On the rigidity of some Hirzebruch genera

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Hirzebruch genera

Ω_U^* is the complex cobordism ring = cobordism ring of (closed) stably complex manifolds

stably complex structure on M = complex structure on $TM \oplus \mathbb{R}^N$ (up to $\oplus \mathbb{C}^k$)

stably complex manifolds M and N are cobordant if $M \sqcup \overline{N} = \partial W$

$$\Omega_U^* = \{\text{stably complex closed manifolds}\}/\sim$$

$$[M] + [N] = [M \sqcup N] \qquad [M] \cdot [N] = [M \times N]$$

Hirzebruch genera

R is a graded commutative $\mathbb{Q}$-algebra

Ω_U^* is the complex cobordism ring

complex Hirzebruch genus is a ring homomorphism $\varphi\colon \Omega_U^* \to R$

complex genera $\varphi\colon \Omega_U^* \to R$ are in the bijection with the power series $f \in R[[x]]$ s. t. $f(x) = x + \dots$ (Hirzebruch)

$f(x) = g^{-1}(x)$, $g(x) = x + \sum_{k\geqslant 1} \frac{\varphi([\mathbb{C}P^k])}{k+1} x^{k+1}$ (Mischenko)

$\varphi([M]) = \langle \prod \frac{x_i}{f(x_i)}(\mathcal{T}M), [M]_{\mathbb{Z}} \rangle$

Equivariant extension

$\Omega^*_{U:T^k}$ is the complex T^k-equivariant cobordism ring

$$\Phi\colon \Omega^*_{U:T^k} \xrightarrow{\text{P–T}} MU^*_{T^k}(pt) \to MU^*(BT^k) = \Omega^*_U[[u_1, \dots, u_k]]$$

Φ is the universal (complex) toric genus. It is injective (Comezaña, Hanke, Löffler).

The equivariant extension of a genus $\varphi\colon \Omega^*_U \to R$ is a composition

$$\varphi^T\colon \Omega^*_{U:T^k} \xrightarrow{\Phi} \Omega^*_U[[u_1, \dots, u_k]] \xrightarrow[u_i \mapsto f(x_i)]{\varphi\colon \Omega^*_U \to R} R[[x_1, \dots, x_k]]$$

Rigidity

A genus $\varphi\colon \Omega_U^* \to R$ is rigid on a T^k-manifold M if $\varphi^T([M]) = const \in R[[x_1, \dots, x_k]]$. In fact this constant is $\varphi([M]) \in R$.

Theorem (Buchstaber–Panov–Ray)

A genus $\varphi\colon \Omega_U^* \to R$ is rigid on M if and only if we have $\varphi(E) = \varphi(M)\varphi(B)$ for any fibre bundle $E \to B$ with fibre M.

Theorem (Buchstaber–Panov–Ray localization formula)

If a T^k-manifold M has only isolated fixed points, then

$$\varphi^T(M) = \sum_{p \in M^T} \sigma(p) \prod_{i=1}^{n} \frac{1}{f(\langle w_i(p), \mathbf{x} \rangle)}$$

Examples

- $\chi_{a,b}\colon \Omega_U^* \to \mathbb{Q}[a,b]$, $f(x) = \frac{e^{ax}-e^{bx}}{ae^{bx}-be^{ax}}$, universal T^k-rigid genus (Musin) , universal $\mathbb{C}P^2$-rigid taking nonzero value on $\mathbb{C}P^2$ (Buchstaber–Bunkova)
- (Oriented) elliptic genus $\varphi_{ell}\colon \Omega_U^* \to \mathbb{Q}[\varepsilon,\delta]$, $f(x) = \mathrm{sn}(x)$

$$(\mathrm{sn}'(x))^2 = 1 - 2\delta(\mathrm{sn}(x))^2 + \varepsilon(\mathrm{sn}(x))^4$$

$\varepsilon = \delta^2$: $\mathrm{sn}(x) = \mathrm{th}(x)$

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- $\chi_{a,b}\colon \Omega_U^* \to \mathbb{Q}[a,b]$, $f(x) = \frac{e^{ax}-e^{bx}}{ae^{bx}-be^{ax}}$, universal T^k-rigid genus (Musin), universal $\mathbb{C}P^2$-rigid taking nonzero value on $\mathbb{C}P^2$ (Buchstaber–Bunkova)
- (Oriented) elliptic genus $\varphi_{ell}\colon \Omega_U^* \to \Omega_{SO}^* \to \mathbb{Q}[\varepsilon,\delta]$, $f(x) = \mathrm{sn}(x)$

$$(\mathrm{sn}'(x))^2 = 1 - 2\delta(\mathrm{sn}(x))^2 + \varepsilon(\mathrm{sn}(x))^4$$

$\varepsilon = \delta^2$: $\mathrm{sn}(x) = \mathrm{th}(x)$
the elliptic genus is the universal $\mathbb{H}P^2$-rigid genus (Kreck–Stolz)

Krichever genus

$$\varphi_{Kr}\colon \Omega_U^* \to \mathbb{Q}[\alpha, b_1, b_2, b_3]$$

$$f_{Kr}(x) = \frac{e^{\alpha x}}{\Phi(x,z)} \in \mathbb{Q}[\alpha, b_1, b_2, b_3][[x]]$$

$$\wp(x) = \frac{1}{x^2} + \frac{1}{20} g_2 x^2 + \frac{1}{28} g_3 x^4 + \dots$$

$$(\wp'(x))^2 = 4(\wp(x))^3 - g_2 \wp(x) - g_3$$

$$\wp(x) = -(\ln \sigma(x))'' \quad \zeta(x) = (\ln \sigma(x))' \quad \sigma(x) \in \mathbb{Q}[g_2, g_3][[x]]$$

$$\Phi(x,z) = \frac{\sigma(z-x)}{\sigma(z)\sigma(x)} e^{\zeta(z)x}$$

$$\frac{1}{\Phi(x,z)} \in \mathbb{Q}[b_1, b_2, b_3][[x]], \;\; b_1 = \wp(z), b_2 = \wp'(z), b_3 = g_2$$

Krichever genus

Theorem (Krichever)

The Krichever genus is rigid on any SU-manifold.

If a genus is rigid and vanishes on $\mathbb{C}P^2$, then it is a Krichever genus (Buchstaber–Bunkova).

Theorem (Buchstaber–Panov–Ray)

The Krichever genus vanishes on any quasitoric SU-manifold.

SU-rigidity

$$\Omega^*_{SU} \otimes \mathbb{Z}[1/2] = \mathbb{Z}[1/2][y_2, y_3, \ldots]$$

$$\Omega^4_{SU} = \mathbb{Z}\langle y_2 \rangle, \;\; \Omega^6_{SU} = \mathbb{Z}\langle y_3 \rangle, \;\; \Omega^8_{SU} = \mathbb{Z}\langle \frac{1}{4} y_2^2, y_4 \rangle,$$

$$\Omega^{10}_{SU} = \mathbb{Z}\langle \frac{1}{2} y_2 y_3, y_5 \rangle \oplus \mathbb{Z}/2$$

$$y_3 = [S^6 = G_2/SU(3)] \qquad T^2 \curvearrowright S^6$$

Theorem (Buchstaber–Panov–Ray)

Let φ be a genus which is rigid on S^6.
1) If $\varphi([S^6]) \neq 0$, then φ is a Krichever genus with $b_2 \neq 0$;
2) If $\varphi([S^6]) = 0$, then $f(x) = e^{\beta x}\widetilde{f}(x)$ for an odd series $\widetilde{f}(x)$.

If $b_2 = 0$, then $f_{Kr} = e^{\alpha x}\mathrm{sn}(x)$.

SU-rigidity

Theorem (Lü–Panov)

Classes y_i with $i \geqslant 5$ can be represented by quasitoric SU-manifolds.

Integer linear combinations of quasitoric SU-manifolds $\widetilde{L}(2k_1, 2k_2+1)$ and $\widetilde{N}(2k_1, 2k_2+1)$.

$\widetilde{L}(2k_1, 2k_2+1)$ is over $\Delta^{2k_1} \times \Delta^{2k_2+1}$, projectivisation of a sum of line bundles over $\mathbb{C}P^{2k_1}$ with a "twisted" stably complex structure

$\widetilde{N}(2k_1, 2k_2+1)$ is over $\Delta^1 \times \Delta^{2k_1} \times \Delta^{2k_2+1}$, projectivisation of a sum of line bundles over $\mathbb{C}P^1 \times \mathbb{C}P^{2k_1}$ with a "twisted" stably complex structure

In particular, $y_5 = [\widetilde{L}(2,3)]$.

SU-rigidity

Theorem

Let φ be a genus which is rigid on S^6 and on $\widetilde{L}(2,3)$. If $\varphi([S^6]) = 0$, then $f(x) = e^{\alpha x}\mathrm{sn}(x)$.

Corollary

The Krichever genus is the universal genus which is rigid on S^6 and $\widetilde{L}(2,3)$. In particular, it is the universal SU-rigid genus.

Witten genus

$$\varphi_W \colon \Omega^*_{SO} \to \mathbb{Q}[\alpha, g_2, g_3]$$

$$f_W(x) = e^{\alpha x^2}\sigma(x)$$

Witten genus is rigid on $\mathbb{O}P^2 = F_4/Spin(9)$ and $\varphi_W([\mathbb{O}P^2]) = 0$.

Theorem

The Witten genus is the universal genus which is rigid and vanishes on $\mathbb{O}P^2$.

The rigidity equation for $\mathbb{O}P^2$ is equivalent to

$$\begin{aligned}0 = f(y_1+y_2)f(y_1-y_2)f(y_3+y_4)f(y_3-y_4)+ \\ + f(y_2-y_3)f(y_2+y_3)f(y_1-y_4)f(y_1+y_4)+ \\ + f(y_2-y_4)f(y_2+y_4)f(y_3-y_1)f(y_1+y_3)\end{aligned}$$

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