The Lie algebra associated with the lower central series of a right-angled Coxeter group

Yakov Veryovkin

Higher School of Economics, Steklov Mathematical Institue RAS, Moscow State University

Workshop on Toric Topology

Canada, Toronto, 19–23 August 2024

1. Polyhedral and graph products

$\mathcal{K}$ a simplicial complex on the set $[m] = \{1, 2, 3, \dots, m\}$, $\varnothing \in \mathcal{K}$.
$I = \{i_1, \dots, i_k\} \in \mathcal{K}$ a simplex.

$(\boldsymbol{X}, \boldsymbol{A}) = \{(X_1, A_1), \dots, (X_m, A_m)\}$ a sequence of pairs of spaces, $A_i \subset X_i$.

Given $I = \{i_1, \dots, i_k\} \subset [m]$, set

$$(\boldsymbol{X}, \boldsymbol{A})^I = Y_1 \times \cdots \times Y_m \qquad \text{where } Y_i = \begin{cases} X_i & \text{if } i \in I, \\ A_i & \text{if } i \notin I. \end{cases}$$

The $\mathcal{K}$-polyhedral product of $(\boldsymbol{X}, \boldsymbol{A})$ is

$$(\boldsymbol{X}, \boldsymbol{A})^{\mathcal{K}} := \bigcup_{I \in \mathcal{K}} (\boldsymbol{X}, \boldsymbol{A})^{I} = \bigcup_{I \in \mathcal{K}} \Big(\prod_{i \in I} X_i \times \prod_{j \notin I} A_j \Big),$$

where the union is taken inside $X_1 \times \cdots \times X_m$.

Notation: $(X, A)^{\mathcal{K}} := (\boldsymbol{X}, \boldsymbol{A})^{\mathcal{K}}$ when all $(X_i, A_i) = (X, A)$;

$\boldsymbol{X}^{\mathcal{K}} := (\boldsymbol{X}, pt)^{\mathcal{K}}$, $X^{\mathcal{K}} := (X, pt)^{\mathcal{K}}$.

Example

Let $(X, A) = (S^1, pt)$, where S^1 is a circle. Then

$$(S^1)^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (S^1)^I \subset (S^1)^m.$$

Example

Let $(X, A) = (S^1, pt)$, where S^1 is a circle. Then

$$(S^1)^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (S^1)^I \subset (S^1)^m.$$

When $\mathcal{K} = \{\varnothing, \{1\}, \ldots, \{m\}\}$ (m disjoint points), the polyhedral product $(S^1)^{\mathcal{K}}$ is the wedge $(S^1)^{\vee m}$ of m circles.

Example

Let $(X, A) = (S^1, pt)$, where S^1 is a circle. Then

$$(S^1)^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (S^1)^I \subset (S^1)^m.$$

When $\mathcal{K} = \{\varnothing, \{1\}, \ldots, \{m\}\}$ (m disjoint points), the polyhedral product $(S^1)^{\mathcal{K}}$ is the wedge $(S^1)^{\vee m}$ of m circles.

When $\mathcal{K}$ consists of all proper subsets of $[m]$ (the boundary $\partial\Delta^{m-1}$ of an $(m-1)$-dimensional simplex), $(S^1)^{\mathcal{K}}$ is the fat wedge of m circles; it is obtained by removing the top-dimensional cell from the m-torus $(S^1)^m$.

Example

Let $(X, A) = (S^1, pt)$, where S^1 is a circle. Then

$$(S^1)^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (S^1)^I \subset (S^1)^m.$$

When $\mathcal{K} = \{\varnothing, \{1\}, \ldots, \{m\}\}$ (m disjoint points), the polyhedral product $(S^1)^{\mathcal{K}}$ is the wedge $(S^1)^{\vee m}$ of m circles.

When $\mathcal{K}$ consists of all proper subsets of $[m]$ (the boundary $\partial\Delta^{m-1}$ of an $(m-1)$-dimensional simplex), $(S^1)^{\mathcal{K}}$ is the fat wedge of m circles; it is obtained by removing the top-dimensional cell from the m-torus $(S^1)^m$.

For a general $\mathcal{K}$ on m vertices, $(S^1)^{\vee m} \subset (S^1)^{\mathcal{K}} \subset (S^1)^m$.

Example

Let $(X, A) = (\mathbb{R}, \mathbb{Z})$. Then

$$\mathcal{L}_{\mathcal{K}} := (\mathbb{R}, \mathbb{Z})^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (\mathbb{R}, \mathbb{Z})^{I} \subset \mathbb{R}^{m}.$$

Example

Let $(X, A) = (\mathbb{R}, \mathbb{Z})$. Then

$$\mathcal{L}_{\mathcal{K}} := (\mathbb{R}, \mathbb{Z})^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (\mathbb{R}, \mathbb{Z})^{I} \subset \mathbb{R}^{m}.$$

When $\mathcal{K}$ consists of m disjoint points, $\mathcal{L}_{\mathcal{K}}$ is a grid in $\mathbb{R}^m$ consisting of all lines parallel to one of the coordinate axis and passing though integer points.

Example

Let $(X, A) = (\mathbb{R}, \mathbb{Z})$. Then

$$\mathcal{L}_{\mathcal{K}} := (\mathbb{R}, \mathbb{Z})^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (\mathbb{R}, \mathbb{Z})^{I} \subset \mathbb{R}^{m}.$$

When $\mathcal{K}$ consists of m disjoint points, $\mathcal{L}_{\mathcal{K}}$ is a grid in $\mathbb{R}^m$ consisting of all lines parallel to one of the coordinate axis and passing though integer points.

When $\mathcal{K} = \partial\Delta^{m-1}$, the complex $\mathcal{L}_{\mathcal{K}}$ is the union of all integer hyperplanes parallel to coordinate hyperplanes.

1. Polyhedral and graph products

Let $\boldsymbol{G} = (G_1, \ldots, G_m)$ a sequence of m discrete groups, $G_i \neq \{1\}$.
$\mathcal{K}$ a simplicial complex on $[m] = \{1, 2, \ldots, m\}$.

1. Polyhedral and graph products

Let $\boldsymbol{G} = (G_1, \ldots, G_m)$ a sequence of m discrete groups, $G_i \neq \{1\}$.

$\mathcal{K}$ a simplicial complex on $[m] = \{1, 2, \ldots, m\}$.

Definition

The graph product of the groups $G_1, \ldots, G_m$ is

$$\boldsymbol{G}^{\mathcal{K}} := \mathop{\bigstar}_{k=1}^{m} G_k \big/ (g_i g_j = g_j g_i \text{ for } g_i \in G_i,\ g_j \in G_j,\ \{i, j\} \in \mathcal{K}),$$

where $\bigstar_{k=1}^{m} G_k$ denotes the free product of the groups G_k.

1. Polyhedral and graph products

Let $\boldsymbol{G} = (G_1, \ldots, G_m)$ a sequence of m discrete groups, $G_i \neq \{1\}$.
$\mathcal{K}$ a simplicial complex on $[m] = \{1, 2, \ldots, m\}$.

Definition

The graph product of the groups $G_1, \ldots, G_m$ is

$$\boldsymbol{G}^{\mathcal{K}} := \mathop{\bigstar}_{k=1}^{m} G_k \big/ (g_i g_j = g_j g_i \text{ for } g_i \in G_i,\ g_j \in G_j,\ \{i, j\} \in \mathcal{K}),$$

where $\bigstar_{k=1}^{m} G_k$ denotes the free product of the groups G_k.

The graph product $\boldsymbol{G}^{\mathcal{K}}$ depends only on the 1-skeleton (graph) of $\mathcal{K}$.

Example

Let $G_i = \mathbb{Z}$. Then $\boldsymbol{G}^{\mathcal{K}}$ is the right-angled Artin group

$$RA_{\mathcal{K}} = F(g_1, \ldots, g_m) \big/ (g_i g_j = g_j g_i \text{ for } \{i, j\} \in \mathcal{K}),$$

where $F(g_1, \ldots, g_m)$ is a free group with m generators.

Example

Let $G_i = \mathbb{Z}$. Then $\boldsymbol{G}^{\mathcal{K}}$ is the right-angled Artin group

$$RA_{\mathcal{K}} = F(g_1, \ldots, g_m)/(g_i g_j = g_j g_i \text{ for } \{i, j\} \in \mathcal{K}),$$

where $F(g_1, \ldots, g_m)$ is a free group with m generators.

When $\mathcal{K}$ is a full simplex, we have $RA_{\mathcal{K}} = \mathbb{Z}^m$. When $\mathcal{K}$ is m points, we obtain a free group of rank m.

Example

Let $G_i = \mathbb{Z}$. Then $\boldsymbol{G}^{\mathcal{K}}$ is the right-angled Artin group

$$RA_{\mathcal{K}} = F(g_1, \ldots, g_m)/(g_i g_j = g_j g_i \text{ for } \{i, j\} \in \mathcal{K}),$$

where $F(g_1, \ldots, g_m)$ is a free group with m generators.

When $\mathcal{K}$ is a full simplex, we have $RA_{\mathcal{K}} = \mathbb{Z}^m$. When $\mathcal{K}$ is m points, we obtain a free group of rank m.

Example

Let $G_i = \mathbb{Z}_2$. Then $\boldsymbol{G}^{\mathcal{K}}$ is the right-angled Coxeter group

$$RC_{\mathcal{K}} = F(g_1, \ldots, g_m)/(g_i^2 = 1,\ g_i g_j = g_j g_i \text{ for } \{i, j\} \in \mathcal{K}).$$

Theorem

Let $RA_{\mathcal{K}}$ be a right-angled Artin group.

1. $\pi_1((S^1)^{\mathcal{K}}) \cong RA_{\mathcal{K}}$.
2. *Both* $(S^1)^{\mathcal{K}}$ *and* $\mathcal{L}_{\mathcal{K}} = (\mathbb{R}, \mathbb{Z})^{\mathcal{K}}$ *are aspherical iff* $\mathcal{K}$ *is flag.*
3. $\pi_i((S^1)^{\mathcal{K}}) \cong \pi_i(\mathcal{L}_{\mathcal{K}})$ *for* $i \geqslant 2$.
4. $\pi_1(\mathcal{L}_{\mathcal{K}})$ *is isomorphic to the commutator subgroup* $RA'_{\mathcal{K}}$.

Theorem

Let $RA_{\mathcal{K}}$ be a right-angled Artin group.

1. $\pi_1((S^1)^{\mathcal{K}}) \cong RA_{\mathcal{K}}$.
2. *Both* $(S^1)^{\mathcal{K}}$ *and* $\mathcal{L}_{\mathcal{K}} = (\mathbb{R}, \mathbb{Z})^{\mathcal{K}}$ *are aspherical iff* $\mathcal{K}$ *is flag.*
3. $\pi_i((S^1)^{\mathcal{K}}) \cong \pi_i(\mathcal{L}_{\mathcal{K}})$ *for* $i \geqslant 2$.
4. $\pi_1(\mathcal{L}_{\mathcal{K}})$ *is isomorphic to the commutator subgroup* $RA'_{\mathcal{K}}$.

Theorem

Let $RC_{\mathcal{K}}$ be a right-angled Coxeter group.

1. $\pi_1((\mathbb{R}P^{\infty})^{\mathcal{K}}) \cong RC_{\mathcal{K}}$.
2. *Both* $(\mathbb{R}P^{\infty})^{\mathcal{K}}$ *and* $\mathcal{R}_{\mathcal{K}} = (D^1, S^0)^{\mathcal{K}}$ *are aspherical iff* $\mathcal{K}$ *is flag.*
3. $\pi_i((\mathbb{R}P^{\infty})^{\mathcal{K}}) \cong \pi_i(\mathcal{R}_{\mathcal{K}})$ *for* $i \geqslant 2$.
4. $\pi_1(\mathcal{R}_{\mathcal{K}})$ *is isomorphic to the commutator subgroup* $RC'_{\mathcal{K}}$.

Example

Let $\mathcal{K}$ be an *m*-cycle (the boundary of an *m*-gon).
A simple argument with Euler characteristic shows that $\mathcal{R}_{\mathcal{K}}$ is homeomorphic to a closed orientable surface of genus $(m-4)2^{m-3}+1$.
(This observation goes back to a 1938 work of Coxeter.)

Example

Let $\mathcal{K}$ be an m-cycle (the boundary of an m-gon).
A simple argument with Euler characteristic shows that $\mathcal{R}_{\mathcal{K}}$ is homeomorphic to a closed orientable surface of genus
$(m-4)2^{m-3}+1$.
(This observation goes back to a 1938 work of Coxeter.)
Therefore, the commutator subgroup of the corresponding right-angled Coxeter group $RC_{\mathcal{K}}$ is a surface group.

Example

Let $\mathcal{K}$ be an m-cycle (the boundary of an m-gon).
A simple argument with Euler characteristic shows that $\mathcal{R}_\mathcal{K}$ is homeomorphic to a closed orientable surface of genus $(m-4)2^{m-3}+1$.
(This observation goes back to a 1938 work of Coxeter.)
Therefore, the commutator subgroup of the corresponding right-angled Coxeter group $RC_\mathcal{K}$ is a surface group.

Similarly, when $|\mathcal{K}| \cong S^2$ (which is equivalent to $\mathcal{K}$ being the boundary of a 3-dimensional simplicial polytope), $\mathcal{R}_\mathcal{K}$ is a 3-dimensional manifold. Therefore, the commutator subgroup of the corresponding $RC_\mathcal{K}$ is a 3-manifold group.

Theorem (Panov-V)

Let $RA_{\mathcal{K}}$ and $RC_{\mathcal{K}}$ be the right-angled Artin and Coxeter groups corresponding to a simplicial complex $\mathcal{K}$.

(a) *The commutator subgroup $RA'_{\mathcal{K}}$ is free if and only if $\mathcal{K}^1$ is a chordal graph.*

(b) *The commutator subgroup $RC'_{\mathcal{K}}$ is free if and only if $\mathcal{K}^1$ is a chordal graph.*

Theorem (Panov-V)

Let $RA_{\mathcal{K}}$ and $RC_{\mathcal{K}}$ be the right-angled Artin and Coxeter groups corresponding to a simplicial complex $\mathcal{K}$.

(a) *The commutator subgroup $RA'_{\mathcal{K}}$ is free if and only if $\mathcal{K}^1$ is a chordal graph.*

(b) *The commutator subgroup $RC'_{\mathcal{K}}$ is free if and only if $\mathcal{K}^1$ is a chordal graph.*

Part (a) is the result of Servatius, Droms and Servatius.

Theorem (Panov-V)

Let $RA_\mathcal{K}$ and $RC_\mathcal{K}$ be the right-angled Artin and Coxeter groups corresponding to a simplicial complex $\mathcal{K}$.

(a) *The commutator subgroup $RA'_\mathcal{K}$ is free if and only if $\mathcal{K}^1$ is a chordal graph.*

(b) *The commutator subgroup $RC'_\mathcal{K}$ is free if and only if $\mathcal{K}^1$ is a chordal graph.*

Part (a) is the result of Servatius, Droms and Servatius.

The difference between (a) and (b) is that the commutator subgroup $RA'_\mathcal{K}$ is infinitely generated, unless $RA_\mathcal{K} = \mathbb{Z}^m$, while the commutator subgroup $RC'_\mathcal{K}$ is finitely generated.

Let G be group. The *commutator* of two elements $a, b \in G$ given by the formula $(a, b) = a^{-1}b^{-1}ab$.

We refer to the following nested commutator of length k

$$(q_{i_1}, q_{i_2}, \dots, q_{i_k}) := (\dots((q_{i_1}, q_{i_2}), q_{i_3}), \dots, q_{i_k}).$$

as the *simple nested commutator* of $q_{i_1}, q_{i_2}, \dots, q_{i_k}$.

Similarly, we define *simple nested Lie commutators*

$$[\mu_{i_1}, \mu_{i_2}, \dots, \mu_{i_k}] := [\dots[[\mu_{i_1}, \mu_{i_2}], \mu_{i_3}], \dots, \mu_{i_k}].$$

For any group G and any three elements $a, b, c \in G$, the following *Hall–Witt identities* hold:

$$\begin{aligned}
&(a, bc) = (a, c)(a, b)(a, b, c), \\
&(ab, c) = (a, c)(a, c, b)(b, c), \\
&(a, b, c)(b, c, a)(c, a, b) = (b, a)(c, a)(c, b)^a(a, b)(a, c)^b(b, c)^a \\
&\qquad\qquad (a, c)(c, a)^b,
\end{aligned} \tag{1}$$

where $a^b = b^{-1}ab$.

Let $H, W \subset G$ be subgroups. Then we define $(H, W) \subset G$ as the subgroup generated by all commutators (h, w), $h \in H, w \in W$. In particular, the *commutator subgroup* G' of the group G is (G, G).

Definition

For any group G, set $\gamma_1(G) = G$ and define inductively $\gamma_{k+1}(G) = (\gamma_k(G), G)$. The resulting sequence of groups $\gamma_1(G), \gamma_2(G), \ldots, \gamma_k(G), \ldots$ is called the *lower central series* of G.

Definition

If $H \subset G$ is normal subgroup, i. e. $H = g^{-1}Hg$ for all $g \in G$, we will use the notation $H \lhd G$.
In particular, $\gamma_{k+1}(G) \lhd \gamma_k(G)$, and the quotient group $\gamma_k(G)/\gamma_{k+1}(G)$ is abelian. Denote $L^k(G) := \gamma_k(G)/\gamma_{k+1}(G)$ and consider the direct sum

$$L(G) := \bigoplus_{k=1}^{+\infty} L^k(G).$$

Given an element $a_k \in \gamma_k(G) \subset G$, we denote by $\overline{a}_k$ its conjugacy class in the quotient group $L^k(G)$. If $a_k \in \gamma_k(G),\ a_l \in \gamma_l(G)$, then $(a_k, a_l) \in \gamma_{k+l}(G)$. Then the Hall–Witt identities imply that $L(G)$ is a graded Lie algebra over $\mathbb{Z}$ (a Lie ring) with Lie bracket $[\overline{a}_k, \overline{a}_l] := \overline{(a_k, a_l)}$. The Lie algebra $L(G)$ is called the Lie algebra associated with the lower central series (or the associated Lie algebra) of G.

Theorem

There is an isomorphism

$$H_k(\mathcal{R}_{\mathcal{K}};\mathbb{Z}) \cong \bigoplus_{J\subset[m]} \widetilde{H}_{k-1}(\mathcal{K}_J)$$

for any $k \geqslant 0$, *where* $\widetilde{H}_{k-1}(\mathcal{K}_J)$ *is the reduced simplicial homology group of* $\mathcal{K}_J$.

Theorem (Panov-V)

Let $RC_{\mathcal{K}}$ be right-angled Coxeter group corresponding to a simplicial complex $\mathcal{K}$ with m vertices. Then the commutator subgroup $RC'_{\mathcal{K}}$ has a finite minimal set of generators consisting of $\sum_{J\subset[m]} \operatorname{rank} \widetilde{H}_0(\mathcal{K}_J)$ nested commutators

$$(g_i, g_j), \quad (g_i, g_j, g_{k_1}), \quad \ldots, \quad (g_i, g_j, g_{k_1}, g_{k_2}, \ldots, g_{k_{m-2}}), \tag{2}$$

where $i < j > k_1 > k_2 > \ldots > k_{\ell-2}$, $k_s \neq i$ for all s, and i is the smallest vertex in a connected component not containing j of the subcomplex $\mathcal{K}_{\{k_1,\ldots,k_{\ell-2},j,i\}}$.

Corollary

The free abelian group $H_1(\mathcal{R}_{\mathcal{K}}) = RC'_{\mathcal{K}}/RC''_{\mathcal{K}}$ of rank $\sum_{J\subset[m]} \operatorname{rank} \widetilde{H}_0(\mathcal{K}_J)$ has a basis consisting of the images of the iterated commutators described in Theorem above.

2. The LCS of a right-angled Coxeter group

Proposition

Let G be a group with generators $g_i, i \in I$. The k-th term $\gamma_k(G)$ of the lower central series is generated by simple nested commutators of length greater than or equal to k in generators and their inverses.

Corollary

Let $RC_{\mathcal{K}}$ be a right-angled Coxeter group with generators g_i. Then the group $\gamma_k(RC_{\mathcal{K}})$ is generated by commutators of length greater than or equal to k in generators g_i.

Proposition

The square of any element of $\gamma_k(RC_{\mathcal{K}})$ is contained in $\gamma_{k+1}(RC_{\mathcal{K}})$.

Proof.

We use γ_k instead of $\gamma_k(RC_{\mathcal{K}})$ in this proof.
Let $a \in \gamma_k$. If $k = 1$, then $a = \prod_{i=1}^{n} g_{k_i}$. If $k > 1$, then $a = \prod_{i=1}^{n} a_i$, where $a_i = (b_i, g_{p_i})$ or $a_i = (g_{p_i}, b_i)$, $b_i \in \gamma_{k-1}$. We use induction on n.
Let $n = 1$. The case $k = 1$ is obvious (because $g_k^2 = 1$). If $k > 1$, then $a = (b, g_i)$ or $a = (g_i, b)$ for some $b \in \gamma_{k-1}$. For $a = (b, g_i)$ we have $a^2 = (b, g_i)(b, g_i) = (g_i, (b, g_i)) \in \gamma_{k+1}$, and for $a = (g_i, b)$ we have $a^2 = (g_i, b)(g_i, b) = (g_i, (g_i, b)) \in \gamma_{k+1}$.
Suppose now the statement is proved for $n - 1$. Let $a = \prod_{i=1}^{n} a_i$ and $a^2 = \prod_{i=1}^{n} a_i \cdot \prod_{i=1}^{n} a_i$. We have:

$$a_1 a_2 \cdots a_n a_1 a_2 \cdots a_n =$$

$$= (a_1^{-1}, (a_2 \cdots a_n)^{-1}) \cdot (a_2 \cdots a_n) a_1^2 (a_2 \cdots a_n)^{-1} \cdot (a_2 \cdots a_n)^2.$$

Clearly, the first factor lies in $\gamma_{2k} \subset \gamma_{k+1}$. The second factor lies in γ_{k+1} as a conjugate to a_1^2 (by induction). The last factor also lies in γ_{k+1} by induction.

$\square$

Corollary

$L(RC_{\mathcal{K}})$ is a Lie algebra over $\mathbb{Z}_2$.

We denote by $FL_{\mathbb{Z}_2}\langle\mu_1, \mu_2, \ldots, \mu_n\rangle$ a free graded Lie algebra over $\mathbb{Z}_2$ with n generators μ_i, where $\deg \mu_i = 1$.
For any simplicial complex $\mathcal{K}$ we consider the *graph Lie algebra* over $\mathbb{Z}_2$:

$$L_{\mathcal{K}} := FL_{\mathbb{Z}_2}\langle\mu_1, \mu_2, \ldots, \mu_n\rangle/([\mu_i, \mu_j] = 0 \text{ for } \{i, j\} \in \mathcal{K}).$$

Clearly, $L_{\mathcal{K}}$ depends only on the 1-skeleton $\mathcal{K}^1$ (a graph), however, as in the case of right-angled Coxeter groups, it is more convenient for us to work with simplicial complexes.

Proposition

There is an epimorphism of Lie algebras $\varphi : L_{\mathcal{K}} \to L(RC_{\mathcal{K}})$.

Proof.

$L(RC_{\mathcal{K}})$ is a Lie algebra over $\mathbb{Z}_2$, generated by the elements $\overline{g}_i \in \gamma_1(RC_{\mathcal{K}})/\gamma_2(RC_{\mathcal{K}}), i = 1, \dots, m$. By definition of a free Lie algebra, we have an epimorphism

$$\widetilde{\varphi}\colon FL_{\mathbb{Z}_2}\langle \mu_1, \mu_2, \dots, \mu_n \rangle \to L(RC_{\mathcal{K}}), \quad \mu_i \mapsto \overline{g}_i.$$

Since there is a relation $[\overline{g}_i, \overline{g}_j] = 0$ for $\{i, j\} \in \mathcal{K}$ in the Lie algebra $L(RC_{\mathcal{K}})$, the epimorphism $\widetilde{\varphi}$ factors through a required epimorphism φ. $\square$

In fact, the homomorphism φ from the proposition above is not injective, and the Lie algebras $L_{\mathcal{K}}$ and $L(RC_{\mathcal{K}})$ are not isomorphic. This distinguishes the case of right-angled Coxeter groups from the case of the right-angled Artin groups, where the associated Lie algebra $L(RA_{\mathcal{K}})$ is isomorphic to the graph Lie algebra over $\mathbb{Z}$.

Example

Let $\mathcal{K}$ consist of two disjoint points, i.e. $\mathcal{K} = \{1, 2\}$. Then $L_{\mathcal{K}} = FL_{\mathbb{Z}_2}\langle\mu_1, \mu_2\rangle = FL_{\mathbb{Z}_2}\langle\mu_1\rangle * FL_{\mathbb{Z}_2}\langle\mu_2\rangle$ (hereinafter $*$ denotes the free product of Lie algebras or groups). The lower central series of $RC_{\mathcal{K}} = \mathbb{Z}_2 * \mathbb{Z}_2$ is as follows: $\gamma_1(RC_{\mathcal{K}}) = \mathbb{Z}_2 * \mathbb{Z}_2$, and for $k \geqslant 2$ we have $\gamma_k(RC_{\mathcal{K}}) \cong \mathbb{Z}$ is an infinite cyclic group generated by the commutator $(g_1, g_2, g_1, \ldots, g_1)$ of length k. Proposition 2 implies that $\gamma_k(RC_{\mathcal{K}})/\gamma_{k+1}(RC_{\mathcal{K}}) = \mathbb{Z}_2$ for $k > 1$, and $\gamma_1(RC_{\mathcal{K}})/\gamma_2(RC_{\mathcal{K}}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Consider the algebra $L(RC_{\mathcal{K}})$. From the arguments above, $L(RC_{\mathcal{K}}) = (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \oplus \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2 \oplus \cdots$. It is easy to see that $L^k(RC_{\mathcal{K}}) \cong L^k_{\mathcal{K}}$ for $k = 1, 2$. However, $L^3_{\mathcal{K}} \cong \mathbb{Z}_2\langle[\mu_1, \mu_2, \mu_1], [\mu_1, \mu_2, \mu_2]\rangle$, while $L^3(RC_{\mathcal{K}}) \cong \mathbb{Z}_2$. Therefore,

$$L^3(RC_{\mathcal{K}}) \cong L^3_{\mathcal{K}}/([\mu_1, \mu_2, \mu_1] = [\mu_1, \mu_2, \mu_2]).$$

It follows that the homomorphism φ is not injective.

Proposition

Let $\mathcal{K}$ consist of two disjoint points. Then

$$L(RC_{\mathcal{K}}) \cong L_{\mathcal{K}} \big/ \big([a, \mu_1] = [a, \mu_2], \;\; [a, \underbrace{\mu_1, \dots, \mu_1}_{2k+1}, a] = 0, \; k \geqslant 0\big),$$

where $a = [\mu_1, \mu_2]$.

Theorem

Let $\mathcal{K}$ be a simplicial complex on $[m]$, let $RC_\mathcal{K}$ be the right-angled Coxeter group corresponding to $\mathcal{K}$, and $L(RC_\mathcal{K})$ its associated Lie algebra. Then:

(a) $L^1(RC_\mathcal{K})$ *has a basis* $\overline{g}_1, \ldots, \overline{g}_m$;

(b) $L^2(RC_\mathcal{K})$ *has a basis consisting of the commutators* $[\overline{g}_i, \overline{g}_j]$ *with* $i < j$ *and* $\{i, j\} \notin \mathcal{K}$;

(c) $L^3(RC_\mathcal{K})$ *has a basis consisting of*

- *the commutators* $[\overline{g}_i, \overline{g}_j, \overline{g}_j]$ *with* $i < j$ *and* $\{i, j\} \notin \mathcal{K}$;
- *the commutators* $[\overline{g}_i, \overline{g}_j, \overline{g}_k]$ *where* $i < j > k$, $i \neq k$ *and* i *is the smallest vertex in a connected component of* $\mathcal{K}_{\{i,j,k\}}$ *not containing* j.

As a consequence, we obtain a description of the first three consecutive quotients of the lower central series for a free product of the groups $\mathbb{Z}_2$.

Corollary

Let $\mathcal{K}$ be a set of m disjoint points, i. e. $RC_{\mathcal{K}} = \mathbb{Z}_2\langle g_1\rangle * \ldots * \mathbb{Z}_2\langle g_m\rangle$. Then:

(a) *$L^1(RC_{\mathcal{K}})$ has a basis $\overline{g}_1, \ldots, \overline{g}_m$;*

(b) *$L^2(RC_{\mathcal{K}})$ has a basis consisting of the commutators $[\overline{g}_i, \overline{g}_j]$ with $i < j$;*

(c) *$L^3(RC_{\mathcal{K}})$ has a basis consisting of*
  - *the commutators $[\overline{g}_i, \overline{g}_j, \overline{g}_j]$ with $i < j$;*
  - *the commutators $[\overline{g}_i, \overline{g}_j, \overline{g}_k]$ with $i < j > k$, $i \neq k$.*

Example

Consider simplicial complexes on 3 vertices.

Let $\mathcal{K} = \underset{1}{\bullet} \quad \underset{2}{\bullet} \quad \underset{3}{\bullet}$. Then $L^3(RC_{\mathcal{K}})$ has a basis consisting of 5 commutators:
$[\overline{g}_1, \overline{g}_2, \overline{g}_2], [\overline{g}_2, \overline{g}_3, \overline{g}_3], [\overline{g}_1, \overline{g}_3, \overline{g}_3], [\overline{g}_1, \overline{g}_3, \overline{g}_2], [\overline{g}_2, \overline{g}_3, \overline{g}_1]$.

Let $\mathcal{K} = \underset{1}{\bullet}\!\!\longrightarrow\!\!\!\!-\underset{2}{\bullet} \quad \underset{3}{\bullet}$. Then $L^3(RC_{\mathcal{K}})$ has a basis consisting of 3 commutators: $[\overline{g}_2, \overline{g}_3, \overline{g}_3], [\overline{g}_1, \overline{g}_3, \overline{g}_3], [\overline{g}_1, \overline{g}_3, \overline{g}_2]$.

Let $\mathcal{K} = \underset{1}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\underset{2}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\underset{3}{\bullet}$. Then $L^3(RC_{\mathcal{K}})$ is generated by the commutator $[\overline{g}_1, \overline{g}_3, \overline{g}_3]$.

Proof of theorem

To simplify the notation we write L^k instead of $L^k(RC_{\mathcal{K}})$ and γ_k instead of $\gamma_k(RC_{\mathcal{K}})$. Statement (a) follows from the fact that

$$L^1 = \gamma_1/\gamma_2 = RC_{\mathcal{K}}/RC'_{\mathcal{K}} = \mathbb{Z}_2^m$$

with basis $\overline{g}_1, \ldots, \overline{g}_m$.
We prove statement (b). Consider the abelianization map

$$\varphi_{\mathrm{ab}} : RC'_{\mathcal{K}} \to RC'_{\mathcal{K}}/RC''_{\mathcal{K}} = \gamma_2/\gamma'_2.$$

The group $RC'_{\mathcal{K}}/RC''_{\mathcal{K}} = H_1(\mathcal{R}_{\mathcal{K}})$ is free abelian (above).
Consider $L^2 = \gamma_2/\gamma_3$. The group L^2 is a $\mathbb{Z}_2$-module (see above), i. e. $L^2 = \mathbb{Z}_2^M$ for some $M \in \mathbb{N}$. We have a sequence of nested normal subgroups

$$\gamma'_2 \lhd \gamma_4 \lhd \gamma_3 \lhd \gamma_2.$$

Consider the exact sequence of abelian groups:

$$\begin{tikzcd} 0 \arrow[r] & \gamma_3/\gamma'_2 \arrow[r, "\psi"] \arrow[d, equal] & \gamma_2/\gamma'_2 \arrow[r] \arrow[d, equal] & \gamma_2/\gamma_3 \arrow[r] \arrow[d, equal] & 0. \\ & \mathbb{Z}^N & \mathbb{Z}^N & \mathbb{Z}_2^M & \end{tikzcd}$$

Recall from Corollary above that the free abelian group $\gamma_2/\gamma_2' = \mathbb{Z}^N$ has a basis consisting of the images of the iterated commutators with all different indices described in Theorem above. The images of the commutators of length $\geqslant 3$ are contained in the subgroup $\gamma_3/\gamma_2' \subset \gamma_2/\gamma_2'$. The group γ_3/γ_2' also contains commutators of length 3 with duplicate indices, i. e. of the form $(g_j, g_i, g_i) = (g_i, g_j)^2$. Therefore, the homomorphism ψ acts by the formula:

$$\psi(\overline{(g_i, g_j, g_{k_1}, g_{k_2}, \ldots, g_{k_{m-2}})}) = \overline{(g_i, g_j, g_{k_1}, g_{k_2}, \ldots, g_{k_{m-2}})}, \quad m \geqslant 3,$$
$$\psi(\overline{(g_j, g_i, g_i)}) = \overline{(g_i, g_j)}^2,$$

where the indices $i, j, k_1, \ldots, k_{m-2}$ are all different. The elements $\overline{(g_j, g_i, g_i)}$ with $i < j$, $\{i, j\} \notin \mathcal{K}$, and the elements $\overline{(g_i, g_j, g_{k_1}, g_{k_2}, \ldots, g_{k_{m-2}})}$, $m \geqslant 3$, with the condition on the indices from theorem above form a basis in a free abelian group γ_3/γ_2'.
It follows that the $\mathbb{Z}_2$-module $L^2 = \gamma_2/\gamma_3$ has a basis consisting of the elements $\overline{(g_i, g_j)} = [\overline{g}_i, \overline{g}_j]$ with $i < j$ and $\{i, j\} \notin \mathcal{K}$, proving (b).

We prove statement (c). Consider $L^3 = \gamma_3/\gamma_4$. The group L^3 is a $\mathbb{Z}_2$-module (see above), i. e. $L^3 = \mathbb{Z}_2^M$ for some $M \in \mathbb{N}$.
Consider the exact sequence of abelian groups:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \gamma_4/\gamma_2' & \xrightarrow{\chi} & \gamma_3/\gamma_2' & \longrightarrow & \gamma_3/\gamma_4 & \longrightarrow & 0. \\ & & \| & & \| & & \| & & \\ & & \mathbb{Z}^N & & \mathbb{Z}^N & & \mathbb{Z}_2^M & & \end{array}$$

For the free abelian group γ_3/γ_2', we will use the basis constructed in the proof of statement (b). Elements of this basis corresponding to commutators of length $\geqslant 4$ are contained in γ_4/γ_2'. The group γ_4/γ_2' also contains commutators of length 4 with repeated indices. These commutators have one of the following nine types, which we divide into two types A and B for convenience:

$$\begin{aligned} A = \{&(g_i, g_j, g_j, g_j), (g_i, g_j, g_j, g_i), (g_i, g_j, g_i, g_j), \\ &(g_i, g_j, g_i, g_i), (g_i, g_j, g_i, g_k), (g_i, g_j, g_j, g_k)\}, \\ B = \{&(g_i, g_j, g_k, g_j), (g_i, g_j, g_k, g_i), (g_i, g_j, g_k, g_k)\}. \end{aligned}$$

Note that

$$(g_i, g_j, g_j, g_j) = ((g_j, g_i) \cdot (g_j, g_i), g_j) =$$
$$= ((g_j, g_i), g_j) \cdot (((g_j, g_i), g_j), (g_j, g_i)) \cdot ((g_j, g_i), g_j) \equiv (g_j, g_i, g_j)^2 \mod \gamma_2',$$

because $(((g_j, g_i), g_j), (g_j, g_i)) \in \gamma_2'$. Here in the second identity we used Hall-Witt commutator identity. A similar decomposition holds for other commutators of type A, for example,

$$(g_i, g_j, g_i, g_k) = (g_j, g_i, g_k)^2 \mod \gamma_2'.$$

Now consider the commutators of type B. We will need the following commutator identities. For any $a, b, c, d \in \gamma_1$ we have:

$$(a, b)(c, d) \equiv (c, d)(a, b) \mod \gamma_2'. \tag{3}$$

It follows that the last of the Hall-Witt identities takes the following form modulo γ_2':

$$(a, b, c)(b, c, a)(c, a, b) \equiv 1 \mod \gamma_2'. \tag{4}$$

Furthermore, the following identity was obtained in (Panov-V):

$$(g_q,(g_p,x))=(g_q,x)(x,(g_p,g_q))(g_q,g_p)(x,g_p)$$
$$(g_p,(g_q,x))(x,g_q)(g_p,g_q)(g_p,x).$$

If $x \in \gamma_2$, then the previous identity and identity (3) imply

$$(g_q,(g_p,x)) \equiv (g_p,(g_q,x)) \mod \gamma_2'. \tag{5}$$

To simplify the notation, we write i instead of g_i. From (1) and (4) we obtain

$$\begin{aligned}(g_i,g_j,g_k,g_i) = (((i,j),k),i) &\equiv ((i,(i,j)),k)^{-1}\cdot((k,i),(i,j))^{-1} \equiv \\ &\equiv (k,(i,(i,j))) = (k,((i,j),i)^{-1}) = (k,(j,i)^{-2}) = \\ &= (k,(j,i)^{-1})\cdot(k,(j,i)^{-1})\cdot((k,(i,j)^{-1}),(i,j)^{-1}) \equiv \\ &\equiv (k,(j,i)^{-1})^2 = (g_i,g_j,g_k)^{-2} \mod \gamma_2',\end{aligned}$$

$$\begin{aligned}(g_i,g_j,g_k,g_j) = (((i,j),k),j) &\equiv ((j,(i,j)),k)^{-1}\cdot((k,j),(i,j))^{-1} \equiv \\ \equiv (k,(j,(i,j))) = (k,((i,j),j)^{-1}) &= (k,(j,i)^{-2}) \equiv (g_i,g_j,g_k)^{-2} \mod \gamma_2'.\end{aligned}$$

The last commutator of type B requires a lengthier calculation:

$$(g_i, g_j, g_k, g_k) \equiv^1 (j, i, k)\cdot(i, j, k)\cdot(k, i, k)\cdot(i, k, k)\cdot((k, j)^i, k)\cdot((j, k)^i, k)\cdot$$
$$\cdot ((i, k)^j, k) \cdot ((k, i)^j, k) \cdot (k, (j, (k, i)))^{-1} \cdot (k, (i, (j, k)))^{-1} \equiv^2$$
$$\equiv^2 (k, (j, (k, i)))^{-1}\cdot(k, (i, (j, k)))^{-1} \equiv^3 (j, (k, (k, i)))^{-1}\cdot(i, (k, (j, k)))^{-1} =$$
$$= (j, (i, k)^{-2})^{-1} \cdot (i, (k, j)^{-2})^{-1} \equiv (k, i, j)^2 \cdot (j, k, i)^2 \equiv$$
$$\equiv (g_i, g_j, g_k)^{-2} \mod \gamma_2'.$$

Here is the identity $\equiv^1$ is obtained with help of the algorithm written by the author in Wolfram Mathematica using commutator identities (1).
The identity $\equiv^2$ follows from the relations $(a, b) \cdot (a^{-1}, b) = (b, a, a^{-1})$ and $(b, a, a^{-1}) \equiv 1 \mod \gamma_2'$, if $a \in \gamma_2$.
The identity $\equiv^3$ follows from (5).

It follows that the homomorphism $\chi\colon \gamma_4/\gamma_2' \to \gamma_3/\gamma_2'$ acts by the formula:

$$\chi(\overline{(g_i, g_j, g_{k_1}, g_{k_2}, \dots, g_{k_{m-2}})}) = \overline{(g_i, g_j, g_{k_1}, g_{k_2}, \dots, g_{k_{m-2}})}, \quad m \geqslant 4,$$
$$\chi(\overline{(g_j, g_i, g_i, g_j)}) = \overline{((g_i, g_j), g_j)}^{2},$$
$$\chi(\overline{(g_j, g_i, g_j, g_k)}) = \overline{((g_i, g_j), g_k)}^{2},$$
$$\chi(\overline{(g_i, g_j, g_k, g_k)}) = \overline{((g_i, g_j), g_k)}^{-2}.$$

where the indices corresponding to a different letters are different. Thus, the $\mathbb{Z}_2$-module $L^3 = \gamma_3/\gamma_4$ has a basis consisting of the elements specified in the theorem.

References

[1] Ya. Veryovkin. *The Lie algebra associated with a right-angled Coxeter group*. Proceedings of the Steklov Institute of Mathematics 305(1), pp 53-62; arXiv:1901.06929.

Thank you for you attention!

This work was supported by the Russian Science Foundation under grant no. 23-11-00143, https://rscf.ru/en/project/23-11-00143/ .