Nondegenerate morphisms of topological graphs

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Nondegenerate morphisms

Definition

Let A and B be C^*-algebras. A **nondegenerate *-homomorphism** $\phi : A \to M(B)$ is a $*$-homomorphism such that $\overline{\phi(A)B} = B$.

Any nondegenerate *-homomorphism extends to a *-homomorphism between the multiplier algebras, i.e. $\phi : A \to M(B)$ extends to a unique $\overline{\phi} : M(A) \to M(B)$.

If $f : X \to Y$ is a continuous map between locally compact spaces, then the induced map is $f_* : C_0(Y) \to C_b(X) \cong M(C_0(X))$ is a nondegenrate $*$-homomorphism.

C*-correspondences

Definition

A **C*-correspondence** over a C^*-algebra A is a right Hilbert A-module X with a left action $\phi : A \to \mathcal{L}(X)$. A C^*-correspondence is **nondegenerate** if $\overline{span}(A \cdot X) = X$.

Definition

A **representation** of a C^*-correspondence X over A on a C^*-algebra B is a pair of maps (ψ, π) - $\psi : X \to B$ is a linear map and $\pi : A \to B$ is a $*$-homomorphism such that the following hold:

1. $\psi(a \cdot \xi) = \pi(a)\psi(\xi)$
2. $\pi(\langle \xi | \eta \rangle) = \psi(\xi)^*\psi(\eta)$

C*-correspondences

Definition

A representation (ψ, π) is **nondegenerate** if $\overline{span}(\pi(A)B) = B = \overline{span}(\psi(X)B)$ and **injective** if π is injective.

Definition

For a C^*-correspondence X over A, define an ideal of A by:

$$\mathcal{J}_X = \{a \in A \mid \phi(a) \in \mathcal{K}(X) \text{ and } ab = 0 \ \forall b \in ker\phi\}$$

Definition

A representation is **covariant** if $\pi(a) = \psi^{(1)}(\phi(a))$ for all $a \in \mathcal{J}_X$, where $\psi^{(1)}(|\xi\rangle \langle\eta|) = \psi(\xi)\psi(\eta)^*$.

Cuntz-Pimsner algebras

Definition

The **Cuntz-Pimsner algebra** $\mathcal{O}_X$ is the C^*-algebra generated by a universal covariant representation (c_X^1, c_A^0) of X.

Example

Let (E^0, E^1, r, s) be a directed graph, let $C_0(E^0)$ be the functions vanishing at and let:

$$X_E = \{\xi : E^1 \to \mathbb{C} \mid v \mapsto \sum_{s(e)=v} |\xi(e)|^2 \text{ is in } C_0(E^0)\}$$

then X_E is a nondegenerate C^*-correspondence over $C_0(E^0)$

Cuntz-Pimsner algebras

The left and right action is given by :

$$(\xi \cdot f)(e) = \xi(e) f(s(e)) \qquad \text{for all } e \in E^1$$

$$(f \cdot \xi)(e) = f(r(e))\xi(e) \qquad \text{for all } e \in E^1$$

$$\langle \xi | \eta \rangle (v) = \sum_{s(e)=v} \overline{\xi(e)} \eta(e) \qquad \text{for all } v \in E^0$$

The Cuntz-Pimsner algebra corresponding to this C^*-correspondence is the graph C^*-algebra corresponding to the directed graph.

Multiplier correspondences

Definition

The **multiplier correspondence** $M(X)$ of a correspondence X is the Banach space $\mathcal{L}(A, X)$ with natural left action by $M(A)$, and right and the inner product $(\langle x|y\rangle = x^*y)$ given by composition of operators.

Example

For the directed graph C^*-correspondence, the multiplier correspondence $M(X_E)$ is given by:

$$\{\phi : E^1 \to \mathbb{C} \mid v \mapsto \sum_{s(e)=v} |\phi(e)|^2 \text{ is in } C_b(E^0)\}$$

Correspondence morphisms

Definition

Let X and Y be nondegenrate C^*-correspondences over A and B respectively. A **correspondence morphism** is a pair of maps (ψ, π), $\psi : X \to M(Y)$ is a linear map; $\pi : A \to M(B)$ a $*$-homomorphism such that:

1. $\psi(a \cdot \xi) = \pi(a) \cdot \psi(\xi)$
2. $\pi(\langle \xi | \eta \rangle) = \langle \psi(\xi) | \psi(\eta) \rangle$

A correspondence morphism is **proper** if $\psi(X) \subseteq Y$ and $\pi(A) \subseteq B$ and **nondegenerate** if π is nondegenerate and $\overline{span}(\psi(X) \cdot Y) = Y$.

Functoriality of correspondence morphisms

Definition

A **decorated** C^*-algebra is a triple (A, C, κ) consisting of C^*-algebras A and C and $\kappa : C \to M(A)$ is a nondegenrate *-homomorphism.

Example

Let C be a C^*-subalgebra of A with the inclusion $\iota : C \to A$. If every approximate unit of A is also an approximate unit for C, then ι is nondegenerate and consequently (A, C, ι) is a decorated C^*-algebra.

Example

Let X be a nondegenerate C^*-correspondence over A with the covariant representation (c_X^1, c_A^0), then $(\mathcal{O}_X, A, c_A^0)$ is a decorated C^*-algebra.

Functoriality of correspondence morphisms

Definition

Let (A, C, κ) be a decorated C^*-algebra, then **relative multiplier algebra** be defined as:

$$M_C(A) = \{x \in M(A) \mid \kappa(C)x \cup x\kappa(C) \subseteq A\}$$

Furthermore if (X, A) is a nondegenerate correspondence and $\kappa; C \to M(A)$ is a nondegenerate homomorphism, then **C-multipliers** of X are

$$M_C(X) = \{x \in M(X) \mid \kappa(C)x \cup x\kappa(C) \subseteq X$$

If an I is an ideal of A, then let

$$M(A; I) = \{x \in M(A) \mid xA \cup Ax \subseteq I\}$$

Functoriality of correspondence morphisms

Definition (Kaliszewski-Quigg-Robertson '13)

Let X and Y be nondegenerate C^*-correspondences over A and B respectively. A correspondence morphism (ψ, π) is a **Cuntz-Pimsner correspondence morphism** from X to Y if:

1. π is a nondegenerate $*$-homomorphism
2. $\psi(X) \subseteq M_B(Y)$
3. $\pi(J_X) \subseteq M(B; J_Y)$
4. The following diagram commutes:

$$\begin{tikzcd} J_X \arrow[r, "\pi|"] \arrow[d, "\phi_A|"'] & M(B; J_Y) \arrow[d, "\overline{\phi}_B|"] \\ K(X) \arrow[r, "\psi^{(1)}"'] & M_B(K(Y)) \end{tikzcd}$$

where $\psi^{(1)}(|\xi\rangle\langle\eta|) = \psi(\xi)\psi(\eta)^*$

Functoriality of correspondence morphisms

Theorem (Kaliszewski-Quigg-Robertson '13)

Let (X, A) and (Y, B) be nondegenerate C^*-correspondences and let (ψ, π) be a nondegenerate Cuntz-Pimsner correspondence morphism from (X, A) to (Y, B). Then there exists a unique $*$-homomorphism $\mathcal{O}_{\psi,\pi}$ such that the following diagram commutes:

$$\begin{tikzcd}
(X, A) \arrow[r, "{(\psi,\pi)}"] \arrow[d, "{(c_1^X, c_0^A)}"'] & (M_B(Y), M(B)) \arrow[d] \\
\mathcal{O}_X \arrow[r, "\mathcal{O}_{\psi,\pi}"'] & M_B(\mathcal{O}_Y)
\end{tikzcd}$$

Moreover, $\mathcal{O}_{\psi,\pi}$ is nondegenerate and is injective if π is as well.

Topological graphs

Definition

A **topological graph** is a quadruple (E^0, E^1, s, r) such that E^0, E^1 are locally compact spaces $s, r : E^1 \to E^0$ are continuous maps such that r is further a local homoeomorphism.

For a topological graph (E^0, E^1, s, r), we can define a C^*-correspondence:

$$(C_0(E^0), X_E, \langle \cdot, \cdot \rangle)$$

Topological graphs

where the Hilbert module structure is given by:

$$\langle \xi, \eta \rangle(v) = \sum_{e \in r^{-1}(v)} \overline{\xi(e)}\eta(e) \qquad \eta, \xi \in C(E^1)$$

$$X_E = \{\xi \in C_b(E^1) \mid \langle \xi, \xi \rangle \in C_0(E^0)\}$$

and the module multiplication is given by:

$$\xi f(e) = \xi(e) f(s(e))$$
$$f\xi(e) = f(r(e))\xi(e)$$

and the left action by $\pi_r : C_0(E^0) \to \mathcal{L}(X_E)$

$$(\pi_r(f)(\xi))(e) = f(r(e))\xi(e)$$

Topological graphs

Definition (Katsura)

Consider the following subsets of E^0

$$E^0_{\mathrm{sce}} = E^0 \setminus \overline{r(E^1)},$$
$$E^0_{\mathrm{fin}} = \{v \in E^0 \ : \exists \text{ a nbhd } V \text{ of } v \text{ such that } r^{-1}(V) \text{ is compact}\},$$
$$E^0_{\mathrm{rg}} = E^0_{\mathrm{fin}} \setminus \overline{E^0_{\mathrm{sce}}}, \qquad E^0_{\mathrm{sg}} = E^0 \setminus E^0_{\mathrm{rg}}.$$

Topological graphs

Example

If E^0, E^1 are discrete, then we have usual directed graphs (or quivers).

Example

Let (X, σ) be a topological dynamical system, i.e. X is a locally compact space, and σ is a homeomorphism of X. Then, $E^0 = E^1 = X, r = \sigma, s = id_X$ is a topological graph.

Topological graphs

Definition

The C^*-algebra of the topological graph ($\mathrm{C}^*(\mathrm{E})$) is the Cuntz-Pimsner algebra of the C^*-correspondence $(X_E, C_0(E^0))$.

Example

If E^0, E^1 are discrete, then $C^*(E)$ is the usual C^*-algebra of a directed graph.

Example

Let (X, σ) be a topological dynamical system, i.e. X is a locally compact space, and σ is a homeomorphism of X. Then with $E^0 = E^1 = X, r = \sigma, s = id_X$, we have $C^*(E) \cong C_0(X) \rtimes_\sigma \mathbb{Z}$.

Morphisms of topological graphs

Definition

A **topological graph morphism** $(m = m^0, m^1) : E \to F$ is a pair of continuous maps $m^0 : E^0 \to F^0$ and $m^1 : E^1 \to F^1$ such that the following diagrams commute:

$$\begin{array}{ccc} E^1 & \xrightarrow{r_E/s_E} & E^0 \\ \Big\downarrow m^1 & & m^0 \Big\downarrow \\ F^1 & \xrightarrow[r_F/s_F]{} & F^0 \end{array}$$

Furthermore, we obtain continuous maps:

$$\theta_s : E^1 \to F^1 \times_{s_F, m^0} E^0, \qquad e \mapsto (m^1(e), s_E(e))$$
$$\theta_s : E^1 \to F^1 \times_{r_F, m^0} E^0, \qquad e \mapsto (m^1(e), r_E(e))$$

Morphisms of topological graphs

Definition

A topological graph morphism $m : E \to F$ is a **factor map** is θ_s is a homeomorphism and θ_r is proper. It is furthermore **regular** if $(m^0)^{-1}(F^0_{rg}) = E^0_{rg}$.

Definition (Katsura '04)

A **proper factor map** $\tilde{m} : E \to F$ is a pair of proper maps $\tilde{m}^0 : E^0 \to F^0$ and $\tilde{m}^1 : E^1 \to F^1$ such that:

1. $r_F(\tilde{m}^1(e)) = \tilde{m}^0(r_E(e))$, $s_F(\tilde{m}^1(e)) = \tilde{m}^0(s_E(e))$
2. If $x \in F^1$ and $v \in E^0$ such that $s_F(x) = \tilde{m}^0(v)$ then there exists a unique element $e \in E^1$ such that $\tilde{m}^1(e) = x$ and $s_E(e) = v$

Morphisms of topological graphs

Proposition

If E is a topological graph with proper r and s maps, and m is a topological graph morphism satisfying (2) (i.e. source bijectivity) condition, then it is a factor map.

Example

Let X and Y be locally compact spaces with homeomorphims σ_X and σ_Y. Then let their topological graph presentation be as above. Furthermore, let $f : X \to Y$ be a continuous map such that $\sigma_Y \circ f = f \circ \sigma_X$. Then, $m = (f, f)$ is a factor map, infact as all vertices are regular in this case, m is a regular factor map. It is proper iff f is a proper map.

Functoriality of correspondence morphisms

Theorem

The assignments

$$ob(TGraph_{rg}) \ni E \mapsto C^*(E) \in ob(MC^*_{\mathbb{T}})$$
$$hom(TGraph_{rg}) \ni m \mapsto \mathcal{O}_{\overline{\mu}^1,\overline{\mu}^0} \in hom(MC^*_{\mathbb{T}})$$

where, $\overline{\mu}^i = \bar{t}^i_E \circ \mu^i$ and μ^i are the induced maps on the correspondences, is a contravariant functor from the category of topological graphs and regular factor maps ($TGraph_{rg}$) to the category of $\mathbb{T}$-C^*-algebras and nondegenerate $\mathbb{T}$-$*$-homomorphisms ($MC^*_{\mathbb{T}}$). Furthermore, $\mathcal{O}_{\overline{\mu}^1,\overline{\mu}^0}$ is nondegenerate and is injective if μ^0 is.

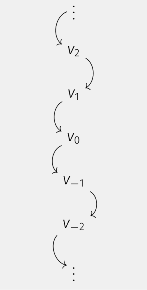
The case of directed graphs

Theorem

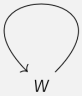
Let $m : E \to F$ be a factor map between directed graphs. Then the induced nondegenerate $*$-homomorphism $\mu : C^*(F) \to M(C^*(E))$ satisfies:

$$P_w \mapsto \sum_{m^0(v)=w} P_v \qquad \text{(strict convergence)}$$

$$S_e \mapsto \sum_{m^1(f)=e} S_f \qquad (C_0(E^0)\text{strict convergence})$$



$$C^*(E_0) \cong \mathcal{K}$$



$$C^*(E_1) \cong C(S^1)$$

The case of directed graphs

Consider a map $(m^0, m^1) : (E_0^0, E_0^1) \to (E_1^0, E_1^1)$, such that $m^0(v_n) = w$ and $m^1(e_n) = f$. This is a regular factor map and correspondingly we have $*$-homomorphism $\mu : C(S^1) \to M(\mathcal{K})$. If E_0 is embedded in $\mathbb{R}$ and E^1 in S^1, this map corresponds to the discretisation of the classical covering map $\mathbb{R} \to \mathbb{R}/2\pi\mathbb{Z} \cong S^1$.