Symplectic Diffeology

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Workshop on Hamiltonian Geometry and Quantization
July 15 - 19, 2024, The Fields Institute, Toronto


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Motivation

Why Symplectic "Diffeology"?

- Symplectic Geometry deals with symplectic (or presymplectic) manifolds.
- Last decades a lot of examples of "symplectic" structures has been built on spaces which are not manifolds: infinite dimension spaces, orbifolds, singular reduction spaces a.s.o.
- We have also to deal with actions of groups that are not Lie groups, because they are infinite dimensional, even acting on manifolds, or not Hausdorff.
- Symplectic reduction introduces quotient spaces that are not manifolds but carry parasymplectic structures.

➠ We'll see how diffeology is a uniformizing framework.

A Few Questions

Action of $C^\infty(M, \mathbf{R})$ on T^*M

- M is a manifold, $q \in M$.
- T^*M is the symplectic cotangent space with:

$$\omega = dp \wedge dq,$$

with $(q, p) \in T^*M$.

- $f \in C^\infty(M, \mathbf{R})$, Abelian group, acts on T^*M by:

$$\underline{f}(q, p) = (q, p + df(q)).$$

Question : Moment Map?

<u>Action of $C^\infty(\Sigma, \mathbf{R})$, $\mathrm{Diff}^+(\Sigma)$, $\Omega^1(\Sigma)$</u>

Σ be a oriented surface and $\Omega^1(\Sigma)$ be the space of 1-forms on Σ. Let $\alpha, \beta \in \Omega^1(\Sigma)$ and $\omega(\alpha, \beta) = \int_\Sigma \alpha \wedge \beta$. Three actions preserve ω, for all $\alpha \in \Omega^1(\Sigma)$:

1. Abelian group: $f \in C^\infty(\Sigma, \mathbf{R})$ acts on $\Omega^1(\Sigma)$: $\underline{f}(\alpha) = \alpha + df$.
2. Diffeo. group $\phi \in \mathrm{Diff}^+(\Sigma)$ acts by: $\underline{\phi}(\alpha) = \phi_*(\alpha)$.
3. Group $\Omega^1(\Sigma)$ acts on iself by: $\beta \in \Omega^1(\Sigma)$, $\underline{\beta}(\alpha) = \alpha + \beta$.

<u>Question</u> : In what sense $\Omega^1(\Sigma)$ is a symplectic space?

<u>Question</u> : Moment Maps?

Moment Map for the Torus

- T^2 is the 2-torus $\mathbf{R}^2/\mathbf{Z}^2$.
- $\omega \in \Omega^2(T^2)$ is the pushforward of $dx \wedge dy$.
- T^2 is a group and act on itself preserving ω, action:

$$\tau, z \in T^2 \quad \underline{\tau}(z) = z + \tau.$$

Question : Moment Map for non-Hamiltonian action?
Question : Generalization to $\mathbf{R}^n/K$, K discrete subgroup?

2-Forms on Orbifolds

- An orbifold is locally a quotient $\mathbf{R}^n/\Gamma$, $\Gamma \subset \mathrm{GL}(n)$ finite.
- The Cone Orbifold $\mathcal{Q}_m = \mathbf{C}/U_m$, U_m m-roots of unity.
- The Corner Orbifold $\mathcal{Q} = [\mathbf{R}/\pm 1]^2$.

Question : Characterization of $\Omega^2(Q_m)$? $\Omega^2(Q)$?

Question : Group actions, Moment maps?

Spaces of geodesic trajectories

- (M, g) is a Riemannian manifold and TM is equipped with the symplectic form

$$\omega = d\epsilon, \text{ with } \epsilon(\delta y) = g(v, \delta x), \text{ for all } y = (x, v) \in TM.$$

- The space of geodesic trajectories is the reduction of the unit tangent bundle

$$UM = \{y \in TM \mid g(v, v) = 1\}$$

by the geodesic flow (the flow of the Reeb vector field).

Question : When $\mathrm{Geod}(M, g)$ is a manifold then the reduction of $\omega \upharpoonright UM$ is symplectic (for example S^2). But, What about the reduction in general? For example with $M = T^2$.

The Infinite Sphere and its Quotient

- $\mathcal{H} = \{Z = (Z_i)_{i=1}^{\infty} \mid \sum_{i=1}^{\infty} \bar{Z}_i \cdot Z_i < \infty\}$, Hermitian product.
- $S^\infty = \{Z \in \mathcal{H} \mid \bar{Z} \cdot Z = 1\}$.
- $\alpha = \frac{1}{2i}[\bar{Z} \cdot dZ - d\bar{Z} \cdot Z]$ Fubini-Study primitive.
- $\mathbf{C}P^\infty = S^\infty / S^1$, the infinite projective space.

Question : Reduction of $d\alpha \restriction S^\infty$ on $\mathbf{C}P^\infty$?

Question : Moment map of $U(\mathcal{H})$? And of S^1?

Question : in what sense is $\mathbf{C}P^\infty$ symplectic?

Immersing S^1 in $\mathbf{R}^2$

- $\mathrm{Imm}(S^1, \mathbf{R}^2) = \{x \in C^\infty(S^1, \mathbf{R}^2) \mid \dot{x}(t) \neq 0\}$.
- $\alpha \in \Omega^1(\mathrm{Imm}(S^1, \mathbf{R}^2))$

$$\alpha(\delta x) = \int_0^{2\pi} \frac{1}{\|\dot{x}(t)\|^2} \langle \ddot{x}(t) | \delta \dot{x}(t) \rangle \, dt.$$

- Group $\mathrm{Diff}(S^1)$ acts on $\mathrm{Imm}(S^1, \mathbf{R}^2)$: $\varphi \in \mathrm{Diff}(S^1)$, $x \in \mathrm{Imm}(S^1, \mathbf{R}^2)$, $\underline{\varphi}(x) = x \circ \varphi^{-1}$.
- α not invariant but $d\alpha$ is: $\underline{\varphi}^*(d\alpha) = d\alpha$.

Question : Moment map of $\mathrm{Diff}(S^1)$? Equivariance?

Question : Souriau's cocycle θ?

Question : "Symplectic" reduction?

Reducing Complex Periodic Functions by a Real flow

- $C^\infty(S^1, \mathbf{C}) \simeq \{f \in C^\infty(\mathbf{R}, \mathbf{C}) \mid f(x+1) = f(x)\}$
- Equivalent to $\mathcal{E} = \{(f_n)_{n\in\mathbf{Z}} \mid f_n \downarrow 0\}$, rapidly decreasing.
- $\epsilon \in \Omega^1(C^\infty(S^1, \mathbf{C}))$, with $\hat{x}(f) = f(x)$:

$$\epsilon(\delta f) = \int_0^1 \bar{f}(x)\delta f(x)\,dx, \quad d\epsilon = \frac{1}{\pi}\int_0^1 \hat{x}^*(\text{surf})\,dx.$$

- Group $\mathbf{R}$ acts on $\mathcal{E}$: $\underline{t}\big((f_n)_{n\in\mathbf{Z}}\big) = \big(e^{2i\pi\alpha_n t} f_n\big)_{n\in\mathbf{Z}}$, with α_n independant on $\mathbf{Q}$.

Question : Orbits of $\mathbf{R}$?

Question : Moment map h of $\mathbf{R}$?

Question : Reduction of $S^\infty_\alpha = h^{-1}(1)$? Quasiprojective space.

Etc.

Diffeology Framework*

*Diffeology, Beijing WPC & Am. Math. Soc. (Rev. 2022)
https://eastred.jp/ja-285869-9787519296087

Category Diffeology I

A diffeology on a set X is given by a choice $\mathcal{D}$ of Euclidean parametrizations (maps from Euclidean domains into X) called plots, satisfying 3 axioms.

- The plots cover X.
- To be a plot is local.
- The composite of a plot by a smooth parametrization of its domain is a plot.

A set X equipped with a diffeology $\mathcal{D}$ is a diffeological space.

- A map $f\colon X \to X'$, where X and X′ are two diffeological spaces, is said to be smooth if the composite of f with a plot of X is a plot of X′.

Category Diffeology II

Diffeological spaces together with smooth maps define the category {Diffeology}. Isomorphisms are called diffeomorphisms. This category is stable for the set theoretic operations:

- Sums: $\coprod_i X_i$.
- Products: $\prod_i X_i$.
- Quotients: $\mathcal{Q} = X/\sim$.
- Subsets: $A \subset X$.

It is a complete and co-complete category.

The set $C^\infty(X, X')$ has a natural functional diffeology which makes {Diffeology} Cartesian closed.

Differential Forms on Diffeological Spaces

- A differential k-form α on a diffeological space X associates with each plot P in X, a smooth form

$$\alpha(\mathrm{P}) \in \Omega^{\mathrm{k}}(\mathrm{dom}(\mathrm{P})),$$

such that, for all smooth parametrization F in dom(P)

$$\alpha(\mathrm{P} \circ \mathrm{F}) = \mathrm{F}^*\big(\alpha(\mathrm{P})\big).$$

- Pullback: $f\colon \mathrm{X} \to \mathrm{X}'$ smooth, and $\alpha' \in \Omega^{\mathrm{k}}(\mathrm{X}')$, then,

$$\Omega^{\mathrm{k}}(\mathrm{X}) \ni f^*(\alpha')\colon \mathrm{P} \mapsto f^*(\alpha')(\mathrm{P}) = \alpha'(f \circ \mathrm{P}).$$

Homotopic Invariance of De Rham Cohomology

Consider two diffeological spaces X and X′, and a smooth path $t \mapsto f_t$ in $C^\infty(X', X)$. let $F : X' \to \mathrm{Paths}(X)$ defined by:

$$F(x') = [t \mapsto f_t(x')].$$

We use the Chain-Homotopy operator

$$\mathcal{K} : \Omega^k(X) \to \Omega^{k-1}(\mathrm{Paths}(X)) \text{ with } \mathcal{K} \circ d + d \circ \mathcal{K} = \hat{1}^* - \hat{0}^*,$$

and $\hat{t}(\gamma) = \gamma(t)$. Let $\alpha \in \Omega^k(X)$ and $d\alpha = 0$. Apply F^*:

$$F^*(\mathcal{K}(d\alpha) + d[\mathcal{K}\alpha]) = F^*(\hat{1}^*(\alpha)) - F^*(\hat{0}^*(\alpha))$$

That gives $d[F^*(\mathcal{K}\alpha)] = f_1^*(\alpha) - f_0^*(\alpha)$, that is:

$$f_1^*(\alpha) = f_0^*(\alpha) + d\beta.$$

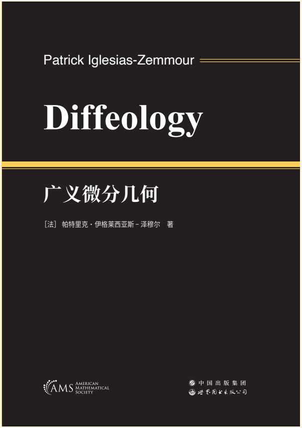
Diffeological Groups And Momenta

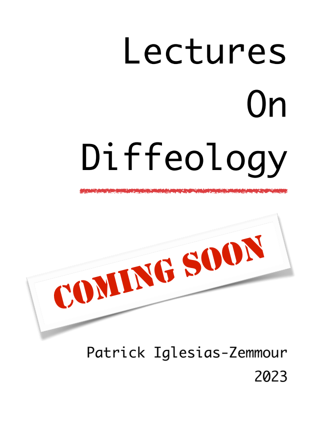
- A diffeological group is a diffeological space with a group law such that the product and the inverse are smooth.
- Every group of diffeomorphisms Diff(X) is a diffeological group, equipped with the functional diffeology.
- A momentum of a diffeological group G is a left-invariant 1-form. We introduce the vector space of momenta:

$$\mathcal{G}^* = \{\alpha \in \Omega^1(G) \mid \forall g \in G, L(g)^*(\alpha) = \alpha\}.$$

With $L(g)\colon g' \mapsto gg'$.

➠ Note that the space of momenta $\mathcal{G}^*$ is not defined by duality with a presumed Lie algebra.





https://www.books.com.tw/products/CN11808406?sloc=main

https://eastred.jp/ja-285869-9787519296087

http://diffeology.net

Symplectic Diffeology

Symplectic geometry applied to mechanics has revealed two fundamental constructions:

- The group of automorphisms, the subgroup of Hamiltonian diffeomorphisms, Hamiltonian vector fields, one-parameter group of Hamiltonian diffeomorphisms, and so on.
- The moment map of groups of automorphisms, that capture a part (or the whole) of the geometry of the symplectic manifold.

These objects have a natural extension in diffeology, and we can build symplectic diffeology around these two constructions.*

*I leave isotropic and co-isotropic subspaces for further investigations.

Moment Map: the **Simplest Case**

- Let us call parasymplectic form on a diffeological space X, any closed 2-form: $\omega \in \Omega^2(X)$ and $d\omega = 0$.
- Let G be a diffeological group acting smoothly on X and preserving ω: $\underline{g}^*(\omega) = \omega$, for all $g \in G$.

➠ Now assume that $\omega = d\alpha$ and $\underline{g}^*\alpha = \alpha$. Let $\hat{x}\colon g \mapsto \underline{g}(x)$ be the orbit map.

- The map $\mu\colon x \mapsto \hat{x}^*(\alpha)$, defined on X is smooth and takes its values in $\mathcal{G}^*$. This is the moment map of ω.

➠ Actually it is **a** moment map. The moment map associated with the primitive α.

Moment Map: the **General Case**

- There exists a chain-homotopy operator

$$\mathcal{K}\colon \Omega^{k}(X) \to \Omega^{k-1}(\mathrm{Paths}(X)),$$

  such that

$$d \circ \mathcal{K} + \mathcal{K} \circ d = \hat{1}^{*} - \hat{0}^{*},$$

  where $\mathrm{Paths}(X) = C^{\infty}(\mathbf{R}, X)$, $\hat{t}(\gamma) = \gamma(t)$, for $t \in \mathbf{R}$.
- Let (X, ω) be a parasymplectic space:

$$\varpi = d\lambda, \text{ with } \lambda = \mathcal{K}\omega \text{ and } \varpi = \hat{1}^{*}(\omega) - \hat{0}^{*}(\omega).$$

- If G preserves ω, then G preserve $\lambda = \mathcal{K}\omega$.
- Hence, we are brought back to the simplest case $\underline{g}^{*}(\lambda) = \lambda$.

The Paths Moment Map

- The paths moment map is defined by:

$$\Psi\colon \mathrm{Paths}(X) \to \mathcal{G}^* \quad \text{with} \quad \Psi(\gamma) = \hat{\gamma}^*(\mathcal{K}\omega).$$

- It is G-equivariant for the coadjoint action. Let $g, k \in G$ and $\alpha \in \mathcal{G}^*$.

$$\mathrm{ad}(g)(k) = gkg^{-1}, \quad \mathrm{Ad}_*(g)(\alpha) = \mathrm{ad}(g)_*(\alpha).$$

  Then

$$\Psi \circ g_* = \mathrm{Ad}_*(g) \circ \Psi \quad \text{with} \quad g_*(\gamma) = \underline{g} \circ \gamma.$$

- It is additive. For γ and γ' juxtaposable:

$$\Psi(\gamma \vee \gamma') = \Psi(\gamma) + \Psi(\gamma'),$$

The Two-Points Moment Map

- The two-points moment map, projection on $X \times X$ of the paths moment map, is defined by:

$$\psi \colon X \times X \to \mathcal{G}^*/\Gamma, \text{ with } \psi(x, x') = \Psi(\gamma),$$

with $x = \gamma(0)$, $x' = \gamma(1)$.

- $\Gamma \subset \mathcal{G}^*$ is made of Ad_*-invariant momenta. It it the Holonomy of the action of G on (X, ω), the obstruction of the action of G for being Hamiltonian.

$$\Gamma = \{\Psi(\ell) \mid \ell \in \mathrm{Loops}(X)\}.$$

- ψ is still G-equivariant and it is a Chasles cocycle:

$$\psi(x, x') + \psi(x', x'') = \psi(x, x'').$$

The One-Point Moment Map

- A one-point moment map is a solution μ of

$$\psi(x, x') = \mu(x') - \mu(x) \text{ with } \mu\colon X \to \mathcal{G}^*/\Gamma.$$

  That is:

$$\mu(x) = \psi(x_0, x) + c, \text{ where } x_0 \in X \text{ and } c \in \mathcal{G}^*/\Gamma.$$

- The moment map μ is θ-affine Ad_*-equivariant:

$$\mu(\underline{g}(x)) = \mathrm{Ad}_*(\mu(x)) + \theta(g), \text{ with}$$
$$\theta(g) = \psi(x_0, \underline{g}(x_0)) - \Delta(c)(g),$$

  $\theta \in H^1(G, \mathcal{G}^*/\Gamma)$, and $\Delta(c)(g)$ is the coboundary $\mathrm{Ad}_*(g)(c) - c$.

From Parasymplectic to Presymplectic – I

A parasymplectic form ω on a manifold M is said to be presymplectic if $\ker(\omega)$ has constant dimension.

- A presymplectic manifold (M, ω) satisfies the passive version of the Darboux theorem : the existence of Darboux charts where ω takes the form

$$[\omega] = \begin{pmatrix} 0_k & 1_k & 0_{k,r} \\ -1_k & 0_k & 0_{k,r} \\ 0_{r,k} & 0_{r,k} & 0_r \end{pmatrix}.$$

- Alternatively, we have the active versions of the Darboux theorem : $\mathrm{Diff}_{\mathrm{loc}}(M, \omega)$ is transitive on M.

The Characteristics of Presymplectic Manifolds

The characteristics of a presymplectic manifold (M, ω) are the integral submanifolds of the linear distribution

$$x \mapsto \ker(\omega).$$

Theorem. *The characteristics of a homogeneous presymplectic manifold* (M, ω), *under the group* G_ω *of automorphisms, are the pre-images of the universal moment map* μ_ω.

This suggests what should be the characteristics of a presymplectic diffeological space. But the condition of homogeneity is too strong and should be weakened, maybe with the introduction of a local moment map (a work in progress).

From Parasymplectic to Presymplectic – II

Diffeology is not very friendly to contravariant objects, and the kernel of a form is contravariant. That's why we define:

Definition. *A parasymplectic diffeological space* (X, ω) *will be said presymplectic if its pseudo-group of local automorphisms* $\mathrm{Diff}_{\mathrm{loc}}(\mathrm{X}, \omega)$ *is transitive. We shall call that condition the Darboux property.*

There is a stronger variant: that X is homogeneous under the group of automorphisms $\mathrm{Diff}(\mathrm{X}, \omega)$.

From Presymplectic to Symplectic

Consider a presymplectic manifold (M, ω).

Theorem. *The presymplectic form* ω *is symplectic if and only if the universal moment map* μ_ω *of the group* $\mathrm{Diff}(M, \omega)$ *is injective.*

Local transitivity and injectivity are necessary as show the example on $\mathbf{R}^2$,

$$\omega = (x^2 + y^2)dx \wedge dy$$

Indeed: The universal moment is injective but $\mathrm{Diff}_{\mathrm{loc}}(M, \omega)$ is not transitive, $(0, 0)$ is fixed.

From Parasymplectic to Symplectic

To sum up, we have a completely geometric characterization of symplectic manifolds:

Theorem. *Let* ω *be a closed 2-form on a manifold* M

$$\omega \in \Omega^2(M) \quad \textit{and} \quad d\omega = 0.$$

The form ω *is symplectic, i.e.* $\ker(\omega) = 0$, *if and only if:*

1. *The local automorphisms* $\mathrm{Diff}_{\mathrm{loc}}(M, \omega)$ *are transitive on* M.
2. *The universal moment map* $\mu_\omega : M \to \mathcal{G}^*_\omega$ *is injective.*

This theorem is the basis of the definition of a symplectic diffeological space.

Symplectic Diffeological Spaces

Consider a parasymplectic diffeological space (X, ω).

Definition. *We shall say that the form* ω *is symplectic if and only if the two conditions are satisfied:*

- *The pseudo-group* $\mathrm{Diff}_{\mathrm{loc}}(X, \omega)$ *is transitive.*
- *The universal moment map* μ_ω *of the group* $\mathrm{Diff}(X, \omega)$ *is injective or at most with diffeologically discrete preimages.*

The pair (X, ω) is then said to be a Symplectic diffeological space.

NOTE. This is a *holistic* extension, based on a structural property of symplectic form on manifold, to avoid the use of kernel distribution. It can be criticized of course.

Symplectically Generated Spaces

According to the previous definition, the cone orbifold $\mathcal{Q}_m$ is not symplectic because its origin is fixed by local automorphisms. This contradicts the heuristic use of the expression "symplectic orbifold" found in the literature.

That is why we need more vocabulary:

Definition. *A parasymplectic space* (X, ω) *is said to be symplectically generated if there exists a generating family* $\mathcal{F}$ *for which* $\omega(F)$ *is symplectic for all* $F \in \mathcal{F}$.

☞ This raises a discussion, and new questions about the relationship between symplectic and symplectically generated, of course. . .

Parasymplectic Orbifolds

Parasymplectic orbifolds are generally not symplectic because of their singularities which are, by definition, the singular orbits of local diffeomorphisms (strata of the Klein stratification).

- The cone orbifold $\mathcal{Q}_m = \mathbf{C}/U_m$ is symplectically generated. It is generated by the projection $\pi : z \mapsto z^m$ for which $\underline{\omega(\pi) = dx \wedge dy}$, with $z = x + iy$.
- The corner orbifold $\mathcal{Q} = [\mathbf{R}/\{\pm 1\}]^2$, which is generated by $\pi : (x, y) \mapsto (x^2, y^2)$, is not symplectically generated. Indeed, $\Omega^2(\mathcal{Q})$ is a module over $C^\infty(\mathcal{Q}, \mathbf{R})$, with generator $\underline{\omega(\pi) = 4xy\, dx \wedge dy}$, which is not symplectic.

Examples of Moment Maps

Action of $C^\infty(M, \mathbf{R})$ on T^*M

- The group $C^\infty(M, \mathbf{R})$ acts on T^*M by $\underline{f}(q, p) = (q, p + df(q))$, $f \in C^\infty(M, \mathbf{R})$. It preserves the 2-form $\omega = dp \wedge dq$.
- The moment map is

$$\mu\colon (q, p) \mapsto d[f \mapsto f(q)].$$

  Note: $[f \mapsto f(q)] \in C^\infty(C^\infty(M, \mathbf{R}), \mathbf{R})$ is not invariant, but its differential is an invariant 1-form on $C^\infty(M, \mathbf{R})$.
- Also $\mu(q, p) = d\delta_q$, where δ_q is the Dirac function. The moment map is the differential of a distribution.

Action of $C^\infty(\Sigma, \mathbf{R})$ on $\Omega^1(\Sigma)$

- The group $C^\infty(\Sigma, \mathbf{R})$ acts on $\Omega^1(\Sigma)$, preserving the 2-form $\omega(\alpha, \beta) = \int_\Sigma \alpha \wedge \beta$.
- For all $f \in C^\infty(\Sigma, \mathbf{R})$, $\alpha \in \Omega^1(\Sigma)$, $\underline{f}(\alpha) = \alpha + df$.
- The moment map is:

$$\mu\colon \alpha \mapsto d\left[f \mapsto \int_\Sigma f\, d\alpha\right].$$

- Again, the moment map is the differential of a distribution: $[d\alpha] : f \mapsto \int_\Sigma f\, d\alpha$. Heuristically, people used to think of $d\alpha$ as the curvature, but here it assumes its true nature.

Action of $\Omega^1(\Sigma)$ on itself

- The group $\Omega^1(\Sigma)$ acts additively on itself, preserving the 2-form $\omega(\alpha, \beta) = \int_\Sigma \alpha \wedge \beta$.
- The moment map is:

$$\mu\colon \alpha \mapsto d\left[\beta \mapsto \int_\Sigma \alpha \wedge \beta\right].$$

- Here again, the moment map is the differential of a distribution.
- The space $\Omega^1(\Sigma)$ is symplectic in the sense above.
    1. The group of automorphisms is transitive.
    2. The moment map μ is injective.

Immersing S^1 in $\mathbf{R}^2$

- We deal with $\omega = d\alpha$, with
$$\alpha(\delta x) = \int_0^{2\pi} \frac{1}{\|\dot{x}(t)\|^2} \langle \ddot{x}(t) | \delta \dot{x}(t) \rangle \, dt, \ \ x \in \mathrm{Imm}(S^1, \mathbf{R}^2).$$
Action of $\mathrm{Diff}^+(S^1)$ on $\mathrm{Imm}(S^1, \mathbf{R}^2)$ by $\underline{\varphi}(x) = x \circ \varphi^{-1}$.
- On the connected component of the standard immersion $t \mapsto (\cos(t), \sin(t))$, the moment map is, up to a constant:
$$\mu(x)(P)_r(\delta r) = \int_0^{2\pi} \left\{ \frac{\|x''(u)\|^2}{\|x'(u)\|^2} - \frac{d^2}{du^2} \log \|x'(u)\|^2 \right\} \delta u \, du.$$
$P \colon r \mapsto \varphi$ is a n-plot of $\mathrm{Diff}_+(S^1)$, $r \in \mathrm{dom}(P)$, $\delta r \in \mathbf{R}^n$, $u = \varphi^{-1}(t)$, where t is the parameter of $x \in \mathrm{Imm}(S^1, \mathbf{R}^2)$, and $\delta u = D(r \mapsto u)(r)(\delta r)$.

Immersing S^1 in $\mathbf{R}^2$

- The affine cocycle (lack of equivariance) of the $\mathrm{Diff}_+(S^1)$ action on $\mathrm{Imm}(S^1, \mathbf{R}^2)$ are cohomologous to θ defined by,

$$\theta(g)(P)_r(\delta r) = \int_0^{2\pi} \frac{3\gamma''(u)^2 - 2\gamma'''(u)\gamma'(u)}{\gamma'(u)^2}\, \delta u\, du,$$

  where $g \in \mathrm{Diff}^+(S^1)$ and $\gamma = g^{-1}$. The integrand of the right-hand side is the Schwarzian derivative.
- The cocycle θ of this integral construction of the moment map in diffeology, extends the Souriau's cocycle of symplectic geometry.

Symplectic Reduction

Reduction of a Contact Manifold

Let us recall that a contact form on a manifold is a differential form λ such that $\ker(\lambda) \cap \ker(d\lambda) = \{0\}$. The characteristics of $d\lambda$ are the integral curves of the Reeb vector field ξ defined uniquely by $\lambda(\xi) = 1$ and $d\lambda(\xi) = 0$.

Theorem. *Let λ be a contact form on a manifold Y. There always exists on the space $\mathcal{S}$ of charateristics of $d\lambda$ a parasymplectic form $\omega \in \Omega^2(\mathcal{S})$ such that*

$$\mathrm{class}^*(\omega) = d\lambda, \ \textit{with}\ \mathrm{class} : Y \to \mathcal{S} = Y/\ker(d\lambda).$$

When $\mathcal{S}$ is a manifold then ω is symplectic. Moreover, if Y has dimension $2n+1$ then $\mathcal{S}$ has dimension $2n$ and is symplectically generated.

Example of the Geodesics of T^2

The spaces of geodesic trajectories are parasymplectic as a particular case of the reduction of a contact manifold. For example, the geodesic trajectories of $\mathrm{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$ are the projections of the affine lines in $\mathbf{R}^2$. The space $\mathrm{Geod}(\mathrm{T}^2)$ is then the quotient of $\mathrm{S}^1 \times \mathbf{R}$ by

$$(u', \rho') \sim (u, \rho) \Leftrightarrow u' = u \text{ and } \rho' = \rho + na + mb,$$

with $u = (a, b)$, the direction vector and $(n, m) \in \mathbf{Z}^2$. It is a quasifold $\mathbf{R}^2/\mathbf{Z}^3$, fibered on S^1 with fiber T_u over u, a rational or irrational torus depending on the rationality of u:

$$\pi : \mathrm{Geod}(\mathrm{T}^2) \to \mathrm{S}^1, \text{ with } \pi^{-1}(u) = \mathrm{T}_u \text{ and } \mathrm{T}_u \simeq \mathbf{R}/(a\mathbf{Z} + b\mathbf{Z}).$$

Singular Reduction in Infinite Dimension

Example of $\mathcal{E} = \{(f_n)_{n\in\mathbf{Z}} \mid f_n \downarrow 0\}$, representing smooth complex periodics functions, with $\omega = \frac{1}{\pi}\int_0^1 \hat{x}^*(\mathrm{surf})\,dx$, and $\mathbf{R}$ acting on $\mathcal{E}$ by $\underline{t}\big((f_n)_{n\in\mathbf{Z}}\big) = \big(e^{2i\pi\alpha_n t} f_n\big)_{n\in\mathbf{Z}}$, with α_n independant on $\mathbf{Q}$.

- Moment maps: $h(f) = \mathrm{E}(f)dt$, $\mathrm{E}(f) = \sum_{n\in\mathbf{Z}} \alpha_n \|f_n\|^2 + c$.
- Let $\mathcal{S}^\infty_\alpha = \mathrm{E}^{-1}(1)$ $(c = 0)$. The singular orbits of $\mathbf{R}$ on $\mathcal{S}^\infty_\alpha$ are the harmonics S^1_k with $f_n = 0$ if $n \neq k$. They are circles of radius $1/\sqrt{\alpha_k}$. The other orbits are principal and diffeomorphic to $\mathbf{R}$.
- Call quasi-projective space the quotient $\mathbf{C}\mathrm{P}^\infty_\alpha = \mathcal{S}^\infty_\alpha/\mathbf{R}$. The form $\omega \restriction \mathcal{S}^\infty_\alpha$ passes to $\mathbf{C}\mathrm{P}^\infty_\alpha$ into a parasympectic form ϖ, despite the infinitely many singular orbits.

Toric Quasifolds as Parasymplectic Spaces

One can embed any space $\mathbf{C}^N$ into $\mathcal{E} = \{(f_n)_{n\in\mathbf{Z}} \mid f_n \downarrow 0\}$ by $(Z_1, \ldots, Z_N) \mapsto (0_\infty, Z_1, \ldots, Z_N, 0_\infty)$, and let $\mathbf{R}$ acting on $\mathbf{C}^N \subset \mathcal{E}$ by some sub-action of $\underline{t}\big((f_n)_{n\in\mathbf{Z}}\big) = \big(e^{2i\pi\alpha_n t} f_n\big)_{n\in\mathbf{Z}}$. By restriction of the infinity dimension case, one gets the constructions of Prato's quasispheres. In this sense $\mathcal{E}$ is the total classifying space for quasiprojective spaces.

This 1-dimensionsional real action can be extended to any irrational action of $\mathbf{R}^k$ on $\mathcal{E}$. This will make $\mathcal{E}$ as the total classifying space for general toric quasifolds, which are symplectically generated diffeological spaces, including toric manifolds and toric orbifolds.

Prequantization

Integration Bundles on Parasymplectic Manifolds

Let (M, ω) be a parasymplectic manifold. Define the group of periods and the torus of periods of ω by:

$$P_\omega = \left\{ \int_\sigma \omega \;\middle|\; \sigma \in H_2(M, \mathbf{Z}) \right\}, \text{ and } T_\omega = \mathbf{R}/\, P_\omega.$$

Theorem. *There exists a* T_ω*-principal bundle* $\pi : Y \to M$*, equipped with a connection form* λ *of curvature* ω*. That is:*

$$\pi^*(\omega) = d\lambda.$$

Such integration bundles are classified, up to equivalence, by $\mathrm{Ext}(H_1(M, \mathbf{Z}), P_\omega)$.

When ω is not integral, $P_\omega \neq a\mathbf{Z}$, then T_ω is an irrational torus and Y is a diffeological space but not a manifold.

Generalized Prequantum Bundles on Parasymplectic Spaces

Let (X, ω) be a simply connected parasymplectic diffeological space. Let $x \in X$ and $\hat{x}$ be the constant loop. Let P_ω and T_ω be the group and the torus of periods of $K\omega \restriction \mathrm{Loops}(X)$:

$$P_\omega = \left\{ \int_\sigma K\omega \;\middle|\; \sigma \in \mathrm{Loops}(\mathrm{Loops}(X, x), \hat{x}) \right\} \text{ and } T_\omega = \mathbf{R}/P_\omega.$$

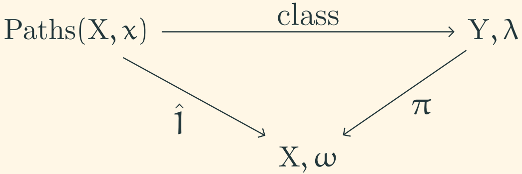
Define on $\mathrm{Paths}(X, x)$ the equivalence relation $\gamma \sim \gamma'$ if:

$$\gamma(1) = \gamma'(1) \text{ and } \int_{\hat{x}}^{\gamma \vee \bar{\gamma}'} K\omega \in P_\omega, \quad \text{with } \bar{\gamma}(t) = \gamma(1 - t).$$

Theorem. *The quotient* $Y = \mathrm{Paths}(X, x)/\sim$ *is a* T_ω*-principal bundle, for the concatenation with loops and projection* $\pi : \mathrm{class}(\gamma) \mapsto \gamma(1)$. *The* 1*-form* $K\omega$ *projects onto* Y *in a connection form* λ *of curvature* ω.

Generalized Prequantum Bundles on Parasymplectic Spaces

In conclusion: For any simply connected parasymplectic diffeological spaces* (finite or infinite dimensional, with or without singularities), there is a (unique up to equivalence in this case) integration bundle which is a quotient of a space of paths. It can also be called a generalized prequantum bundle.



*The non simply connected case is a work in progress.

Global Analysis

Hamiltonian Diffeomorphisms

Theorem. *For any (connected) parasymplectic space* (X, ω), *there exists a largest connected subgroup* $\mathrm{Ham}(X, \omega)$ *in* $G_\omega = \mathrm{Diff}(X, \omega)$ *whose holonomy is trivial. This is the group of Hamiltonian diffeomorphisms.*

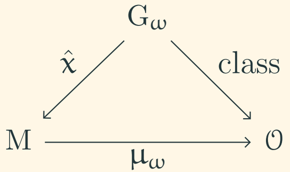
Proof. Let $\tilde{G}^\circ_\omega$ be the universal covering of the identity component of G_ω. Every element γ of the holonomy group Γ_ω is a closed 1-form on G_ω. Let $k(\gamma)$ be the real homomorphism on $\tilde{G}^\circ_\omega$ such that $\pi^*(\gamma) = d[k(\gamma)]$. Let

$$\hat{H}_\omega = \bigcap_{\gamma \in \Gamma_\omega} \ker\big(k(\gamma)\big), \text{ then } \mathrm{Ham}(X, \omega) = \pi(\hat{H}^\circ_\omega),$$

with $\pi : \tilde{G}^\circ_\omega \to G^\circ_\omega$. $\square$

Symplectic Manifolds are Coadjoint Orbits

The group $G_\omega = \mathrm{Diff}(M, \omega)$ of a symplectic manifold is transitive. The orbit map $\hat{x} : \varphi \mapsto \varphi(x)$, $x \in M$, is a principal fiber bundle with group G_ω^x, in particular a subduction.



The universal moment map μ_ω is injective (ω symplectic). Let $\mathcal{O} = G_\omega/\mathrm{Stab}(\epsilon)$, with $\epsilon = \mu_\omega(x)$ and $\mathrm{class} : G_\omega \to \mathcal{O}$.

Then, $\mu_\omega : M \to \mathcal{O}$ is an equivariant diffeomorphism for the, possibly affine, coadjoint action $\mathrm{Ad}_* + \theta_\omega$.

Example: the torus T^2 is an affine coadjoint orbit of itself.

Symplectic Manifolds are (Linear) Coadjoint Orbits

An integration bundle (Y, λ) of (M, ω) produces a central extension of the Hamiltonian diffeomorphisms:

$$\mathbf{1} \longrightarrow T_\omega \longrightarrow \mathrm{Aut}(Y, \lambda)^\circ \xrightarrow{\mathrm{pr}} \mathrm{Ham}(X, \omega) \longrightarrow \mathbf{1}$$

$\mathcal{A}^*$ and $\mathcal{H}^*_\omega$ denote the respective spaces of momenta. From here one gets the commutative diagram of moment maps:

$$\begin{array}{ccc} Y & \xrightarrow{\mu_Y} & \mathcal{A}^* \\ {\scriptstyle \pi}\downarrow & \overset{\bar{\mu}_M}{\nearrow} & \uparrow{\scriptstyle \mathrm{pr}^*} \\ M & \xrightarrow[\mu_M]{} & \mathcal{H}^*_\omega \end{array}$$

The moment map $\bar{\mu}_M$, which is the projection of the moment map μ_Y, identifies M with a linear-coadjoint (non affine) orbit of $\mathrm{Aut}(Y, \lambda)$.


That's all Folks!

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