Feynman Formula for Discrete-time Quantum Walks

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Discrete-time Quantum Walks

Definition

$$\psi_{n+1} = W\psi_n = SC\psi_n \quad \text{where} \quad \psi_0 \in \ell_2(\mathbb{Z} \times \{\pm 1\})$$

Coin operator(C): $e_x e_z \mapsto e_x C_x e_z$. Shift operator(S): $e_x e_z \mapsto e_{x+z} e_z$

Introduced by Aharonov, Davidovich, and Zagury 1993, promoted by Ambainis et al. 2001. A deterministic dynamics: Meyer 1996.

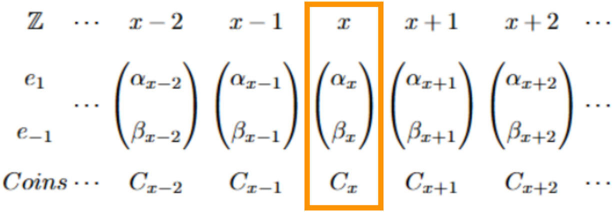

Figure: A line of $\mathbb{C}^2$ vectors

Visualization

Coin operator

$$
\begin{array}{cccccccc}
\mathbb{Z} & \cdots & x-2 & x-1 & x & x+1 & x+2 & \cdots \\
\begin{array}{c} e_1 \\ e_{-1} \end{array} & \cdots & \begin{pmatrix} \tilde{\alpha}_{x-2} \\ \tilde{\beta}_{x-2} \end{pmatrix} & \begin{pmatrix} \tilde{\alpha}_{x-1} \\ \tilde{\beta}_{x-1} \end{pmatrix} & \begin{pmatrix} \tilde{\alpha}_{x} \\ \tilde{\beta}_{x} \end{pmatrix} & \begin{pmatrix} \tilde{\alpha}_{x+1} \\ \tilde{\beta}_{x+1} \end{pmatrix} & \begin{pmatrix} \tilde{\alpha}_{x+2} \\ \tilde{\beta}_{x+2} \end{pmatrix} & \cdots \\
Coins & \cdots & C_{x-2} & C_{x-1} & C_x & C_{x+1} & C_{x+2} & \cdots
\end{array}
$$

Shift operator

$$
\begin{array}{cccccccc}
\mathbb{Z} & \cdots & x-2 & x-1 & x & x+1 & x+2 & \cdots \\
\begin{array}{c} e_1 \\ e_{-1} \end{array} & \cdots & \begin{pmatrix} \tilde{\alpha}_{x-3} \\ \tilde{\beta}_{x-1} \end{pmatrix} & \begin{pmatrix} \tilde{\alpha}_{x-2} \\ \tilde{\beta}_{x} \end{pmatrix} & \begin{pmatrix} \tilde{\alpha}_{x-1} \\ \tilde{\beta}_{x+1} \end{pmatrix} & \begin{pmatrix} \tilde{\alpha}_{x} \\ \tilde{\beta}_{x+2} \end{pmatrix} & \begin{pmatrix} \tilde{\alpha}_{x+1} \\ \tilde{\beta}_{x+3} \end{pmatrix} & \cdots \\
Coins & \cdots & C_{x-2} & C_{x-1} & C_x & C_{x+1} & C_{x+2} & \cdots
\end{array}
$$

Our Contribution

Theorem (Feynman Formula, Fouque, Ichiba and Lam 2025)

Let $C = e^{i\theta\sigma_1}$ *with initial state* ψ_0. *Then*

$$\psi_n(x,z) = e^{n\theta}\mathbb{E}(i^{S_n}\psi_0(X_n, Z_n) \mid (S_0, Z_0, X_0) = (0, z, x))$$

where $S_n = \sum_{j=1}^{n} N_j \pmod 4$, $Z_n = z(-1)^{S_n}$, $X_n = x - \sum_{j=0}^{n-1} Z_j$

- Novel relation between quantum processes and stochastic processes, in particular, Markov additive processes.
- This comes from the Feynman path formula:

$$\psi_n(x,z) = \sum_{k_1,\cdots,k_n=0}^{\infty} \frac{(i\theta)^{k_1+\cdots+k_n}}{k_1!\cdots k_n!}\psi_0(x - z\sum_{j=0}^{n-1}(-1)^{k_1+\cdots+k_j}, z(-1)^{k_1+\cdots+k_n})$$

Remarks

- The random variables $\mathbb{P}(\Xi_n = x) \sim |\psi_n(x,1)|^2 + |\psi_n(x,-1)|^2$ does not give a process but a flow of marginals.
- Problem of mimicking: continuous-time (conservative diffusion by Carlen 1984); and discrete-time (Montero 2017).

Our Feynman formula is useful for asymptotic analysis of quantum walks!

Asymptotics 1: Ballistic weak limit

- Define the random variables Ξ_n on $\mathbb{Z}$ by

$$\mathbb{P}(\Xi_n = x) = |\psi_n(x,1)|^2 + |\psi_n(x,-1)|^2,$$

Konno 2005, Grimmett, Janson, and Scudo 2004 showed that with the coin $C = e^{i\theta\sigma_1}$, we have the weak convergence $\frac{1}{n}\Xi_n \Rightarrow f$ where

$$f(y) = \frac{1}{\pi}\frac{\sin(\theta)}{\sqrt{\cos^2(\theta) - y^2}(1-y^2)}\mathbb{1}_{|y|\leq\cos\theta}$$

with $\psi_0 = \frac{1}{\sqrt{2}}\mathbb{1}_{0,1} + \frac{1}{\sqrt{2}}\mathbb{1}_{0,-1}$.

- Diffusive scaling $\frac{S_n}{\sqrt{n}}$ VS hyperbolic scaling $\frac{\Xi_n}{n}$
- Localized behavior VS bimodal behavior $\Rightarrow$ Quantum walk tends to have momentum!

Visualization - Ballistic weak limit

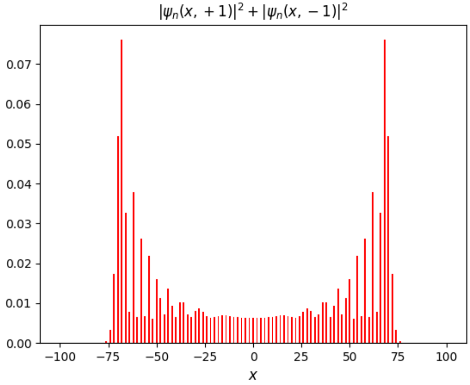

Figure: The random variable Ξ_n with $n = 100$

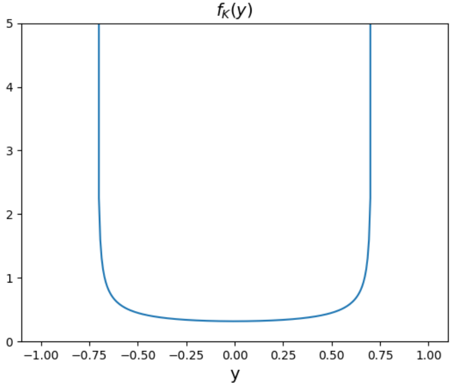

Figure: Limiting density

Re-deriving bimodal weak limit

- Define $T_n = \sum_{j=0}^{n-1}(-1)^{S_j}$. Put $\psi_0^{\pm} = \mathbb{1}_{0,\pm 1}$. Then for $a \in (-1,1)$ and $an \in \mathbb{Z}$

$$\psi_n^+(an,z) = e^{n\theta}[\mathbb{P}(\frac{T_n}{n} = a, S_n = 0) - \mathbb{P}(\frac{T_n}{n} = a, S_n = 2)]$$

$$\psi_n^-(an,z) = ie^{n\theta}[\mathbb{P}(\frac{T_n}{n} = a, S_n = 1) - \mathbb{P}(\frac{T_n}{n} = a, S_n = 3)]$$

- This is a difference of small probabilities.
- Tilted kernel technique: Express each probabilities in terms of integrals of the spectrum of a one-parameter 4×4 matrix.
- Effect of large deviation of (S_n, T_n) is cancelled.

Asymptotics 2: Continuous-time limit

- Recall $\psi_n(x,z) = e^{n\theta}\mathbb{E}(i^{S_n}\psi_0(x - zT_n, z(-1)^{S_n}))$.
- Poisson randomness $\theta \mapsto \epsilon\lambda$; then:

$$S^{\epsilon}_{[t/\epsilon]} \Rightarrow N_t, \qquad \epsilon T^{\epsilon}_{[t/\epsilon]} \Rightarrow T_t := \int_0^t (-1)^{N_s} ds,$$

  T_t : the telegraph process, hyperbolic continuous-time limit of correlated walk (Kac 1974)
- Also scaling the initial state $\psi_0^{\epsilon}(x,z) := f(\epsilon x, z)$ governed by some $f \in L_2(\mathbb{R})$ (technical assumption) gives

$$\psi^{\pm}(t,y) = \lim_{\epsilon\to 0} \psi^{\epsilon}_{[t/\epsilon]}([y/\epsilon], \pm 1) = e^{ty}\mathbb{E}(i^{N_t} f(x - (\pm 1)\cdot \int_0^t T_s ds, (\pm 1)\cdot(-1)^{N_t}))$$

Quantum transport PDE

- Easy to show that

$$\begin{cases} \partial_t \psi_+ = -\partial_y \psi_+ + i\lambda\psi_- \\ \partial_t \psi_- = \partial_y \psi_- + i\lambda\psi_+ \end{cases}$$

- Quantum transport equation (Molfetta and Debbasch 2012) or Dirac equations (Maeda et al. 2018).
- Monte Carlo method on transport equations (useful for continuous-time but not discrete-time as suggested by Kac 1974)

Visualization by Monte Carlo

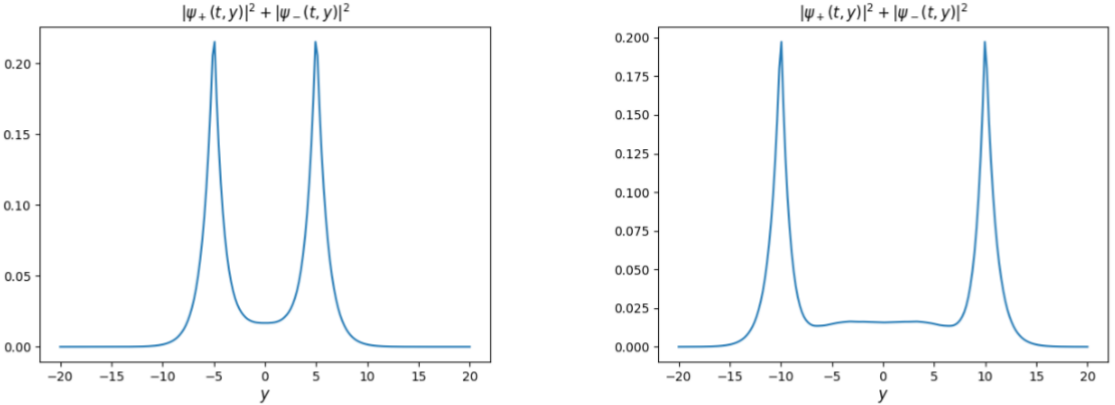

Figure: Left: t = 5, right t = 10

Summary

- Feynman formula:

$$\psi_n(x,z) = e^{n\theta}\mathbb{E}(i^{S_n}\psi_0(X_n, Z_n) \mid (S_0, Z_0, X_0) = (0, z, x))$$

  Valid also for general unitary coins, inhomogeneous coins.
- Useful in deriving asymptotics
  1. Relation between quantum walks asymptotics and LDP of classical processes
  2. Probabilistic formulation for quantum transport PDE
- On-going works include:
  1. Other variants: Higher-dimensional quantum walks, non-linear, open quantum walks Attal et al. 2012.
  2. Financial application: telegraph process (Ratanov and Kolesnik 2022) and quantum walks (Orrell 2021).
- More references in our paper.

Thank you!

Happy to take questions

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- **arXiv**: https://arxiv.org/abs/2510.12038

Appendix - Tilted kernel

1. $P_k(s,s') = e^{ik(-1)^s} p(s,s')$
2. If $p(s,s') = \begin{bmatrix} p_0 & p_1 & p_2 & p_3 \\ p_3 & p_0 & p_1 & p_2 \\ p_2 & p_3 & p_0 & p_1 \\ p_1 & p_2 & p_3 & p_0 \end{bmatrix}$ then

$$P_k = \begin{bmatrix} e^{ik} p_0 & e^{ik} p_1 & e^{ik} p_2 & e^{ik} p_3 \\ e^{-ik} p_3 & e^{-ik} p_0 & e^{-ik} p_1 & e^{-ik} p_2 \\ e^{ik} p_2 & e^{ik} p_3 & e^{ik} p_0 & e^{ik} p_1 \\ e^{-ik} p_1 & e^{-ik} p_2 & e^{-ik} p_3 & e^{-ik} p_0 \end{bmatrix}$$

Appendix - Main decomposition

Eigenvalues are grouped in

$$A(k) = e^{\theta} \begin{bmatrix} e^{ik} \cosh\theta & e^{ik} \sinh\theta \\ e^{-ik} \sinh\theta & e^{-ik} \cosh\theta \end{bmatrix}, \quad B(k) = e^{\theta} \begin{bmatrix} e^{ik} \cos\theta & e^{ik} \sin\theta \\ -e^{-ik} \sin\theta & e^{-ik} \cos\theta \end{bmatrix}$$

- Only eigen-values form $B(k)$ contribute to the quantum walk amplitute. The weak limit comes from its asymptotic analysis (see for instance Olver 1997).
- Because of the difference, the contribution of $A(k)$ got cancelled.
- Asymptotic analysis of the eigen-value from $A(k)$ (the Perron-Frobenius eigenvalues of the tilted kernel) famously gives Large deviation of the Markov additive model (S_n, T_n) (see for instance Dembo and Zeitouni 2010).

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