm{pr}^L_{d-1} \otimes 1}"] \\
 & \underset{i=1}{\overset{d}{\bigotimes}} C(\bar{\mathbb{D}}) \Big/ \underset{i=1}{\overset{d}{\bigotimes}} C_0(\mathbb{D}) &
\end{tikzcd}$$

Schubert Cycle:

$$
\begin{tikzcd}
& C(\mathbb{CP}^1_q) = \mathcal{T} \underset{(\sigma_2,\iota_2)}{\oplus} \mathbb{C} \arrow[dl, hook, "\color{red}{\mathrm{pr}_2^L}"] \arrow[dr, two heads, "\mathrm{pr}_2^R"'] & \\
\mathcal{T} \arrow[dr, two heads, "\sigma_2"] & & \mathbb{C}(X^0) = \mathbb{C}\{1\} \arrow[dl, hook', "\iota_2"', "\color{red}{1\mapsto 1}"] \\
& C(\mathbb{S}^1) = \mathcal{T}/\mathcal{K} &
\end{tikzcd}
$$

Induction:

$$
\begin{tikzcd}
& C(e_q(\sigma_1,\sigma_2)) \arrow[dl, hook, "\mathrm{pr}_d^L"'] \arrow[dr, two heads, "\mathrm{pr}_n^R"'] & \\
\bigotimes_{i=1}^{d} \mathcal{T} \arrow[dr, two heads, "\sigma_d"] & & C(X_q^{2d-2}) \subset \bigotimes_{i=1}^{d-2} C(\bar{\mathbb{D}}) \arrow[dl, hook', "\iota_d"', "\color{red}{\mathrm{pr}_{d-1}^L \otimes 1}"] \\
& \bigotimes_{i=1}^{d} \mathcal{T} \Big/ \bigotimes_{i=1}^{d} \mathcal{K} &
\end{tikzcd}
$$

Caution: $C(e_q(\sigma_1,\sigma_2)) \underset{C(X_q^{d-2})}{\bigoplus} C(e_q(\lambda_1,\lambda_2)) \subset \bigotimes_{i=1}^{d} \mathcal{T}$ **?**

Schochet Spectral Sequence: $E^1_{j,k} \cong E^\infty_{j,k}$

$$(\varnothing) \longleftarrow \left(\ydiagram{1}\right) \longleftarrow \left(\ydiagram{1,1}\right) \oplus \left(\ydiagram{2}\right) \longleftarrow \left(\ydiagram{2,1}\right) \longleftarrow \left(\ydiagram{2,2}\right)$$

$$\begin{array}{cccccccccc}
\mathbb{Z} \xleftarrow{\mathrm{d}_1} & 0 \xleftarrow{\mathrm{d}_1} & \mathbb{Z} \xleftarrow{\mathrm{d}_1} & 0 \xleftarrow{\mathrm{d}_1} & \mathbb{Z}\oplus\mathbb{Z} \xleftarrow{\mathrm{d}_1} & 0 \xleftarrow{\mathrm{d}_1} & \mathbb{Z} \xleftarrow{\mathrm{d}_1} & 0 \xleftarrow{\mathrm{d}_1} & \mathbb{Z} \xleftarrow{\mathrm{d}_1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbb{Z} \xleftarrow{\mathrm{d}_1} & 0 \xleftarrow{\mathrm{d}_1} & \mathbb{Z} \xleftarrow{\mathrm{d}_1} & 0 \xleftarrow{\mathrm{d}_1} & \mathbb{Z}\oplus\mathbb{Z} \xleftarrow{\mathrm{d}_1} & 0 \xleftarrow{\mathrm{d}_1} & \mathbb{Z} \xleftarrow{\mathrm{d}_1} & 0 \xleftarrow{\mathrm{d}_1} & \mathbb{Z} \xleftarrow{\mathrm{d}_1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbb{Z} \xleftarrow{\mathrm{d}_1} & 0 \xleftarrow{\mathrm{d}_1} & \mathbb{Z} \xleftarrow{\mathrm{d}_1} & 0 \xleftarrow{\mathrm{d}_1} & \mathbb{Z}\oplus\mathbb{Z} \xleftarrow{\mathrm{d}_1} & 0 \xleftarrow{\mathrm{d}_1} & \mathbb{Z} \xleftarrow{\mathrm{d}_1} & 0 \xleftarrow{\mathrm{d}_1} & \mathbb{Z} \xleftarrow{\mathrm{d}_1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

$$\Rightarrow \quad K_0(C(\mathrm{Gr}_q(2,4))) = \mathbb{Z}\oplus\mathbb{Z}\oplus(\mathbb{Z}\oplus\mathbb{Z})\oplus\mathbb{Z}\oplus\mathbb{Z}, \quad K_1(C(\mathrm{Gr}_q(2,4))) = 0$$