Quantitative Stability in Optimal Transport: From Classical Potentials to Stable Transport Maps

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Lecture 04 roadmap

We focus on the quadratic cost $c(x, y) = \frac{1}{2}|x - y|^2$, or $c(x, y) = -x \cdot y$.

1. **Stability mechanism.**
2. **PDE expectations.**
3. **Obstructions.**
4. **Gigli's estimate.**
5. **Modern quantitative stability.** Delalande - Merigot

Warm-up: concavity and stability of maximizers

Let F be a concave functional, and suppose that it satisfies a quantitative concavity estimate (amounts to $\nabla^2 F \leq -\lambda I$ with $\lambda > 0$) $\longrightarrow$ $F(x) + \frac{|x|^2}{2}\lambda$ concave

$$F((1-t)a + tb) \geq (1-t)F(a) + tF(b) + \frac{\lambda}{2}t(1-t)|a-b|^2.$$

If a and b are both almost maximizers $F(a), F(b) \approx \max F$, then they must be close. Indeed, taking $t = 1/2$,

$$\frac{\lambda}{8}|a-b|^2 \leq \underbrace{F\left(\frac{a+b}{2}\right)}_{\lesssim \max F} - \frac{1}{2}F(a) - \frac{1}{2}F(b) \lesssim 0$$

Stability of optimizers comes from quantitative concavity.

PDE expectation

Stability $\Rightarrow$ uniqueness

Recall the formal linearization

$$-\Delta u \approx \rho - \mu.$$

The variational formulation

$$F_f(u) = -\frac{1}{2}\int \overbrace{|\nabla u|^2}^{\text{quadratic}} - \int \overbrace{u\underbrace{(\rho - \mu)}_{f}}^{\text{linear}}$$

suggests concavity with $\lambda = 1$ w.r.t. H^1 (semi)-norm:

u_f optimizer for $F_f(u)$

u_g optimizer

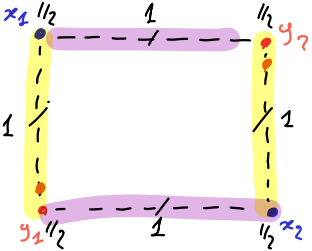
$$\int |\overset{a}{\overset{=}{\nabla u_f}} - \overset{b}{\overset{=}{\nabla u_g}}|^2 \lesssim \|f - g\|^2_{\dot{H}^{-1}}.$$

Since, formally, $f = \rho - \mu$, $g = \rho - \nu$, this would give

$$\int |T_\mu - T_\nu|^2 \, d\rho \lesssim \|\mu - \nu\|^2_{\dot{H}^{-1}} \approx W_2^2(\mu, \nu).$$

The PDE dream: target stability should imply map stability.

Reality check: nonunique optimal plans



$$c(x,y) = \frac{|x-y|^2}{2}$$

$$\rho = \frac{1}{2}\delta_{x_1} + \frac{1}{2}\delta_{x_2}$$
$$\mu = \frac{1}{2}\delta_{y_1} + \frac{1}{2}\delta_{y_2} \qquad \in \mathcal{P}(\mathbb{R}^2)$$

$$\forall \pi \qquad \int c(x,y)\,d\pi = \frac{1}{2}$$

Downgrade: stability of potentials?

After the above example, map stability is too much to ask for.

So let us first ask for stability of potentials:

$u(x) \leq v(y) + c(x,y)$

$\boxed{u(x)+a} \leq \boxed{v(y)+a} + c(x,y)$

$\boxed{\text{g fixed}}$ $\mu_n \to \mu \implies u_{\mu_n} \to u_\mu$?

Potentials are always defined up to additive constants. If the contact structure splits into disconnected components,

$$\Omega = A \cup B, \qquad A \cap B = \emptyset,$$

and no optimal contacts connect A with B, then constants may drift independently on A and B.

This is the same obstruction as in PDE:

$$\boxed{\text{Poincaré inequality fails on disconnected domains.}}$$

Stability and Brenier's theorem

For quadratic cost, Brenier's theorem gives existence and uniqueness of the map if

$$\rho \ll \mathcal{L}^d.$$

There is a convex function Φ_μ such that

$$T_\mu = \nabla \Phi_\mu, \qquad (T_\mu)_\# \rho = \mu.$$

In our notation,

$$\boxed{\Phi_\mu(x) = \frac{1}{2}|x|^2 - u_\mu(x),}$$

so that

$$T_\mu(x) = x - \nabla u_\mu(x).$$

Thus qualitative stability can be asked at two levels:

$$\mu \approx \nu \quad \Longrightarrow \quad u_\mu \approx u_\nu \quad \text{and} \quad \nabla u_\mu \approx \nabla u_\nu.$$

Potential stability is weaker; map stability requires gradient stability.

Potential and map stability metrics

First, fix a normalization, for instance

$$\int u_\mu \, d\rho = 0.$$

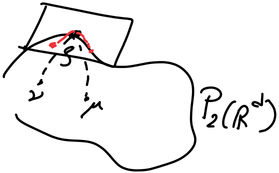
Then potential stability asks for estimates such as

$$\|u_\mu - u_\nu\|_{L^2(\rho)} \leq C\, W_2(\mu,\nu)^\alpha.$$

$\underline{\alpha > 0}$ $\alpha = 1$?

Gradient stability gives map stability because

$$T_\mu - T_\nu = -\nabla u_\mu + \nabla u_\nu.$$



Define the linearized transport distance

$$W_{2,\rho}(\mu,\nu) := \|T_\mu - T_\nu\|_{L^2(\rho)} = \|\nabla u_\mu - \nabla u_\nu\|_{L^2(\rho)}.$$

The quantitative stability question is:

$$W_{2,\rho}(\mu,\nu) \leq C\, W_2(\mu,\nu)^\beta.$$

Gigli's estimate: statement

Let $T_\mu = \nabla\Phi_\mu$ be the Brenier map from ρ to μ.

Assume that Φ_μ has a quantitative upper curvature bound (with $\lambda > 0$):

$$\phi_\mu(x) + \nabla\phi_\mu(x)(z-x) \leq \Phi_\mu(z) \leq \Phi_\mu(x) + \nabla\Phi_\mu(x)\cdot(z-x) + \frac{\lambda}{2}|z-x|^2. \qquad \forall z, x$$

Equivalently, $D^2\Phi_\mu \leq \lambda I$.

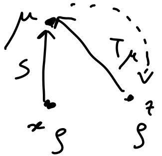
Let S be any other map such that $S_\# \rho = \mu$. Then

$$\int |S - T_\mu|^2 \, d\rho \leq \lambda \left[\int |S(x) - x|^2 \, d\rho(x) - \int |T_\mu(x) - x|^2 \, d\rho(x) \right].$$

Gigli, *On Hölder continuity-in-time of the optimal transport map towards measures along a curve*, Proc. Edinburgh Math. Soc. 54 (2011), 401–409. Li–Nochetto, *Quantitative Stability and Error Estimates for Optimal Transport Plans*, Found. Comput. Math. 22 (2022), 1797–1833.

Open problem: find (useful) extension to general p-cost. $p>1$

Proof of Gigli's estimate



Since Φ is convex and $D^2\Phi \leq \lambda I$, its gradient is $1/\lambda$-cocoercive:

$$(T(x) - T(z)) \cdot (x - z) \geq \frac{1}{\lambda}|T(x) - T(z)|^2, \qquad T = \nabla\Phi.$$

Let $S_{\#}\rho = T_{\#}\rho = \mu$. Couple x and z (gluing lemma) so that $S(x) = T(z)$.Then

$\boxed{T^{-1} \circ S(x) = z}$ ρ-a.e. x

$$(T(x) - S(x)) \cdot (x - z) \geq \frac{1}{\lambda}|T(x) - S(x)|^2.$$

Integrating,

$$\frac{1}{\lambda}\int |T - S|^2 \, d\rho \leq \int (T - S) \cdot (x - z).$$

Since $S(x) = T(z)$,

$$x - z = (x - T(x)) - (z - T(z)) + (T(x) - S(x)).$$

After rearranging and using that S and T have the same law, this gives

$$\int |S - T|^2 \, d\rho \leq \lambda \left[\int |S(x) - x|^2 \, d\rho - \int |T(x) - x|^2 \, d\rho \right].$$

$\leq$ - - - -

Exercise can you find a proof that uses only potentials!

From stability to Hölder continuity

$W_2(\mu_s,\mu_t) = |t-s|\, W_2(\mu_0,\mu_1)$

Gigli's motivation was the study of Hölder regularity of Wasserstein curves (e.g. geodesics) $(\mu_t)_t$. Let T_t be the Brenier map from a fixed ρ to a moving target μ_t.

If the corresponding Brenier potentials are uniformly λ-smooth,

$$\|T_t - T_s\|^2_{L^2(\rho)} \lesssim \text{excess cost of a competitor.}$$

Using a competitor built from an optimal coupling between μ_s and μ_t, one obtains schematically

$$\|T_t - T_s\|^2_{L^2(\rho)} \lesssim W_2(\mu_s,\mu_t) = |t-s| \quad \Rightarrow \quad \|T_t - T_s\|_{L^2(\rho)} \lesssim |t-s|^{1/2}.$$

This is the origin of the 1/2-Hölder estimate.

Limitation of Gigli's estimate

Gigli's estimate gives a clean stability mechanism but requires a regularity assumption on the Brenier potential:

$$T = \nabla\Phi, \qquad D^2\Phi \le \lambda I.$$

Such bounds may follow from regularity theory for the Monge–Ampère equation,

$$\rho(x) = \mu(\nabla\Phi(x)) \det D^2\Phi(x),$$

but only under strong assumptions on both measures:

$$\boxed{\rho, \mu \text{ regular, positive, and supported on suitable convex sets.}}$$

This is not well adapted to irregular targets, for instance empirical measures.

Why asymmetric assumptions are natural

In many applications the source is regular:

$$\rho = \text{uniform measure, Gaussian measure, latent distribution.}$$

The target may be rough:

$$\mu = \text{empirical data distribution.}$$

For instance, in generative modeling one often transports

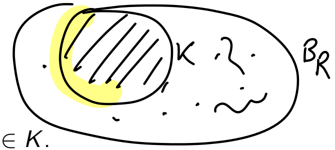
$$\text{simple source} \quad \longrightarrow \quad \text{complex data law.}$$

So one wants stability under target perturbations:

$$\mu \approx \nu \quad \Longrightarrow \quad T_\mu \approx T_\nu,$$

without assuming that μ, ν have smooth positive densities.

Delalande–Mérigot: asymmetric stability



Let ρ be supported on a compact convex set $K \subset \mathbb{R}^d$, with

$$0 < m \leq \rho(x) \leq M < \infty \quad \text{for a.e. } x \in K.$$

For each target μ, let Φ_μ be the (zero-mean) Brenier potential and $T_\mu = \nabla\Phi_\mu$ the Brenier map from $\rho\,dx$ to μ.

On large classes of target measures, for instance measures supported in a fixed ball B_R, we have Hölder stability

- of potentials:

$$\|\Phi_\mu - \Phi_\nu\|_{L^2(\rho)} \leq C\, W_2(\mu,\nu)^{1/2}.$$

- of maps: for some $\beta = \beta(d) > 0$,

$$\|T_\mu - T_\nu\|_{L^2(\rho)} \leq C\, W_2(\mu,\nu)^{\cancel{\beta}}, \quad 1/6$$

where $C = C(\rho, K, R) < \infty$.

A regular source gives quantitative stability even for rough targets.

Delalande–Mérigot, *Quantitative Stability of Optimal Transport Maps under Variations of the Target Measure*; Mischler–Trevisan, *Quantitative stability in optimal transport for general power costs*. Chazal–Delalande–Mérigot, *Quantitative stability of optimal transport maps and linearization of the 2-Wasserstein space*.

Entropic regularization and a second partition function

Assume for simplicity ρ uniform on K, and finite $Y = \{y_1, \ldots, y_n\}$, so $v : Y \to \mathbb{R}$ is actually $v \in \mathbb{R}^n$. For fixed $\varepsilon > 0$, define

$$Q^{\varepsilon} v(x) = -\varepsilon \log \sum_{i=1}^{n} \exp\left(-\frac{v_i + c(x, y_i)}{\varepsilon}\right).$$

$\longrightarrow$ $\boxed{v \mapsto Q^{\varepsilon} v(x) \text{ is concave}}$

The entropic Kantorovich functional is

$$\mathcal{D}^{\varepsilon}(v) = \int_K Q^{\varepsilon} v \, d\rho \cancel{\left(-\sum_{i=1}^{n} v_i \mu_i\right)}.$$

$v \mapsto \mathcal{D}^{\varepsilon}(v)$ is concave

Introduce a second parameter $\beta > 0$ and the auxiliary partition function

$$\mathcal{Z}_{\beta}^{\varepsilon}(v) := \int_K \exp\left(\beta \, Q^{\varepsilon} v(x)\right) d\rho(x).$$

Key claim: by Prékopa–Leindler inequality,

$$v \longmapsto \log \mathcal{Z}_{\beta}^{\varepsilon}(v) \quad \text{is concave}$$

$\boxed{\varepsilon \text{ regularizes OT; } \beta \text{ is tuned to extract coercivity.}}$

From Prékopa–Leindler to variance coercivity

Let $v_t = (1-t)v_0 + tv_1$, $\partial_t v_t = v_1 - v_0$. Let $\gamma_{v_t}^{x,\varepsilon}$ be the Gibbs kernel associated with $\mathrm{Q}^\varepsilon v_t(x)$, and set

$$m_t(x) := \int_Y \partial_t v_t(y)\, d\gamma_{v_t}^{x,\varepsilon}(y).$$

Then

$$\frac{d}{dt}\mathrm{Q}^\varepsilon v_t(x) = m_t(x) \quad \text{and} \quad \frac{d^2}{dt^2}\mathrm{Q}^\varepsilon v_t(x) = -\frac{1}{\varepsilon}\mathrm{Var}_{\gamma_{v_t}^{x,\varepsilon}}(\partial_t v_t).$$

$\varepsilon \downarrow 0$??

Introduce the Gibbs measure $\rho_{\beta,t}^\varepsilon(x) := \exp\left(\beta \mathrm{Q}^\varepsilon v_t(x)\right)/\mathcal{Z}_\beta^\varepsilon(v_t)$.

By computing $\frac{d^2}{dt^2}\log \mathcal{Z}_\beta^\varepsilon(v_t) \le 0$, we get $\beta\, \mathrm{Var}_{\rho_{\beta,t}^\varepsilon}(m_t) \le \frac{1}{\varepsilon}\int_K \mathrm{Var}_{\gamma_{v_t}^{x,\varepsilon}}(\partial_t v_t)\, d\rho_{\beta,t}^\varepsilon(x)$.

Tuning suitably β we get $\rho_{\beta,t}^\varepsilon \approx 1$ yielding

$$\boxed{\mathrm{Var}_K(m_t) \le C\left\langle \partial_t v_t, -\nabla^2\mathcal{D}^\varepsilon(v_t)\partial_t v_t\right\rangle,}$$

strict convexity estimate!

Soft potential stability

Since

$$\mathrm{Q}^\varepsilon v_1 - \mathrm{Q}^\varepsilon v_0 = \int_0^1 \frac{d}{dt}\mathrm{Q}^\varepsilon v_t \, dt = \int_0^1 m_t \, dt,$$

convexity of the variance gives

$$\mathrm{Var}_\rho\left(\mathrm{Q}^\varepsilon v_1 - \mathrm{Q}^\varepsilon v_0\right) \leq \int_0^1 \mathrm{Var}_\rho(m_t)\, dt$$

Concavity estimate

$$\leq C \int_0^1 \left\langle v_1 - v_0, -\nabla^2 \mathcal{D}^\varepsilon(v_t)\partial_t v_t \right\rangle dt$$

$$= C\left\langle \nabla\mathcal{D}^\varepsilon(v_0) - \nabla\mathcal{D}^\varepsilon(v_1), v_1 - v_0 \right\rangle$$

If v_μ and v_ν are optimal, then $\nabla\mathcal{D}^\varepsilon(v_\mu) = \mu$, $\nabla\mathcal{D}^\varepsilon(v_\nu) = \nu$. We conclude

$$\boxed{\mathrm{Var}_\rho\left(\mathrm{Q}^\varepsilon v_\nu - \mathrm{Q}^\varepsilon v_\mu\right) \leq C\left\langle \mu - \nu, v_\nu - v_\mu \right\rangle} \lesssim W_1(\mu,\nu)$$

where the implicit constant does not depend on $\varepsilon \downarrow 0$ nor $n = \sharp Y \to \infty$.

From potential stability to map stability

For the quadratic cost,

$$T_\mu = \nabla\Phi_\mu, \qquad \Phi_\mu(x) = \frac{1}{2}|x|^2 - u_\mu(x).$$

Thus

$$\|T_\mu - T_\nu\|_{L^2(\rho)} = \|\nabla\Phi_\mu - \nabla\Phi_\nu\|_{L^2(\rho)}.$$

Potential stability gives, after normalization,

$$\|\Phi_\mu - \Phi_\nu\|_{L^2(\rho)} = \|u_\mu - u_\nu\|_{L^2(\rho)} \le C\, W_1(\mu,\nu)^{1/2}.$$

The missing step is an interpolation inequality for convex functions: (Gagliardo-Nierenberg inequality)

$$\|\nabla\Phi_\mu - \nabla\Phi_\nu\|_{L^2(\rho)} \le C\|\Phi_\mu - \Phi_\nu\|_{L^2(\rho)}^{1/3}.$$

Therefore

$$\|T_\mu - T_\nu\|_{L^2(\rho)} \le C\, W_1(\mu,\nu)^{1/6}.$$

Convexity plus Gagliardo–Nirenberg interpolation upgrades potentials to maps.

Recent developments

Recent work pushes the quadratic stability theory beyond the original convex-domain setting.

- **Gluing methods.** Letrouit–Mérigot prove bi-Hölder stability under more general assumptions, using local variance inequalities and gluing arguments.
- **Riemannian costs.** Kitagawa–Letrouit–Mérigot extend quantitative stability to squared Riemannian distance costs.
- **Sharp exponents.** Recent work of Mérigot obtains stability with sharp exponent $1/4$ under minimal assumptions for the quadratic cost.

Further reading: *Lectures on quantitative stability of optimal transport*; Letrouit–Mérigot, *Gluing methods for quantitative stability of optimal transport maps*; Mérigot, *Sharp stability of Brenier maps via quantitative regularity of potentials*.

Final picture

We started with a price inequality:

$$\boxed{u(x) \leq v(y) + c(x, y).}$$

We ended with a stability question:

$$\boxed{\mu \approx \nu \quad \Longrightarrow \quad T_\mu \approx T_\nu.}$$

In between, we have seen entropy, tropical/idempotent analysis, non-smooth Poincaré lemmas, Poisson and p-Poisson equations, Hamilton-Jacobi semigroups. . .

Potentials link transport with geometry and PDE.

References for Lecture 04

- N. Gigli, *On Hölder continuity-in-time of the optimal transport map towards measures along a curve*, Proc. Edinburgh Math. Soc., 2011.
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