Torus Actions in topology and Geometry

-Graduate School on Toric Topology-

Fields Institute

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The Davis-Januszkiewicz Construction Reinterpreted and Generalized

REFERENCES FOR THE INTRODUCTION

M. Davis and T. Januszkiewicz,
Convex polytopes, Coxeter orbifolds and torus actions,
Duke Math. J., **62**, (1991), no. 2, 417–451.

V. Buchstaber and T. Panov,
Toric Topology,
AMS Mathematical Surveys and Monographs, **204** (2015).

... and we get:

$$\mathbb{C}/S^1 \;\cong\; \mathbb{R}^+$$

Here, the right hand side is a combinatorial object.

This extends to:

$$\mathbb{C}^n/(S^1)^n \;\cong\; (\mathbb{R}^+)^n$$

and is called “the standard action”.

"Definition":

A toric manifold M^{2n} is a manifold such that:

(i) the real n-dimensional torus T^n acts on the manifold

(ii) M^{2n} is covered by local charts $\mathbb{C}^n$, satisfying the property the action of the torus restricts to the standard action

(iii) the quotient M^{2n}/T^n has the structure of a *simple* polytope P^n.

⁕ Under the T^n action, each copy of $\mathbb{C}^n$ must project to an $\mathbb{R}^n_+$ neighborhood of a vertex of P^n.

Here, *simple* means that P^n has the property that at each vertex, exactly n facets intersect.

The topological approach to non-singular projective toric varieties, (and more general spaces), requires two ingredients:

(i) A simple polytope P^n of dimension n having a set of m facets

$$\mathcal{F} = \{F_1, F_2, \ldots, F_m\}.$$

When P^n is simple, every codimension-l face F can be written uniquely as

$$F = F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_l}$$

where the F_{i_j} are the facets containing F.

(ii) A *characteristic* function

$$\lambda\colon \mathcal{F} \longrightarrow \mathbb{Z}^n$$

which assigns an integer vector to each facet of the simple polytope P^n.

It can be considered as an $(n \times m)$-matrix,

$$\lambda\colon \mathbb{Z}^m \longrightarrow \mathbb{Z}^n$$

with integer entries and columns indexed by the facets of the polytope P^n.

We require λ to satisfy a *regularity* condition: if

$$F = F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_l}$$

then the vectors

$$\{\lambda(F_{i_1}), \lambda(F_{i_2}), \ldots, \lambda(F_{i_l})\}$$

must span an l-dimensional submodule of $\mathbb{Z}^n$ which is a direct summand.

(All $n \times n$ minors of λ corresponding to the vertices of P^n are required to be ± 1.)

Next, regarding $\mathbb{R}^n$ as the Lie algebra of T^n, the map λ is used to associate to each codimension-l face F of P^n a rank-l subgroup $G_F \subset T^n$.

Specifically for the facet F_{i_j}, writing

$$\lambda(F_{i_j}) = (\lambda_{1i_j}, \lambda_{2i_j}, \ldots, \lambda_{ni_j})$$

gives G_F as the subgroup in $(S^1)^n$

$$\left\{\left(e^{2\pi i(\lambda_{1i_1}t_1+\lambda_{1i_2}t_2+\cdots+\lambda_{1i_l}t_l)}, \ldots, e^{2\pi i(\lambda_{ni_1}t_1+\lambda_{ni_2}t_2+\cdots+\lambda_{ni_l}t_l)}\right)\right\}$$

where $t_i \in \mathbb{R}$, $i = 1, 2, \ldots, l$.

Finally, let $p \in P^n$ and $F(p)$ be the unique face with p in its relative interior.

Define an equivalence relation $\sim_\lambda$ on $T^n \times P^n$ by

$$(g, p) \sim_\lambda (h, q)$$

if and only if:

(i) $p = q$

(ii) $g^{-1}h \in G_{F(p)} \cong T^l$.

$$M^{2n} \cong M^{2n}(\lambda) = T^n \times P^n / \sim_\lambda$$

AN EXAMPLE

$\mathbb{C}P^2$ is constructed from:

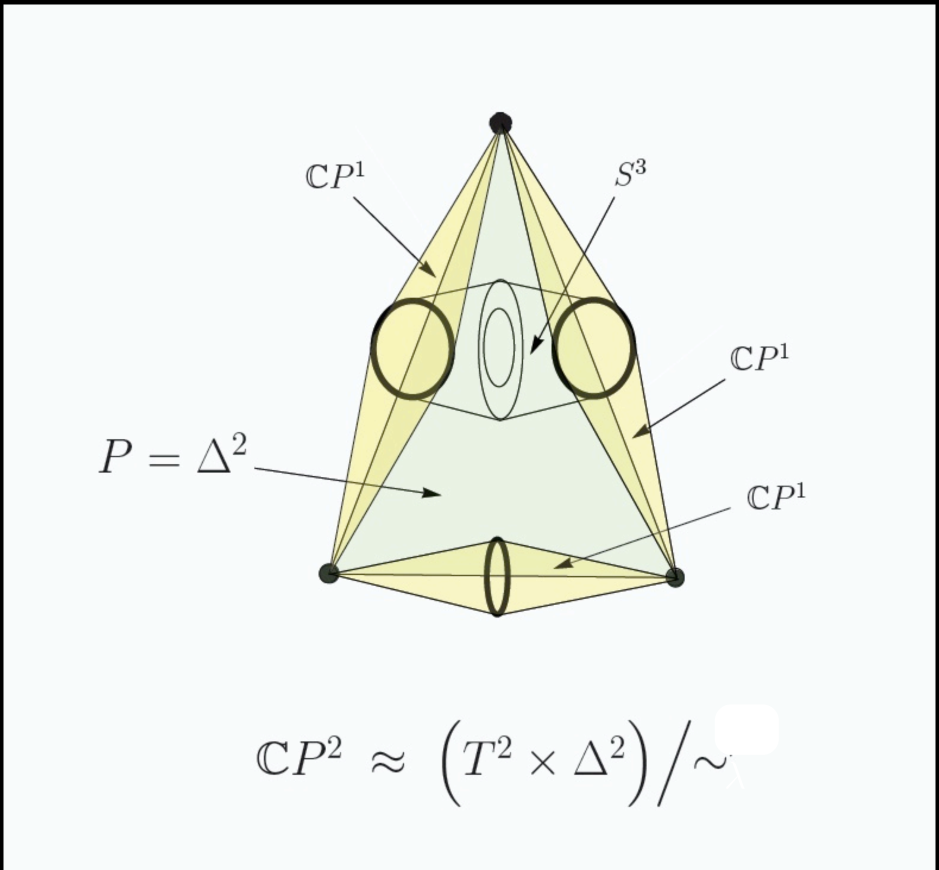
(i) P^2 is the two-simplex which has dimension $n = 2$ and $m = 3$ facets

(ii) The matrix λ is

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

PICTURE

RECALL $\mathbb{C}P^2$



The diagram presents $\mathbb{C}P^2$ as the cone on S^3 attached to $\mathbb{C}P^1$ via the Hopf map

$$S^3 \longrightarrow S^2$$

by "collapsing" by the action of the diagonal circle

A SECOND CONSTRUCTION

Davis and Januszkiewicz constructed a second space by

$$\mathcal{L} = T^m \times P^n / \sim_\theta$$

where here $\sim_\theta$ does not involve the characteristic λ.

Specifically, λ has been replaced by

$$\theta \colon \mathcal{F} \longrightarrow \mathbb{Z}^m$$

where $\theta(F_i) = \underline{e}_i \in \mathbb{Z}^m$, so that the facets of P^n index "co-ordinate circles" in the torus T^m.

An equivalence relation $\sim_\theta$ is defined 0n $T^m \times P^n$ by exact analogy with $\sim_\lambda$ on $T^n \times P^n$.

It's easy to see that the case when P^2 is the two-simplex we get

$$\begin{aligned}\mathcal{L} &= T^m \times P^n / \sim_\theta \\ &= T^3 \times \Delta^2 / \sim_\theta \\ &= S^1 * (S^1 * S^1) \\ &= S^5\end{aligned}$$

PICTURE

A SIMPLICIAL COMPLEX ASSOCIATED TO P

Let K_P denote the simplicial complex which is dual to the boundary of a simple polytope P^n having m facets.

The duality here is in the sense that the facets of P^n correspond to the vertices of K_P.

A set of vertices in K_P is a simplex if and only if the corresponding facets in P^n all intersect.

In the Buchstaber-Panov formalism, we have

$$\underbrace{\mathcal{L} \;=\; T^m \times P^n/\!\sim_\theta}_{\text{a quotient}} \quad \cong \quad \underbrace{Z(K_P;(D^2,S^1))}_{\text{a colimt}}$$

Here

$$Z(K_P;(D^2,S^1)) \;\subset\; D^2 \times D^2 \times \cdots \times D^2 \quad (m-\text{factors})$$

is a natural subspace invariant under the action of T^m.

The two toric constructions are related by a quotient map given by the free action of

$$\ker\lambda \;\cong\; T^{m-n} \;\subset\; T^m$$

on

$$\mathcal{L} \;=\; T^m \times P^n/{\sim_\theta},$$

(courtesy of the regularity condition),

and a commutative diagram

$$\begin{array}{ccccc}
T^m \times P^n/{\sim_\theta} & \longrightarrow & T^n \times P^n/{\sim_\lambda} & \xrightarrow{\cong} & (T^m \times P^n/{\sim_\theta})/\ker\lambda \\
\big\downarrow & & & & \big\downarrow{\scriptstyle\cong} \\
Z(K_P;(D^2,S^1)) & & \longrightarrow & & Z(K_P;(D^2,S^1))/\ker\lambda
\end{array}$$

in which we are mindful of the linear extension

$$\begin{array}{ccc}
\mathfrak{F} & \xrightarrow{\theta} & \mathbb{Z}^m \\
\big\downarrow{\scriptstyle =} & & \big\downarrow{\scriptstyle\lambda} \\
\mathfrak{F} & \xrightarrow{\lambda} & \mathbb{Z}^n
\end{array}$$

THE MAIN REFERENCE FOR ALL THAT FOLLOWS

A. Bahri, M. Bendersky, F. Cohen and S. Gitler

A generalization of the Davis-Januszkiewicz construction and applications to toric manifolds and iterated polyhedral products.

Perspectives in Lie Theory,
F. Callegaro, G. Carnovale, F. Caselli, C. De Concini, A. De Sole (editors),

Springer INdAM series, (2017), 369–388.

(a version is up on the arXiv)

Again, let P^n be simple polytope. Next, we replace each of the circles in T^m by spaces $X_1, X_2, \ldots, X_m$ indexed by the facets of P^n.

An equivalence relation $\sim_1$ on the product

$$X_1 \times X_2 \times \cdots \times X_m \times P^n$$

as follows:

$$(x_1, x_2, \ldots, x_m, p) \;\sim_1\; (y_1, y_2, \ldots, y_m, q)$$

if and only if:

(i) $p = q$ and

(ii) when p is in the relative interior of the face

$$F(p) = F_{j_1} \cap F_{j_2} \cap \cdots F_{j_k}$$

given as the intersection of the k facets which are complementary to $\{F_{i_1}, F_{i_2}, \ldots, F_{i_{m-k}}\}$, then

$$x_{i_s} = y_{i_s} \quad \text{for all} \quad s \in \{1, 2, \ldots, m-k\}$$

Equivalence classes are denoted by the symbol

$$\big[(x_1, x_2, \ldots, x_m, p)\big]_1$$

For the case $X_i = S^1$ for all $i = 1, 2, \ldots, m$ we have

$$\begin{aligned} X_1 \times X_2 \times \cdots \times X_m \times P^n / \sim_1 \\ &= S^1 \times S^1 \times \cdots \times S^1 \times P^n / \sim_1 \\ &= T^m \times P^n / \sim_1 \\ &= T^m \times P^n / \sim_\theta \\ &= \mathcal{L} \end{aligned}$$

The original Davis-Januszkiewicz space. Here, for the relation $\sim_\theta$ we have used the torus T^m action on itself in the usual way.

In fact:

$$X_1 \times X_2 \times \cdots \times X_m \times P^n / {\sim_1} \;\cong\; Z\big(K_P; (\underline{CX}, \underline{X})\big)$$

Suppose now that S^1 acts *freely* on each of the **spaces**

$$X_1, X_2, \ldots, X_m$$

giving a natural free action of T^m on

$$X_1 \times X_2 \times \cdots \times X_m$$

in the obvious way.

Recall now that the function θ indexes the "coordinate" circles in T^m by the facets of P^n. This associates each space X_i to a facet F_i.

So, an intersection of k facets in P^n determines a projection

$$T^m \longrightarrow T^{m-k}$$

and by this projection, T^m acts on the product

$$X_{i_1} \times X_{i_2} \times \cdots \times X_{i_{m-k}}.$$

Next, let λ be a characteristic map specified for the polytope P^n, so that

$$\ker\lambda \cong T^{m-n} \subset T^m.$$

For $k \leq n$, there is the induced action of $\ker\lambda$ on the product

$$X_{i_1} \times X_{i_2} \times \cdots \times X_{i_{m-k}}$$

and a well-defined projection $\pi_{i_1, i_2, \ldots, i_{m-k}}$

$$\begin{array}{ccc} X_1 \times X_2 \times \cdots \times X_m / \ker\lambda & \longrightarrow & \mathrm{X_{i_1} \times X_{i_2} \times \cdots \times X_{i_{m-k}}} / \ker\lambda \\ [x_1, x_2, \ldots, x_m]_\lambda & \mapsto & [x_{i_1}, x_{i_2}, \ldots, x_{i_{m-k}}]_\lambda \end{array}$$

corresponding to each intersection of k facets.

We define next an equivalence relation $\sim_2$ on the product

$$\big(X_1 \times X_2 \times \cdots \times X_m/\ker\lambda\big) \times \mathrm{P^n}$$

as follows:

$$\big([x_1, x_2, \ldots, x_m]_\lambda, p\big) \sim_2 \big([y_1, y_2, \ldots, y_m]_\lambda, q\big)$$

if and only if:

(i) $p = q$ and

(ii) when p is in the relative interior of the face

$$F(p) = F_{j_1} \cap F_{j_2} \cap \cdots F_{j_k}$$

given as the intersection of the k facets which are complementary to $\{F_{i_1}, F_{i_2}, \ldots, F_{i_{m-k}}\}$, then

$$\pi_{i_1,i_2,\ldots,i_{m-k}}([x_1, x_2, \ldots, x_m]_\lambda) = \pi_{i_1,i_2,\ldots,i_{m-k}}([y_1, y_2, \ldots, y_m]_\lambda).$$

Equivalence classes of points in

$$\big(X_1 \times X_2 \times \cdots \times X_m/\ker\lambda\big) \times P^n/{\sim_2}$$

are denoted by the symbol

$$\big[\big([x_1, x_2, \ldots, x_m]_\lambda, p\big)\big]_2$$

The group $\ker\lambda$ acts on the space

$$(X_1 \times X_2 \times \cdots \times X_m) \times P^n/{\sim_1}$$

by

$$t \cdot [(x_1, x_2, \ldots, x_m, p)]_1 \;=\; [t \cdot (x_1, x_2, \ldots, x_m), p)]_1.$$

Property (ii) in the yellow construction ensures that the action is well defined.

The next homeomorphism makes things worthwhile.

$$\big(X_1{\times}X_2{\times}\cdots{\times}X_m/\ker\lambda\big){\times}P^n/{\sim_2} \xrightarrow{\cong} \big((X_1{\times}X_2{\times}\cdots{\times}X_m){\times}P^n/{\sim_1}\big)/\ker\lambda$$

$$\big[([x_1, x_2, \ldots, x_m]_\lambda, p)\big]_2 \;\mapsto\; \big[[(x_1, x_2, \ldots, x_m, p)]_1\big]_\lambda.$$

MORE IS TRUE

The regularity condition on the characteristic map λ ensures that

$$\ker\lambda \;\cong\; T^{m-n}$$

acts freely on $Z\Big(K_P; (\underline{CX_i},\ \underline{X_i})\Big)$.

There is a commutative diagram:

$$\begin{array}{ccc}
X_1 \times X_2 \times \cdots \times X_m \times P^n/{\sim_1} & \longrightarrow & \big(X_1 \times X_2 \times \cdots \times X_m/\ker\lambda\big) \times P^n/{\sim_2} \\
\downarrow\cong & & \cong\downarrow \\
Z\big(K_P; (\underline{CX},\ \underline{X})\big) & \longrightarrow & Z\big(K_P; (\underline{CX},\ \underline{X})\big)/\ker\lambda
\end{array}$$

$\boxed{\text{ASIDE}}$

If we consider the case

$$X_i = Z\big(K_{P_i}; (D^2, S^1)\big)$$

with S^1 acting by the free diagonal action, we very quickly enter the realm of the Ayzenberg's iterated polyhedral product constructions.

(That's for another course.)

ANOTHER REALITY CHECK

Our first example, reinterprets the basic construction. Let λ be a characteristic map for a simple polytope P^n.

Set $X_i = S^1$ for each $i = 1, 2, \ldots, m$.

Let S^1 act on itself in the usual way.

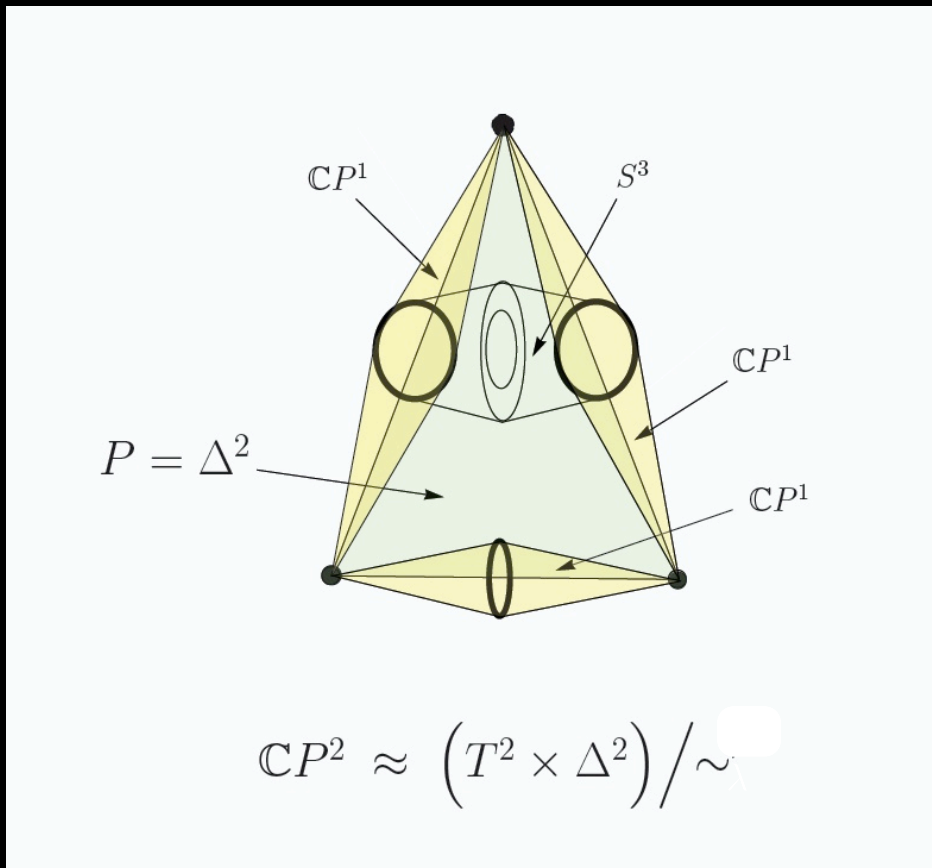
This reformulation sees the Davis-Januszkiewicz construction as

$$T^n \times P^n/\sim_\lambda = M^{2n} \cong (T^m/\mathrm{ker}\lambda) \times P^n \Big/ \sim_2$$

Notice here that the equivalence relation $\sim_2$ does not involve the map λ directly. Moreover, we have

$$(T^m/\mathrm{ker}\lambda) \times P^n \Big/ \sim_2 \quad \cong \quad Z(K_P; (D^2, S^1)) \Big/ \mathrm{ker}\lambda.$$

RECALL $\mathbb{C}P^2$



Here, $P^n = \Delta^2$ the two-simplex

The diagram presents $\mathbb{C}P^2$ as the cone on S^3 attached to $\mathbb{C}P^1$ via the Hopf map

$$S^3 \longrightarrow S^2$$

by "collapsing" by the action of the diagonal circle

The ingredients are:

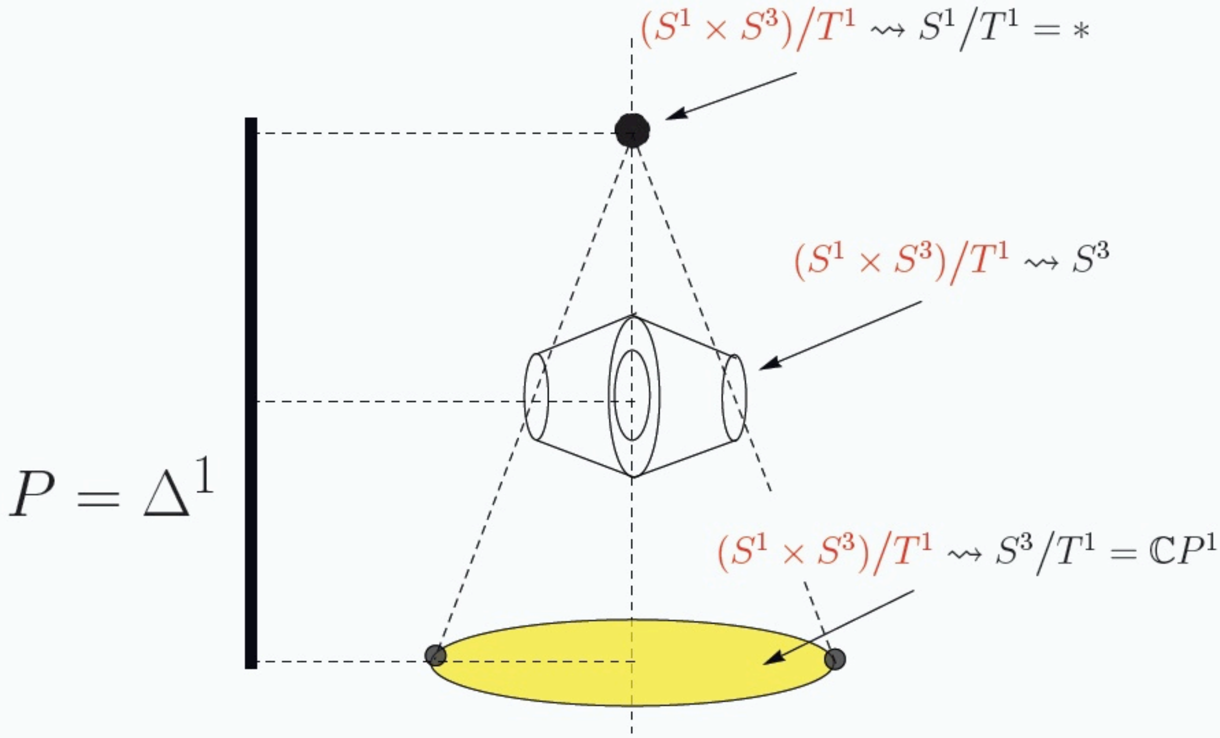
(i) $P^n = \Delta^1$ a one-simplex. Here $n = 1$ and $m = 2$.

(ii) $X_1 = S^1$ and $X_2 = S^3$ with the usual free S^1 action.

(iii) the characteristic map $\lambda\colon \mathbb{Z}^2 \longrightarrow \mathbb{Z}$ is given by the matrix $[1, -1]$.

(iv) $\ker\lambda \cong T^1$ sits inside T^2 as $t \mapsto (t, t^{-1})$.

$\mathbb{C}P^2$ FROM THE NEW POINT OF VIEW



$$\mathbb{C}P^2 \approx \Big((S^1 \times S^3)/T^1 \times \Delta^1 \Big) \Big/ \sim_2$$