Voltage and derived quantum graphs

Mariusz Tobolski

Uniwersytet Wrocławski

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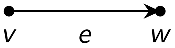
Plan of the talk

1. Classical voltage and derived graphs
2. Quantum sets and quantum graphs
3. Voltage and derived quantum graphs
4. Simple quantum graphs over $M_2(\mathbb{C})$ as derived quantum graphs
5. On a class of simple quantum graphs over $M_3(\mathbb{C})$

The talk is based on the joint work with Björn Schäfer (points 1–4) and Filip Rabiega (point 5).

A directed graph is a quadruple (E^0, E^1, r, s), where

- E^0 is the set of **vertices**,
- E^1 is the set of **edges**,
- $s : E^1 \to E^0$ is the **source** map,
- $r : E^1 \to E^0$ is the **range** map.



$$s(e) = v \qquad \text{and} \qquad r(e) = w$$

Voltage graphs

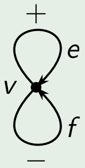
Let Γ be a group and let $E = (E^0, E^1, s, r)$ be a directed graph.

Definition (Gross, 1974)

A **voltage graph** consists of a directed graph E and a map $\alpha : E^1 \to \Gamma$, called a voltage assignment. We denote a voltage graph by (E, Γ, α).

Example

Let $\Gamma = \mathbb{Z}_2 = \{1, -1\}$. Consider the following voltage graph:



$$\alpha(e) = +1$$

$$\alpha(f) = -1$$

Derived graphs (aka skew-product graphs)

Definition (Gross–Tucker, 1974)

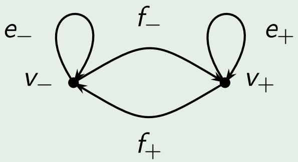
Let (E, Γ, α) be a voltage graph. A **derived graph** E_α is defined as follows

$$E^0_\alpha := E^0 \times \Gamma, \qquad E^1_\alpha := E^1 \times \Gamma,$$

$$s_\alpha(e, g) = (s(e), g), \qquad r_\alpha(e, g) = (r(e), \alpha(e)g)\,.$$

Example

Take the voltage graph from the previous example. Its derived graph is



Quantum sets

History: **Wang** (1998), **Banica** (2002), **Coecke–Pavlovic–Vicary** (2013).

Definition (Musto–Reutter–Verdon, 2018)

A **finite quantum set** $\mathbb{X}$ consists of a finite-dimensional C*-algebra $C(\mathbb{X})$ and a faithful positive functional $\psi_{\mathbb{X}}$ on $C(\mathbb{X})$ such that

$$mm^{\dagger} = \mathrm{Id}_{C(\mathbb{X})}\,,$$

where $m : \ell^2(\mathbb{X}) \otimes \ell^2(\mathbb{X}) \to \ell^2(\mathbb{X})$ is the multiplication map and $m^{\dagger}$ is its adjoint. Here $\ell^2(\mathbb{X})$ denotes the GNS space given by $\psi_{\mathbb{X}}$.

Example

1. $(C(X), \psi_X)$, where X is a finite set and ψ_X is the counting measure.
2. $(M_n(\mathbb{C}), n\mathrm{Tr})$, where Tr is the standard trace.
3. $(C(\mathbb{G}), Nh_{\mathbb{G}})$, where $\mathbb{G}$ is a finite quantum group, $h_{\mathbb{G}}$ is the Haar state, and $N = \dim C(\mathbb{G})$.

Quantum adjacency matrix

History: **Duan–Severini–Winter** (2010), **Kuperberg–Weaver** (2012).

Definition (Musto–Reutter–Verdon, 2018)

A **quantum graph** on a finite quantum set $\mathbb{X}$ is given by a linear map $A : \ell^2(\mathbb{X}) \to \ell^2(\mathbb{X})$ satisfying the conditions

$$m(A \otimes A)m^\dagger = A \qquad \text{and} \qquad A(a^*) = A(a)^* \text{ for all } a \in C(\mathbb{X}).$$

The map A is called a **quantum adjacency matrix**.

Example

If $\mathbb{X} = X_n$ is a finite set of n elements then A is an $n \times n$ matrix and the condition $m(A \otimes A)m^\dagger = A$ implies that its entries are either 0 or 1.

Such matrices (with more conditions) are used to define **Cuntz–Krieger** algebras and can be thought of as **adjacency matrices of a directed graphs** with n vertices and no multiple edges.

Simple quantum graphs

Definition

A quantum graph on $\mathbb{X}$ given by a quantum adjacency matrix A is

- **undirected** if $A = A^\dagger$,
- **without loops** if $m(A \otimes \mathrm{id}_{C(\mathbb{X})})m^\dagger = 0$,
- **simple** if it is undirected and without loops.

Theorem (Matsuda, Gromada, 2021)

Up to isomorphism, there are three nonempty simple quantum graphs on $(M_2(\mathbb{C}), 2Tr)$ with quantum adjacency matrices (in the standard basis):

1. $A_1 = \sigma_1 \otimes \sigma_1^* + \sigma_2 \otimes \sigma_2^* + \sigma_3 \otimes \sigma_3^*$,
2. $A_2 = \sigma_2 \otimes \sigma_2^* + \sigma_3 \otimes \sigma_3^*$,
3. $A_3 = \sigma_3 \otimes \sigma_3^*$,

where σ_1, σ_2, σ_3 are the Pauli matrices.

How to define voltage and derived quantum graphs?

- Quantum graphs generalize only graphs with **no multiple edges**.
- Even if the derived graph has no multiple edges, the voltage graph might have them.

As usual, we have to **reformulate** the construction of Gross and Tucker to obtain a noncommutative generalization.

- Derived graphs are defined using products but in noncommutative geometry it often better to use **crossed products**.
- We assume that the quantum (multi)graph with voltages in Γ has also an action of the dual group $\hat{\Gamma}$ and we build the derived quantum graph on the crossed product quantum set coming from this action.
- This way we can obtain genuine quantum graphs from classical graphs equipped with an action of the dual group!

Crossed-product quantum sets

Definition (Schäfer–T, 2025)

Let $\mathbb{X}$ be a finite quantum set and let γ be an action of a finite group Γ on $C(\mathbb{X})$. The **crossed product finite quantum set** $\mathbb{X} \rtimes_\gamma \Gamma$ consists of the crossed product C*-algebra $C(\mathbb{X}) \rtimes_\gamma \Gamma$ and the functional

$$\psi_\rtimes := \psi_\mathbb{X} \circ E_\rtimes,$$

where $E_\rtimes$ is the conditional expectation on $C(\mathbb{X}) \rtimes_\gamma \Gamma$ multiplied by $|\Gamma|$.

Example

Consider the set $X_n = \{1, 2, \ldots, n\}$ with an action γ of $\mathbb{Z}_n$ on X_n given by a cyclic permutation of elements. Then

$$C(X_n) \rtimes_\gamma \mathbb{Z}_n = \mathbb{C}^n \rtimes_\gamma \mathbb{Z}_n \cong M_n(\mathbb{C})$$

and under this isomorphism $\psi_\rtimes$ agrees with $n\mathrm{Tr}$.

Voltage quantum graph

Definition (Schäfer–T, 2025)

Let $\mathbb{X}$ be a finite quantum set, let Γ be a finite abelian group, and let $\hat{\gamma}$ be an action of $\hat{\Gamma}$ on $C(\mathbb{X})$. A **voltage quantum graph** on $\mathbb{X}$ is a family of quantum adjacency matrices $(A_g)_{g\in\Gamma}$ on $\mathbb{X}$ such that

$$A_g \circ \hat{\gamma}_\chi = \hat{\gamma}_\chi \circ A_g \qquad \text{for all } g \in \Gamma \text{ and } \chi \in \hat{\Gamma}.$$

Remarks:

- To recover the classical situation, one has to reformulate the notion of a voltage graph.
- We generalize voltage graphs in which there a no multiple edges with the same voltage, source, and range. The dual action is trivial here.
- The map $\sum_{g\in\Gamma} A_g$ gives rise to a quantum multigraph in the sense of Gromada.

Quantum graphs on crossed product quantum sets

Let $\mathbb{X}$ be a quantum set, let Γ be a finite abelian group, and let $\hat{\gamma}$ be an action of $\hat{\Gamma}$ on $\mathbb{X}$. For every $g \in \Gamma$, define

$$Q_g : \ell^2(\mathbb{X} \rtimes_{\hat{\gamma}} \hat{\Gamma}) \to \ell^2(\mathbb{X} \rtimes_{\hat{\gamma}} \hat{\Gamma}), \qquad a \mapsto \frac{1}{|\Gamma|} \sum_{\chi \in \hat{\Gamma}} \overline{\chi(g)} \langle u_\chi, a \rangle_{\psi_\rtimes} u_\chi \,,$$

where $(u_\chi)_{\chi \in \hat{\Gamma}}$ is the unitary representation of $\hat{\Gamma}$ on $C(\mathbb{X}) \rtimes_{\hat{\gamma}} \hat{\Gamma}$.

Proposition (Schäfer–T, 2025)

For every $g \in \Gamma$, Q_g is a quantum adjacency matrix on $\mathbb{X} \rtimes_{\hat{\gamma}} \hat{\Gamma}$.

Derived quantum graph

Let $\mathbb{X}$, Γ, $\mathbb{X} \rtimes_{\hat{\gamma}} \hat{\Gamma}$, and $(Q_g)_{g\in\Gamma}$ be as before.

Definition (Schäfer–T, 2025)

Given a voltage graph $(A_g)_{g\in\Gamma}$ on $\mathbb{X}$, we define its **derived quantum graph** on $\mathbb{X} \rtimes_{\hat{\gamma}} \hat{\Gamma}$ by specifing the quantum adjacency matrix

$$A := \sum_{g\in\Gamma} m_{\rtimes}(Q_g \otimes E_{\rtimes}^{\dagger} A_g E_{\rtimes}) m_{\rtimes}^{\dagger}.$$

Again, here $E_{\rtimes}$ is the conditional expectation on the crossed product multiplied by $|\Gamma|$.

Proposition (Schäfer–T, 2025)

The map A defined above is a quantum adjacency matrix on $\mathbb{X} \rtimes_{\hat{\gamma}} \hat{\Gamma}$.

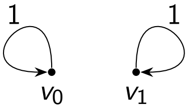
Why we call it a derived quantum graph?

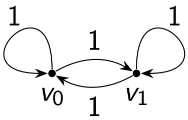
1. In the classical case, the quantum graph given by Q_g corresponds to the graph with vertex space $E^0 \times \Gamma$ that contains all edges of the form
$$(v, h) \to (w, hg) \qquad \text{for } v, w \in E^0 \text{ and } h \in \Gamma.$$
2. Similarly, the quantum graph $E_{\rtimes}^{\dagger} A_g E_{\rtimes}$ corresponds to the graph with vertex space $E^0 \times \Gamma$ that contains all edges of the form
$$(v, h) \to (w, k) \qquad \text{for } v \xrightarrow{A_g} w \in V \text{ and } h, k \in \Gamma.$$
Here $v \xrightarrow{A_g} w$ indicates that there is an edge from v to w in the graph corresponding to the adjacency matrix A_g.
3. Finally, $m(Q_g \otimes E_{\rtimes}^{\dagger} A_g E_{\rtimes})m^{\dagger}$ corresponds to the intersection of the above graphs. Taking all such intersections recover the derived graph.

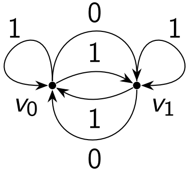
Examples: Simple quantum graphs on $M_2(\mathbb{C})$

Proposition (Schäfer–T, 2025)

Up to isomorphism, all simple quantum graphs on $(M_2, 2\mathrm{Tr})$ can be obtained as derived quantum graphs of classical $\mathbb{Z}_2$-voltage graphs on two vertices.







Examples: Some simple quantum graphs on $M_3(\mathbb{C})$

In contrast with the 2×2 case, one can prove that there are **infinitely** many non-isomorphic simple quantum graphs on $M_3(\mathbb{C})$. We consider a class of such graphs constructed using the **clock** and **shift** matrices

$$P := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad Q := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$$

satisfying the Weyl canonical commutation relation

$$QP = \omega PQ.$$

Here $\omega := \exp(2\pi i/3)$. The matrices

$$\sigma_{j,k} := P^j Q^k,$$

are often called the **generalized Pauli matrices**.

Examples: Simple quantum graphs on $M_3(\mathbb{C})$

Proposition (Rabiega–T, 2026)

Up to isomorphism, there are four nonempty simple quantum graphs on $M_3(\mathbb{C})$ generated by the matrices P and Q with quantum adjacency matrices:

1. $A_1 = \sigma_{0,1} \otimes \sigma_{0,1}^* + \sigma_{0,2} \otimes \sigma_{0,2}^*$
2. $A_2 = \sigma_{1,0} \otimes \sigma_{1,0}^* + \sigma_{2,0} \otimes \sigma_{2,0}^* + A_1$
3. $A_3 = \sigma_{1,2} \otimes \sigma_{1,2}^* + \sigma_{2,1} \otimes \sigma_{2,1}^* + A_2$
4. $A_4 = \sigma_{1,1} \otimes \sigma_{1,1}^* + \sigma_{2,2} \otimes \sigma_{2,2}^* + A_3$

All the above quantum graphs can be realized as derived quantum graphs of classical $\mathbb{Z}_3$-voltage graphs on three vertices.