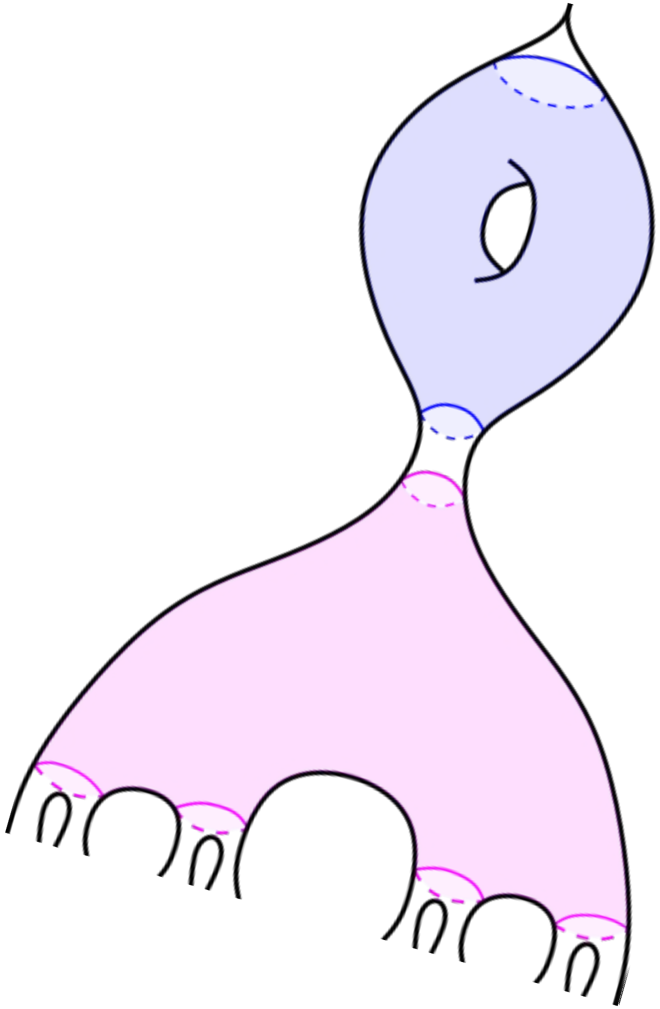
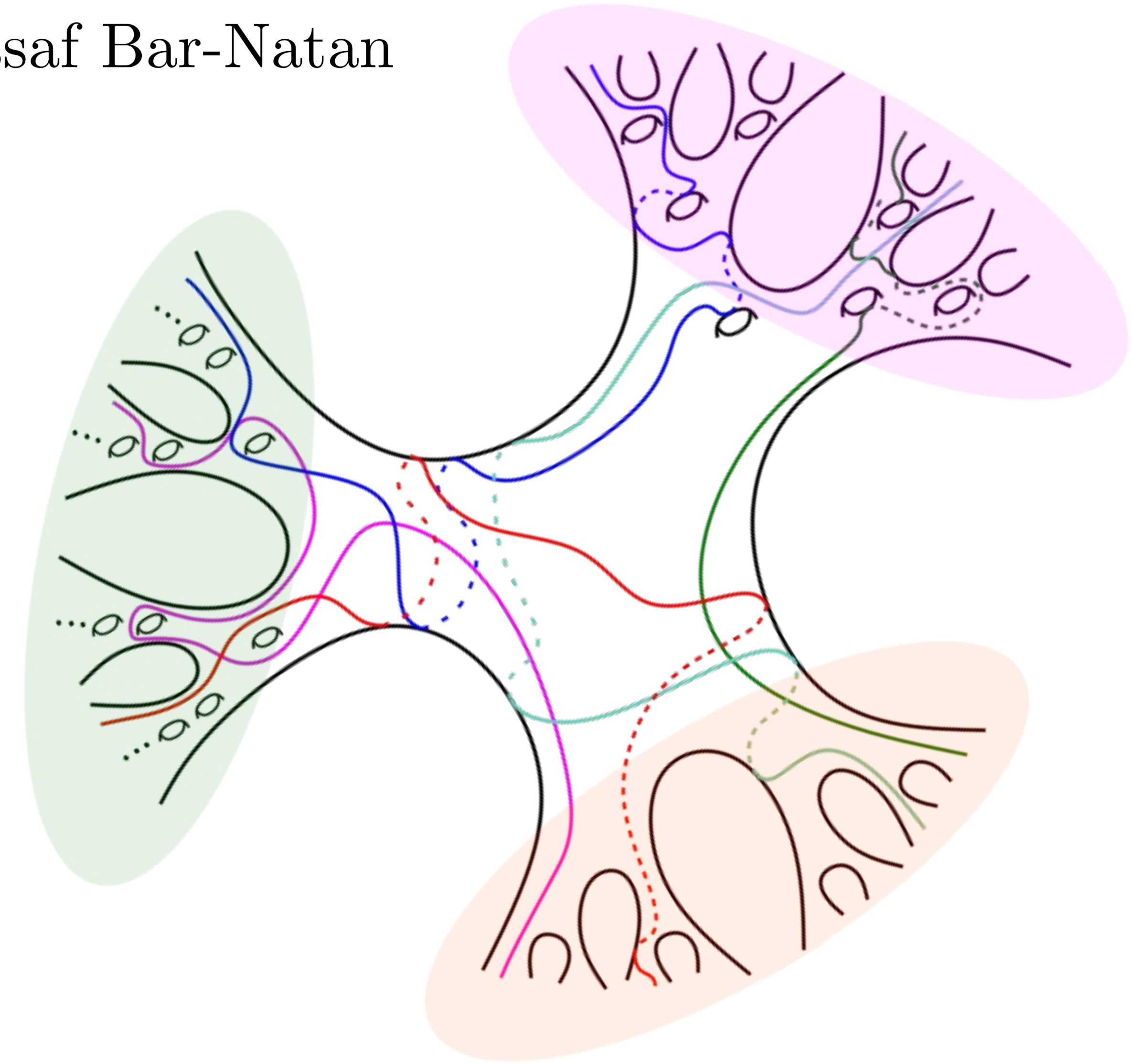
The Grand Arc Graph

Joint with Assaf Bar-Natan

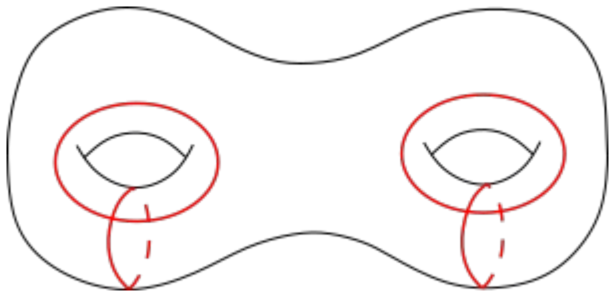




Yvon Verberne - University of Western Ontario

Mapping class groups

S is finite-type if the fundamental group is finitely generated



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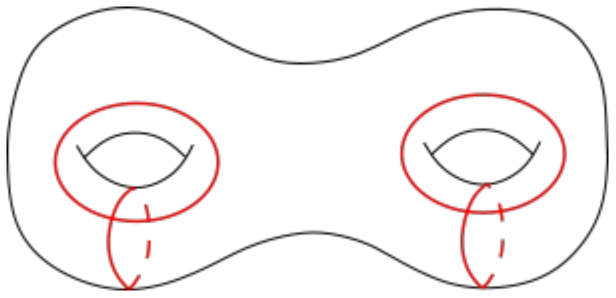
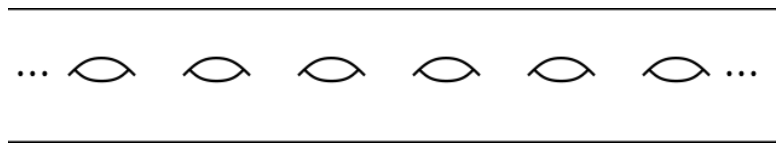


Σ is infinite-type if the fundamental group is infinitely generated



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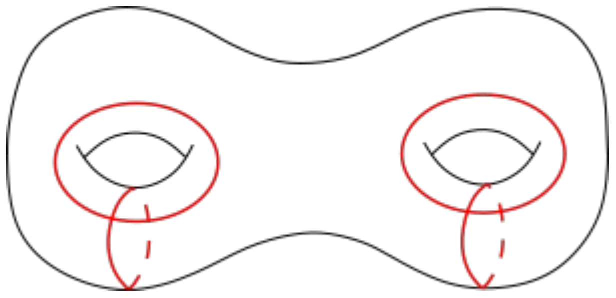

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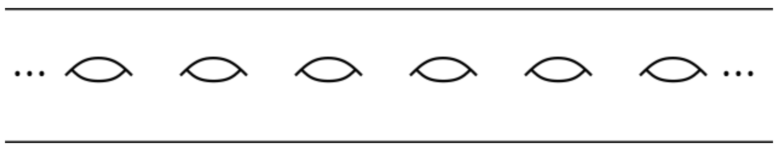

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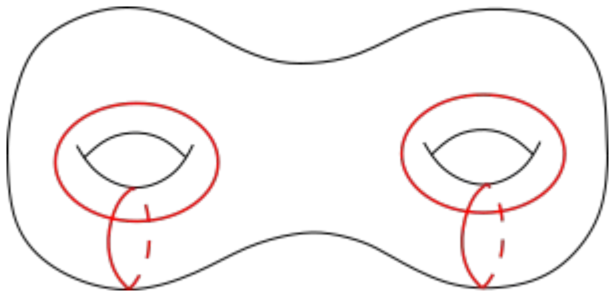

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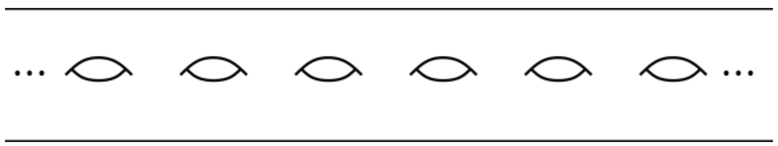

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Mapping class groups of infinite type surfaces are called big mapping class groups

Why study infinite type surfaces?

- Connections to complex dynamics

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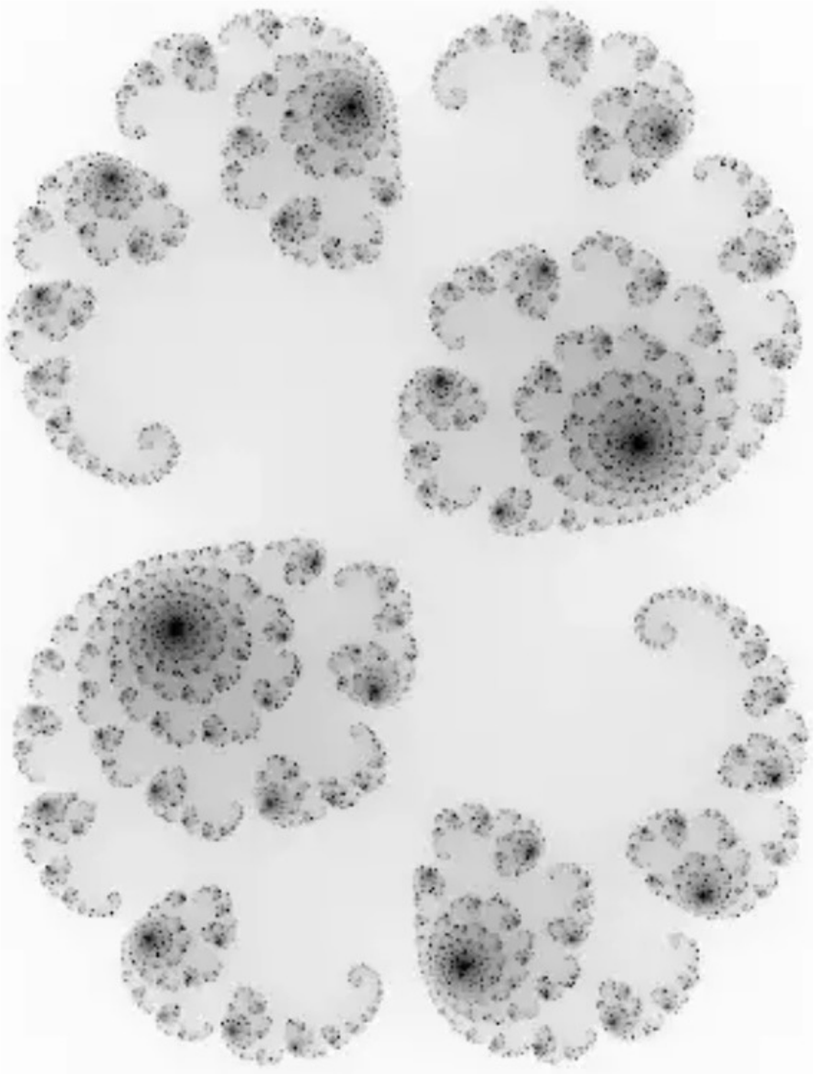

Julia set when $c = 0.285 + 0.01i$

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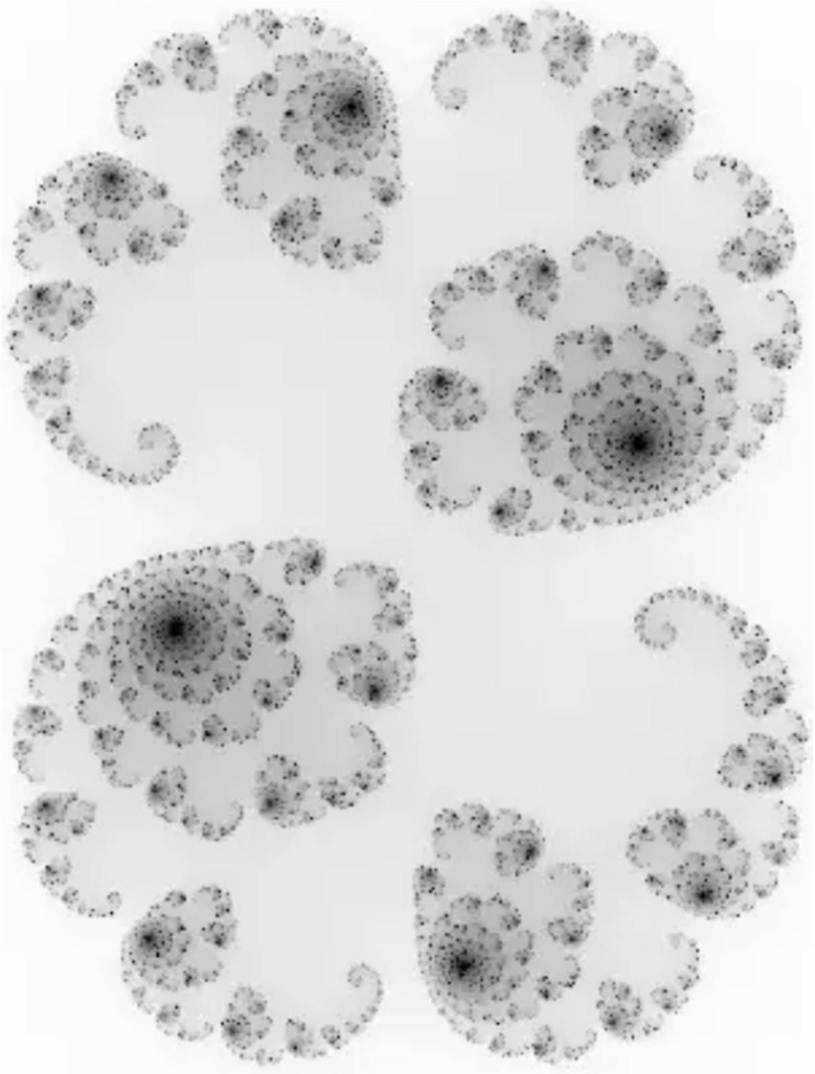
Vary the parameter $c \in \mathbb{C}$



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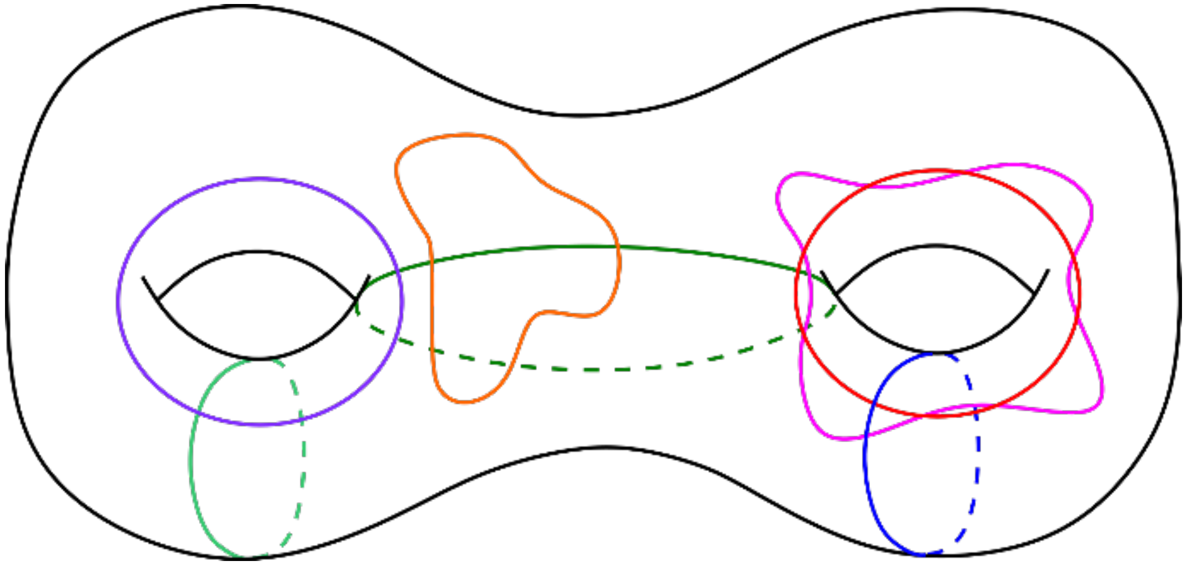
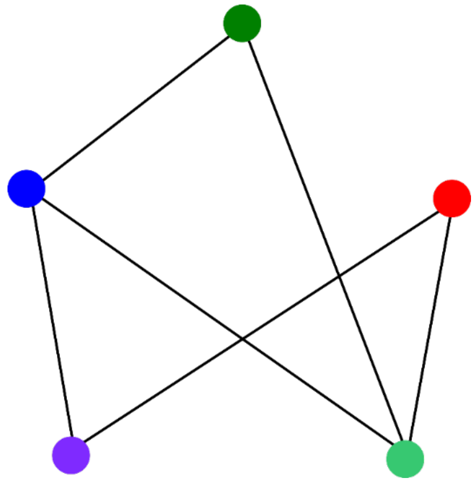
Curve Graph (Harvey)

Vertices: Homotopy classes of essential simple closed curves

Edges: Disjointness





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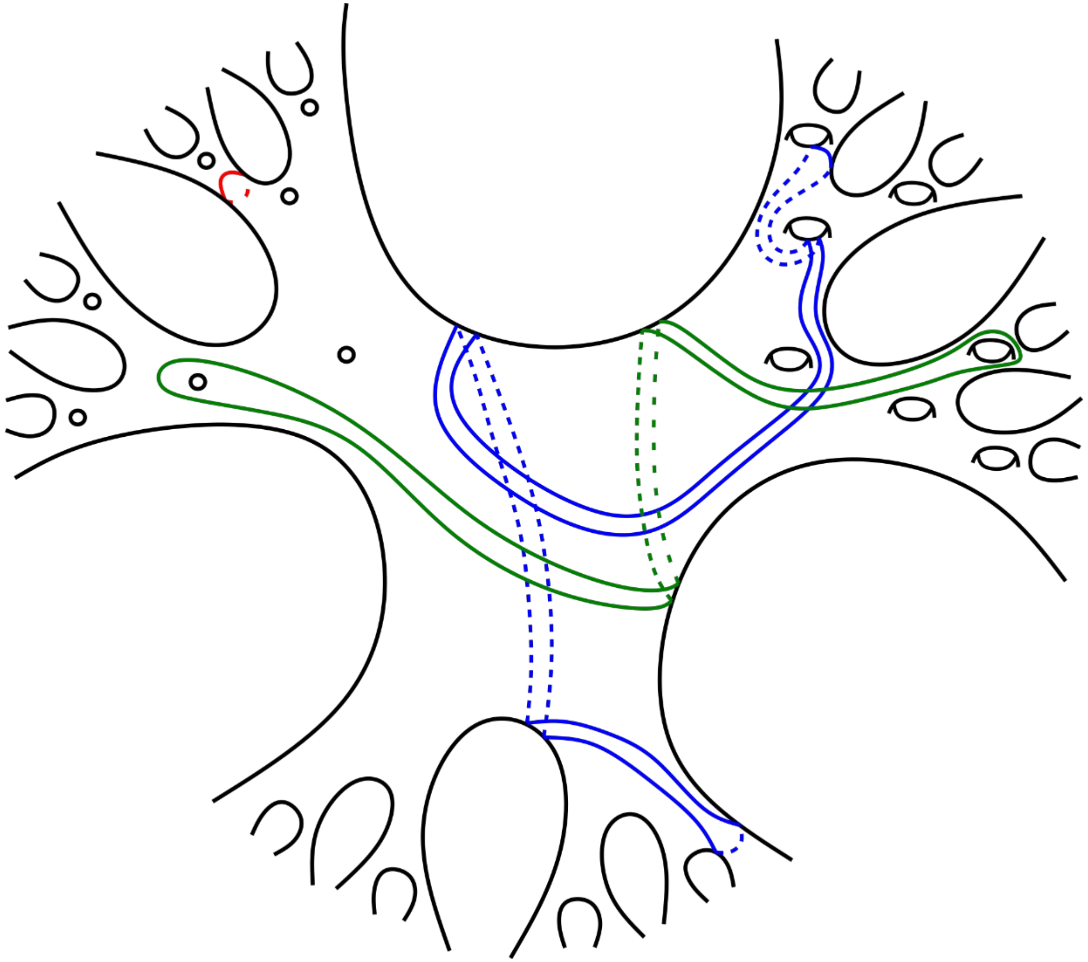

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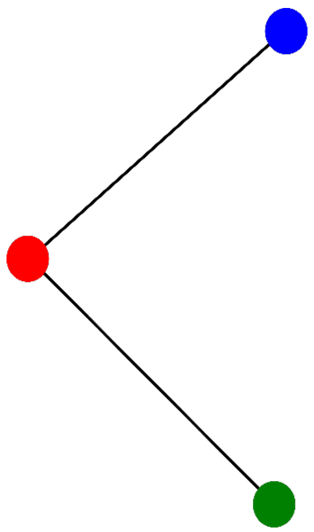
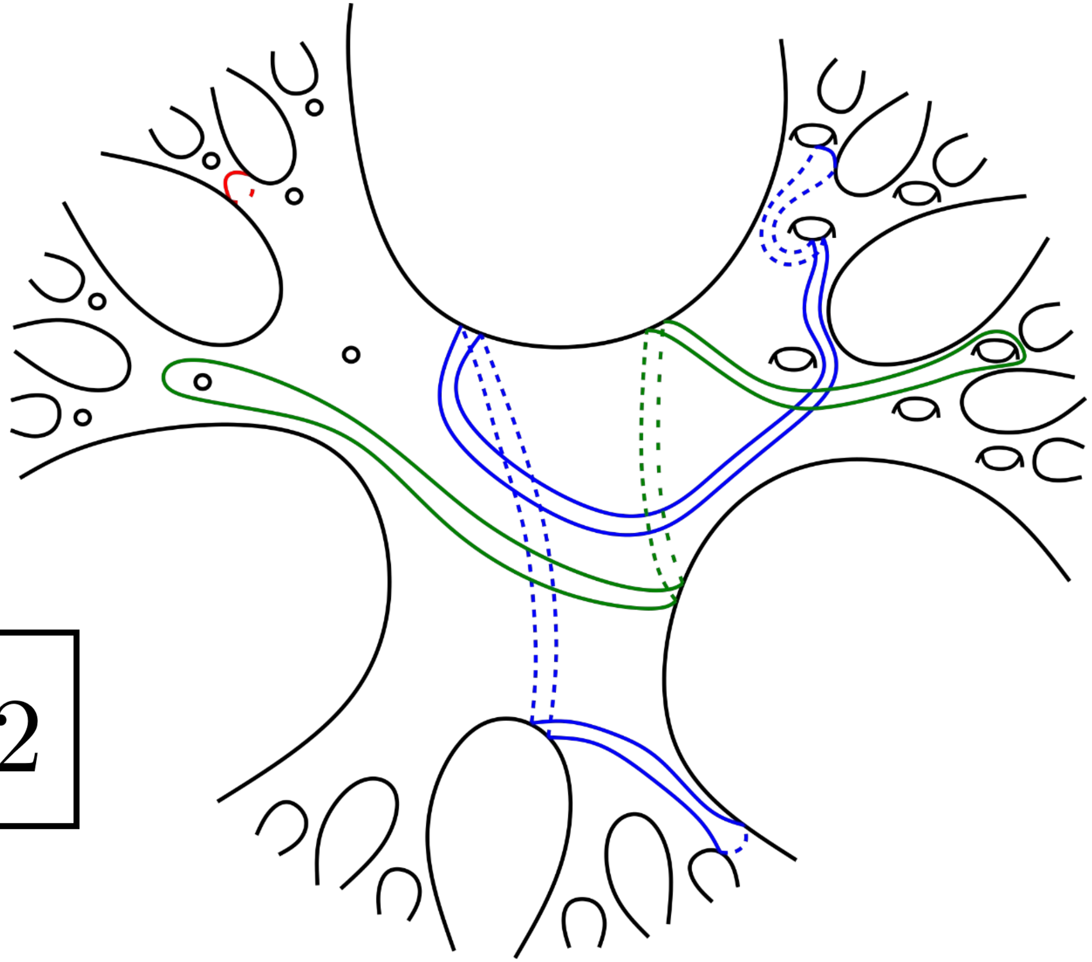




$$\mathrm{diam}(\mathcal{C}(\Sigma)) = 2$$

Question (AIM Workshop Problem 2.1):

What combinatorial objects are "good" analogues of the curve graph, either uniformly for all infinite-type surfaces or for some class of infinite-type surfaces?

Theorem (Bar-Natan – V.): For a large class of surfaces, the grand arc graph is connected, hyperbolic, has infinite diameter, and MCG(Σ) acts continuously on visible boundary.

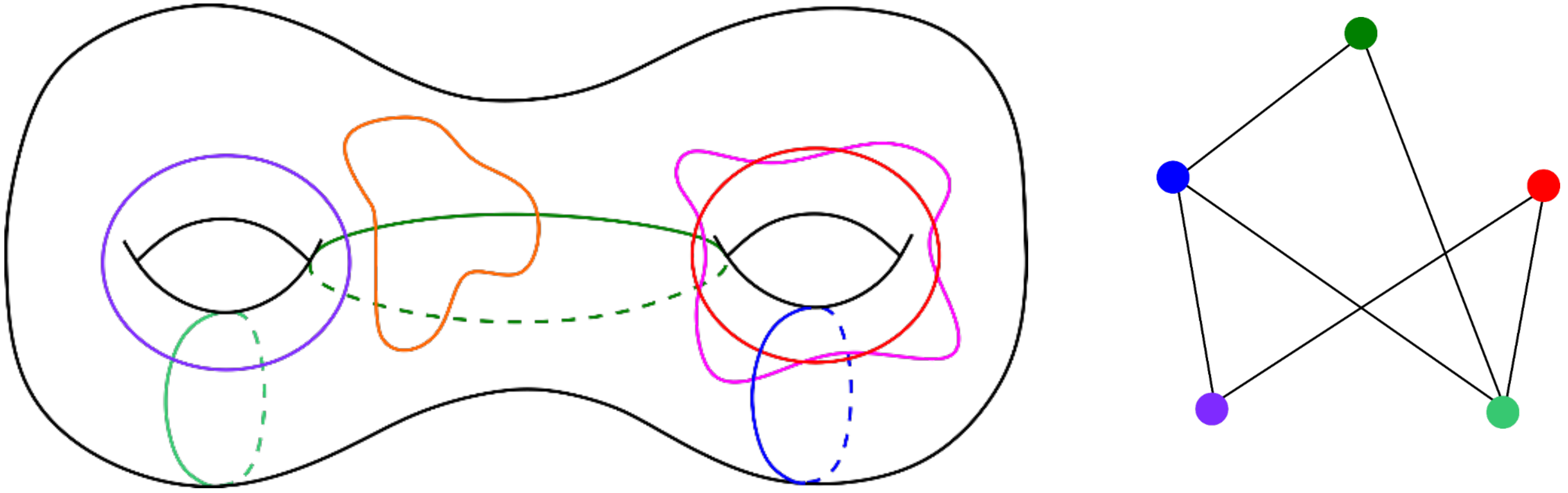
Background

Finite-Type Surfaces

Curve Graph (Harvey)

Vertices: Homotopy classes of essential simple closed curves

Edges: Disjointness



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Ivanov(1997): For $g \geq 3$, the natural map

$$\mathrm{MCG}(S_g) \to \mathrm{Aut}(C(S_g))$$

is an isomorphism.

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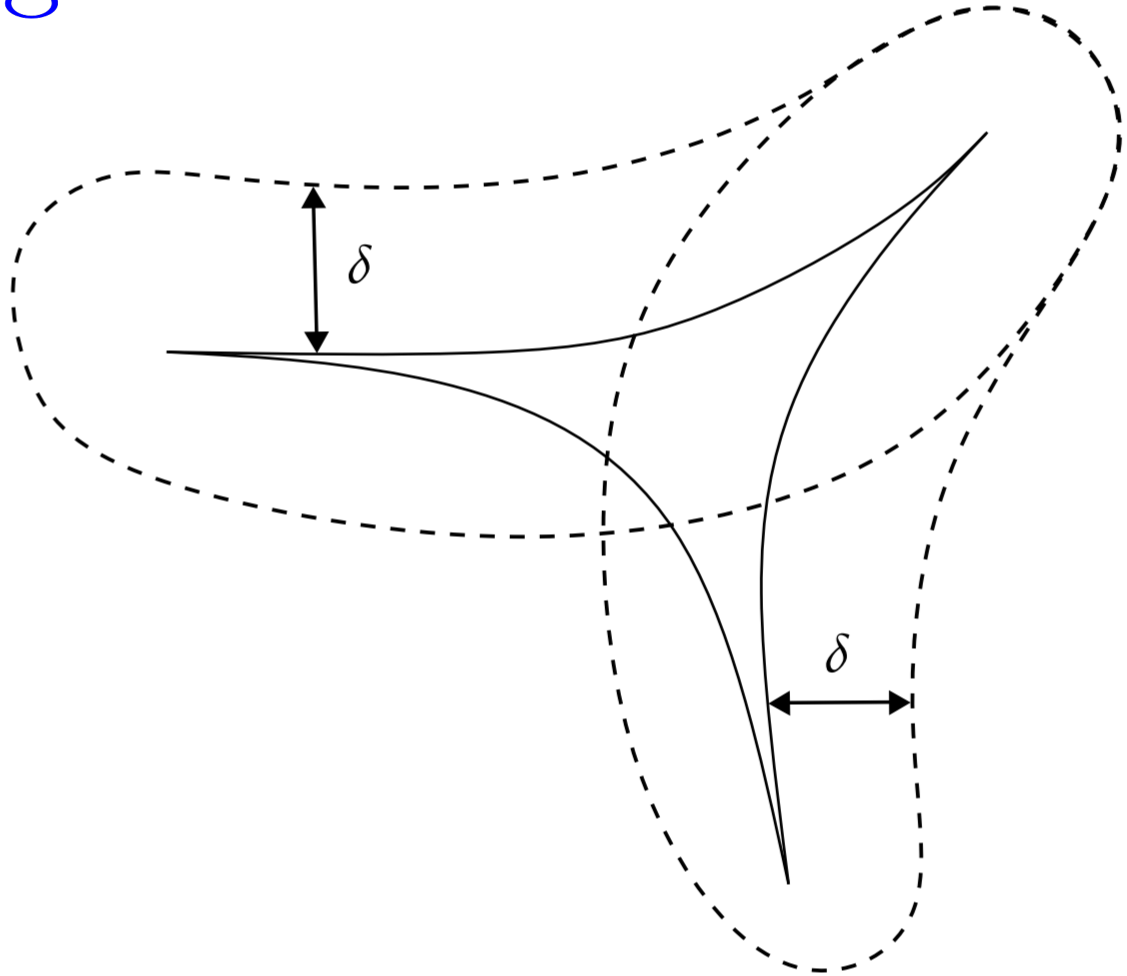
$\rightsquigarrow \mathcal{C}(S)$ a combinatorial tool to study $\mathrm{MCG}(S)$

Thin Triangles Condition:



A geodesic metric space is Gromov hyperbolic if it satisfies the thin triangle condition.

Curve Graph (Harvey)

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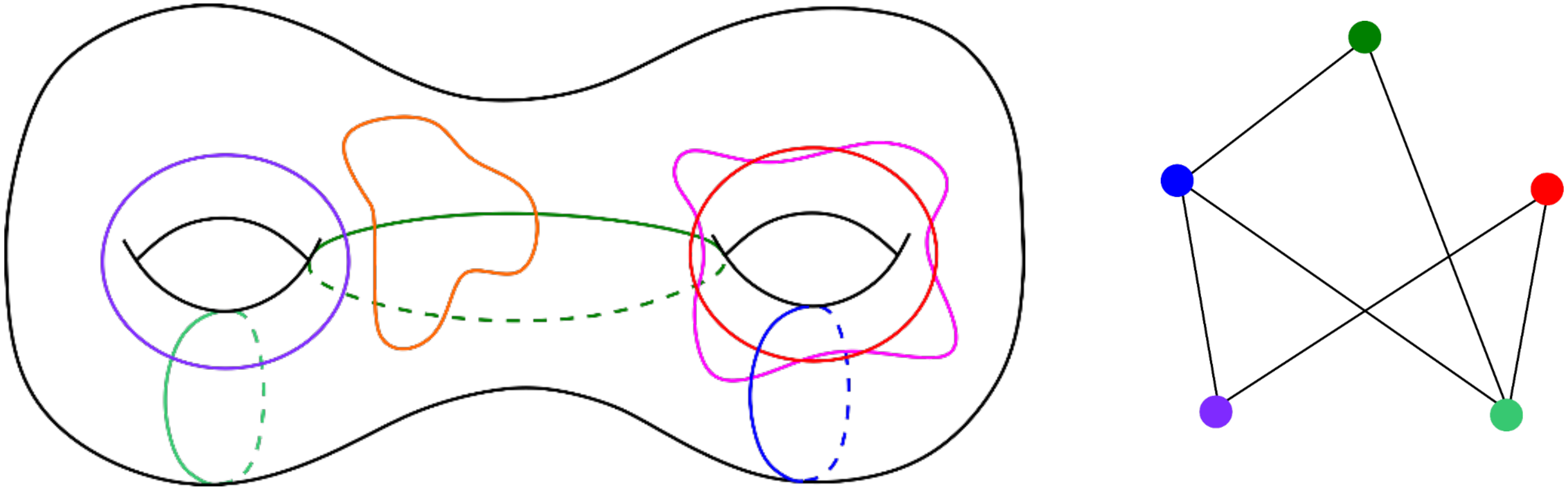

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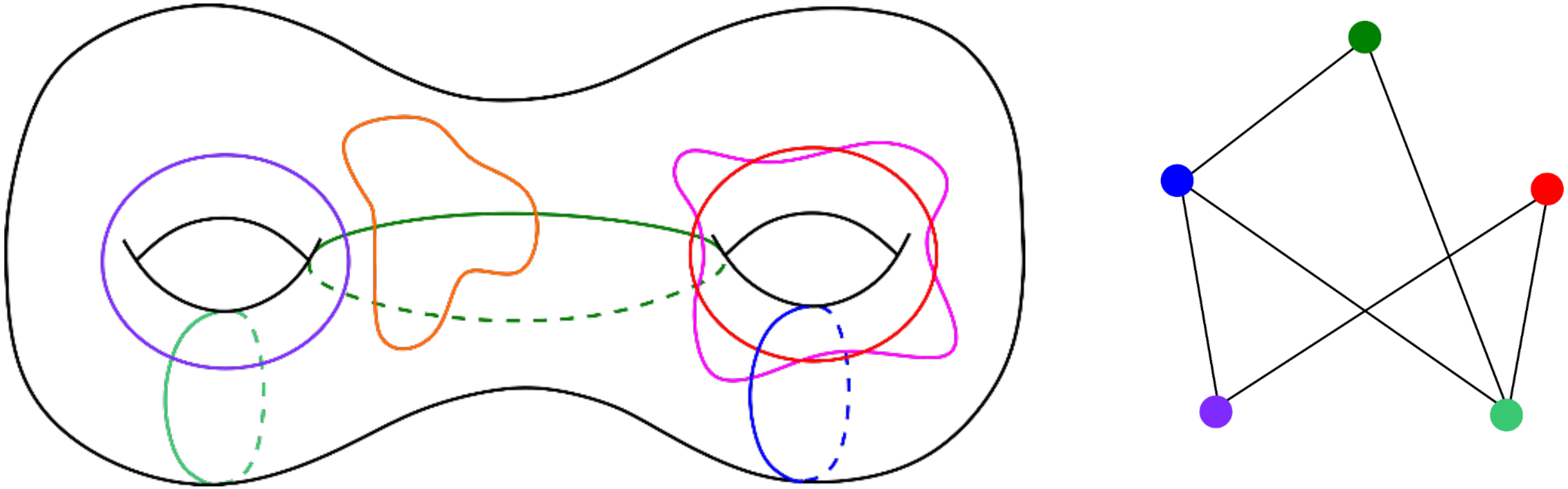

Masur–Minsky: The curve graph is Gromov hyperbolic.

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Consequence: The curve graph is infinite diameter.

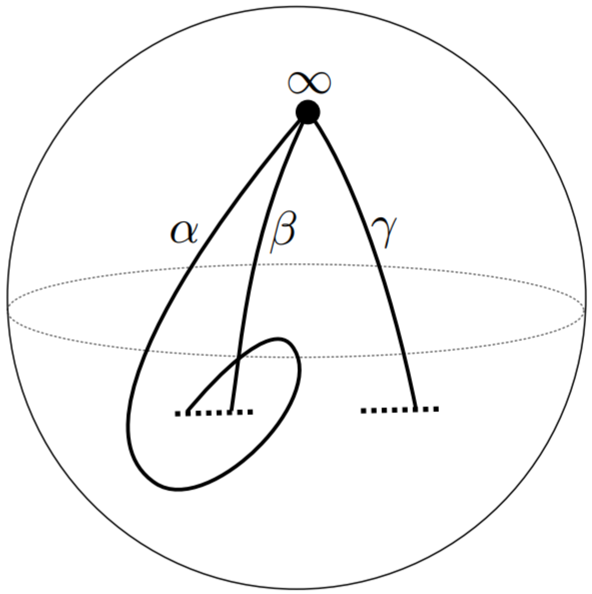
Background

Infinite-Type Surfaces

Ray Graph (Calegari)

Vertices: Isotopy classes of proper rays, with interior in the complement of K, from a point in K to infinity

Edges: Disjointness



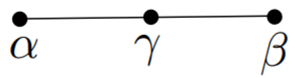


Image by J. Bavard

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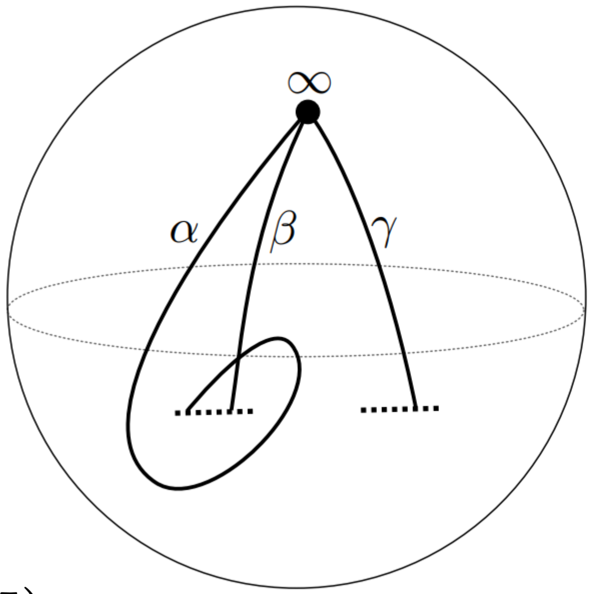

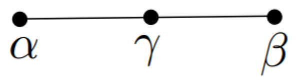

Theorem (Bavard): The ray graph has infinite diameter, is Gromov hyperbolic, and there exists an element of $\mathrm{MCG}(\mathbb{R}^2 \setminus K)$ which acts by translation on a geodesic axis of the ray graph.

Image by J. Bavard

$\mathcal{A}(\Sigma, P)$ (Aramayona–Fossas–Parlier)

P - set of isolated punctures

Vertices: Isotopy classes of arcs with both endpoints in P

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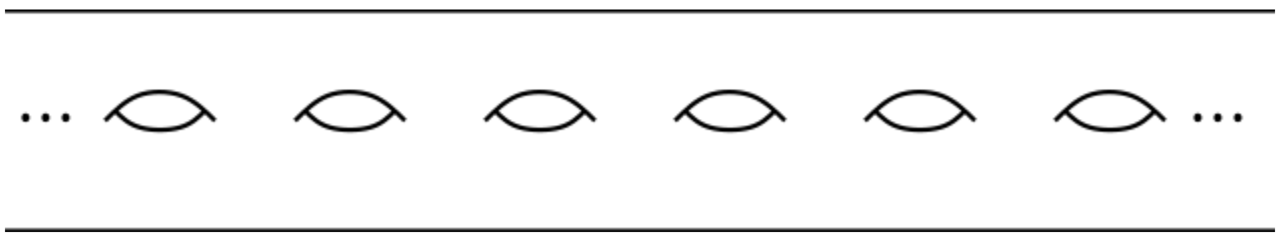
Theorem (Aramayona–Fossas–Parlier): For P finite, $\mathcal{A}(\Sigma, P)$ is connected, has infinite diameter, and is 7-hyperbolic

Ends

An end is a way of exiting every compact set of the surface.



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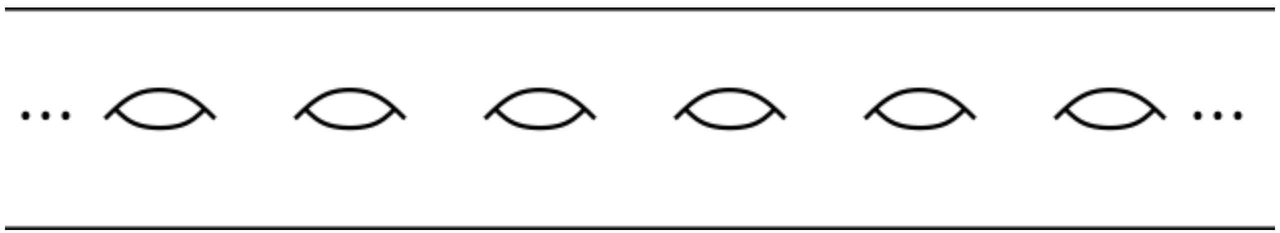

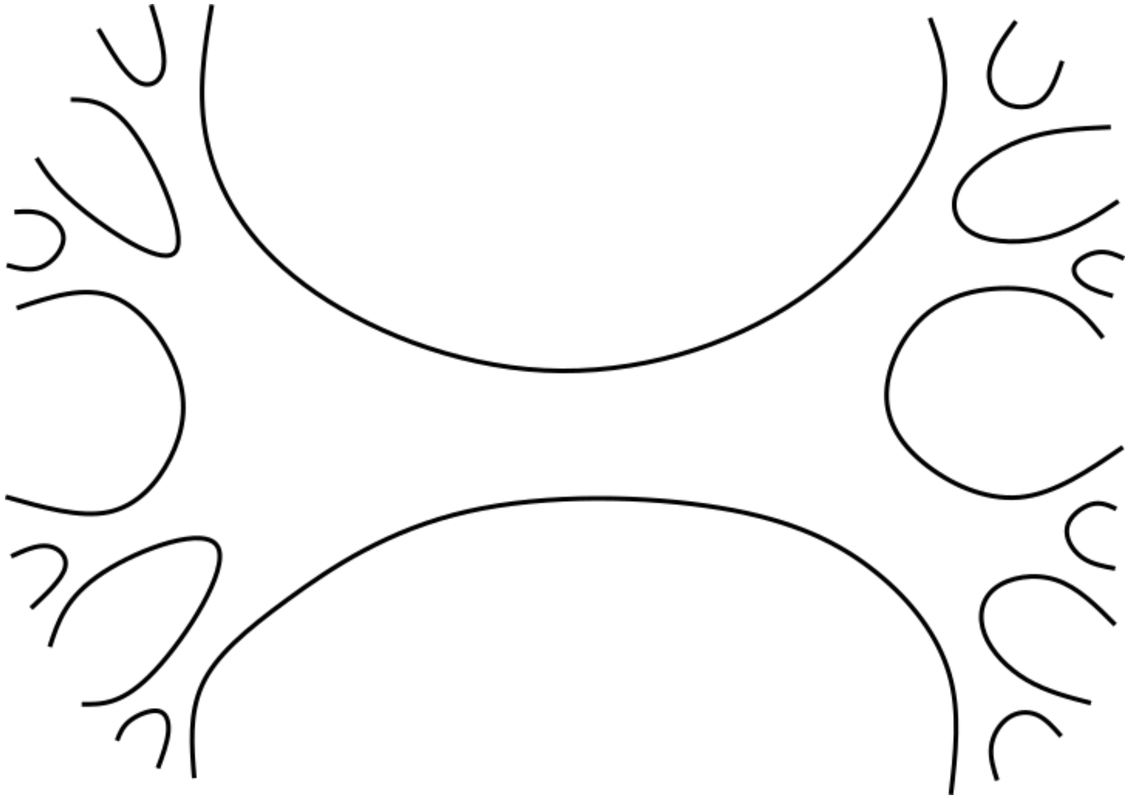

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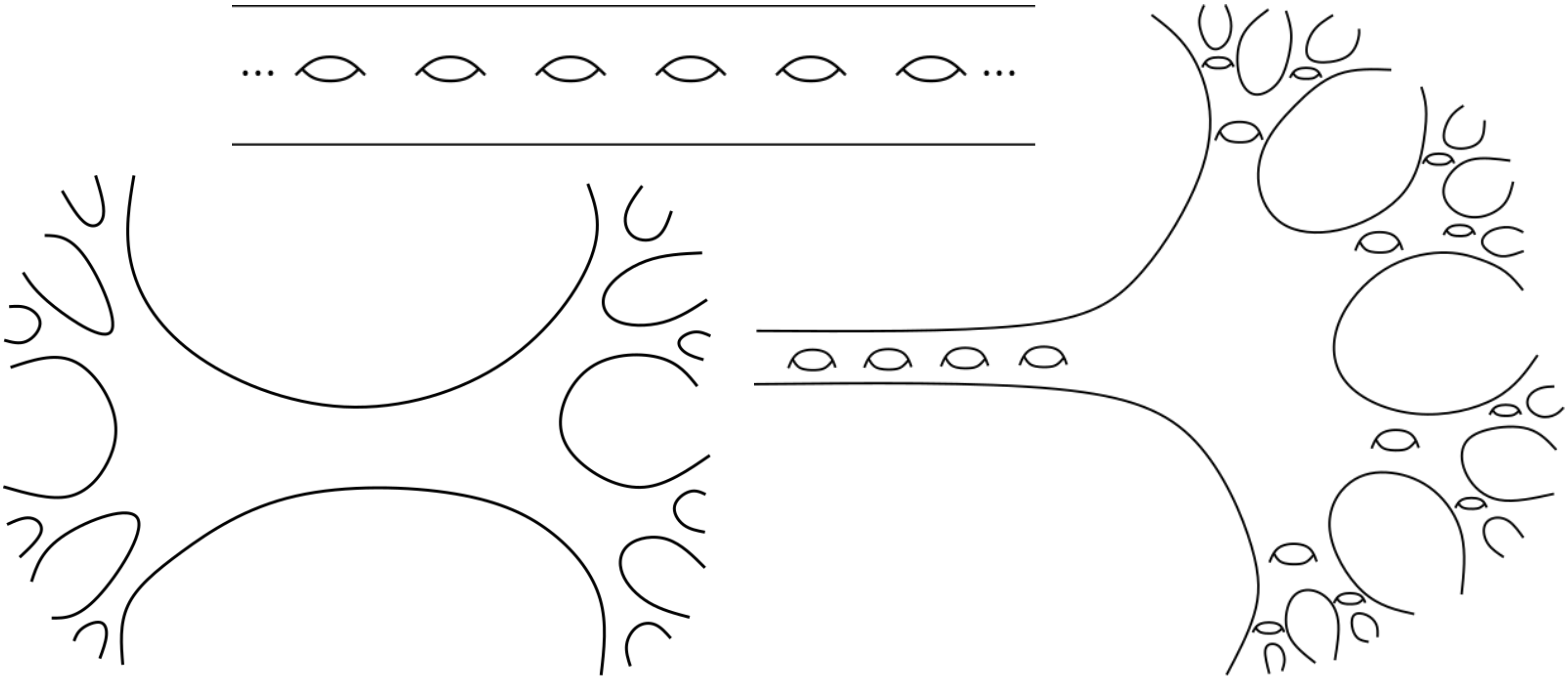

$\mathrm{Sep}_2(\Sigma, \mathcal{P})$ (Durham–Fanoni–Vlamis)

$\mathcal{P}$: Finite collection of pairwise closed subsets of $\mathrm{Ends}(S)$

Vertices: Separating curves such that:

1. Set of ends of each component of $S \setminus c$ contains two elements of $\mathcal{P}$
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Theorem (Durham–Fanoni–Vlamis): $\text{Sep}_2(\Sigma, \mathcal{P})$ is connected and infinite diameter. If each element of $\mathcal{P}$ is a singleton, $\text{Sep}_2(\Sigma, \mathcal{P})$ is δ-hyperbolic.

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One-cut subsurface: complementary component of a separating loop.

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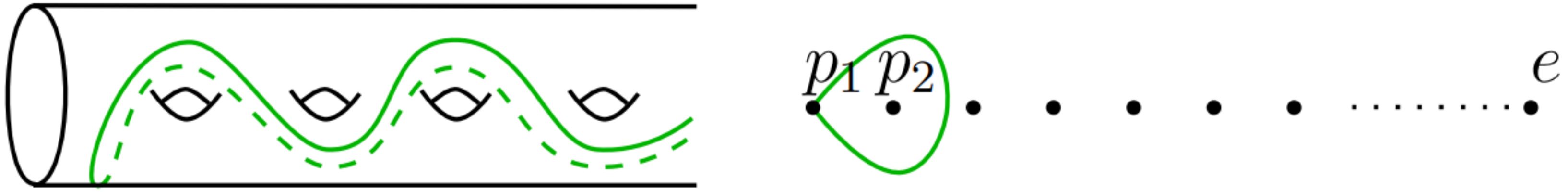
One-cut homeomorphic subsurface: A one-cut subsurface which is homeomorphic to the full surface



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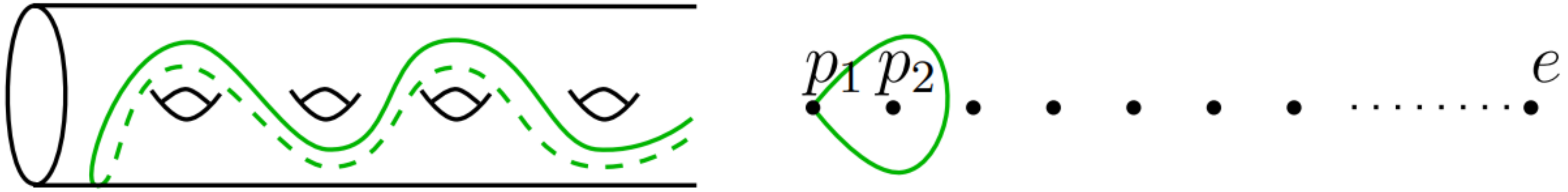

Image by Fanoni–Ghaswala–McLeay

An arc joining distinct ends is omnipresent if it intersects every one-cut homeomorphic subsurface.

Omnipresent Arc Graph (Fanoni–Ghaswala–McLeay)

Arc Graph, $\mathcal{A}(\Sigma)$ Vertices: isotopy classes of essential arcs

Edges: Disjointess

Omnipresent arc graph: subgraph of $\mathcal{A}(\Sigma)$ spanned by all omnipresent arcs

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Theorem (Fanoni–Ghaswala–McLeay): For any stable surface Σ with at least three finite-orbit ends, the omnipresent arc graph is a connected δ-hyperbolic graph on which $\mathrm{MCG}(\Sigma)$ acts with unbounded orbits

The Grand Arc Graph

Mann–Rafi

Partial order: $x \preccurlyeq y$ if for any neighborhood U of y, there exists a neighborhood V of x and $f \in \mathrm{MCG}(\Sigma)$ such that $f(V) \subset U$

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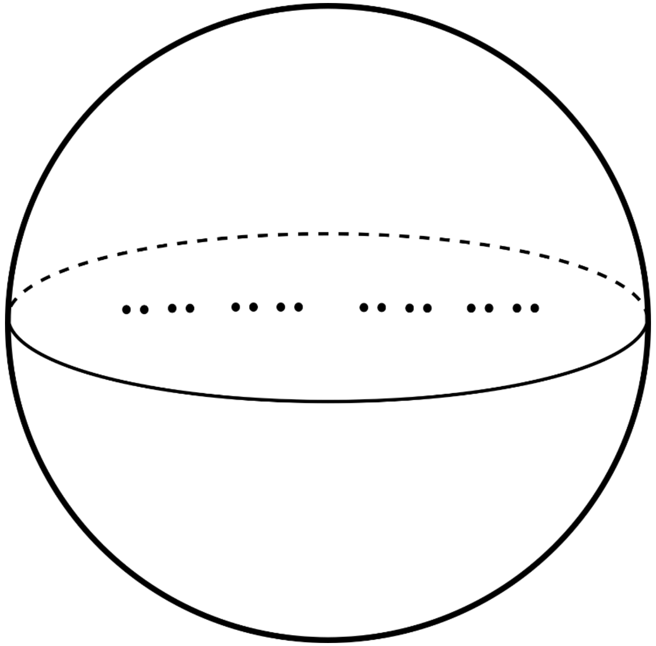

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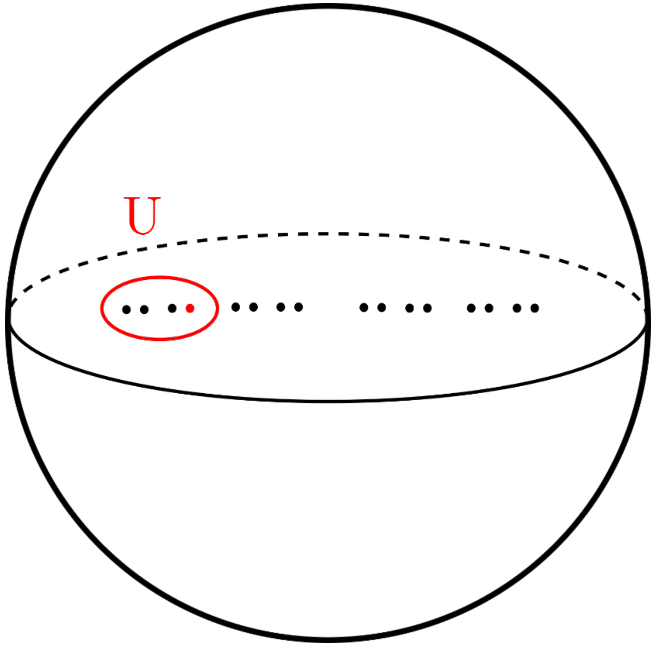

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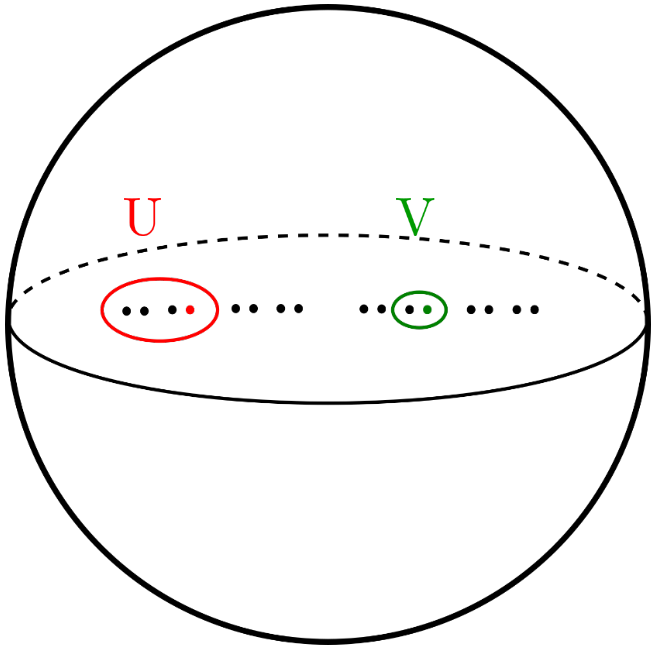

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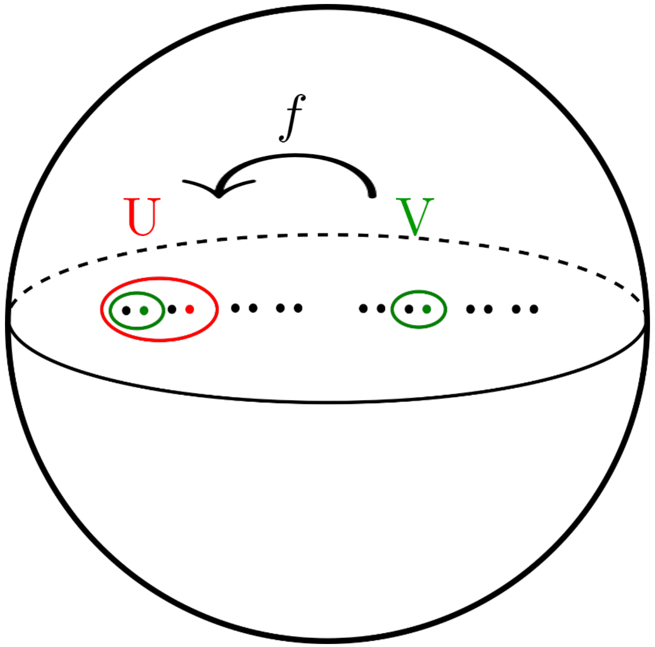

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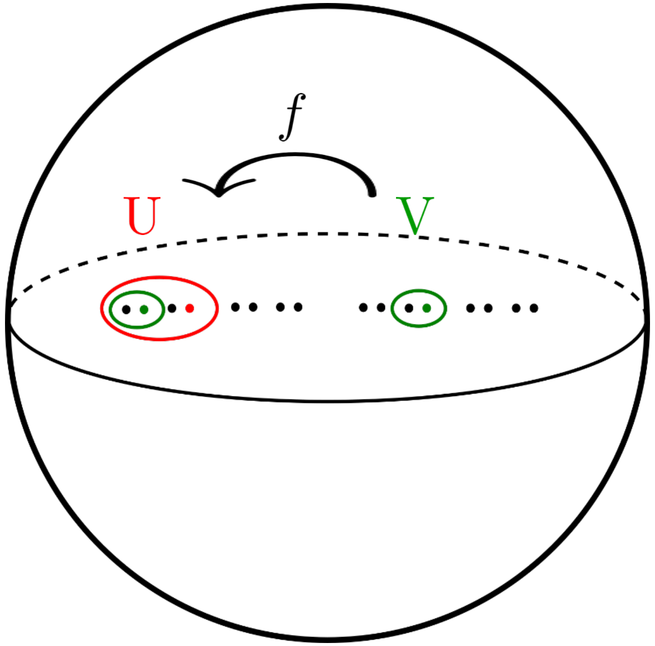
$x \sim y$ if $x \preccurlyeq y$ and $y \preccurlyeq x$



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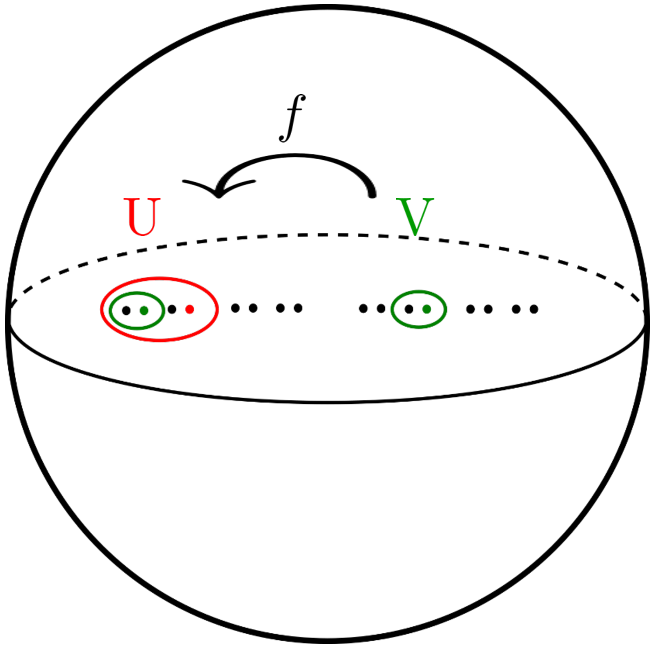

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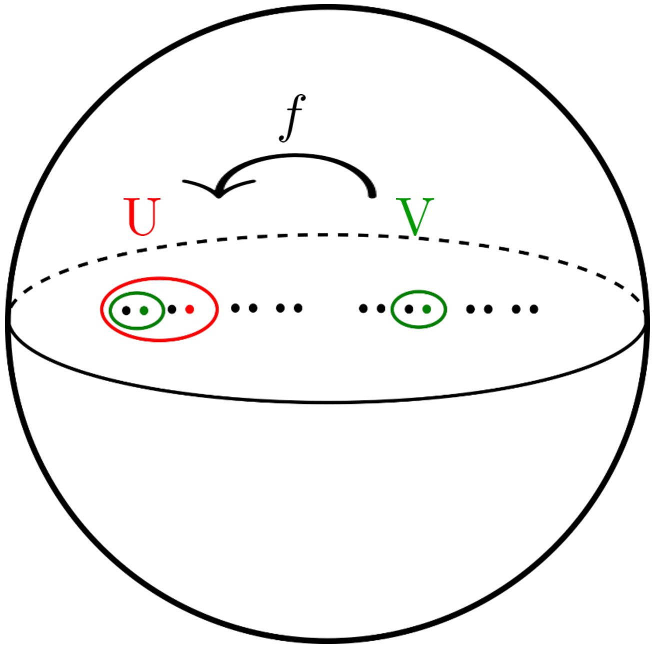

$E(x) = \{y | y \sim x\}$

Theorem (Mann–Rafi): The partial order has maximal elements. Furthermore, for every maximal element x, $E(x)$ is either finite or a Cantor set.

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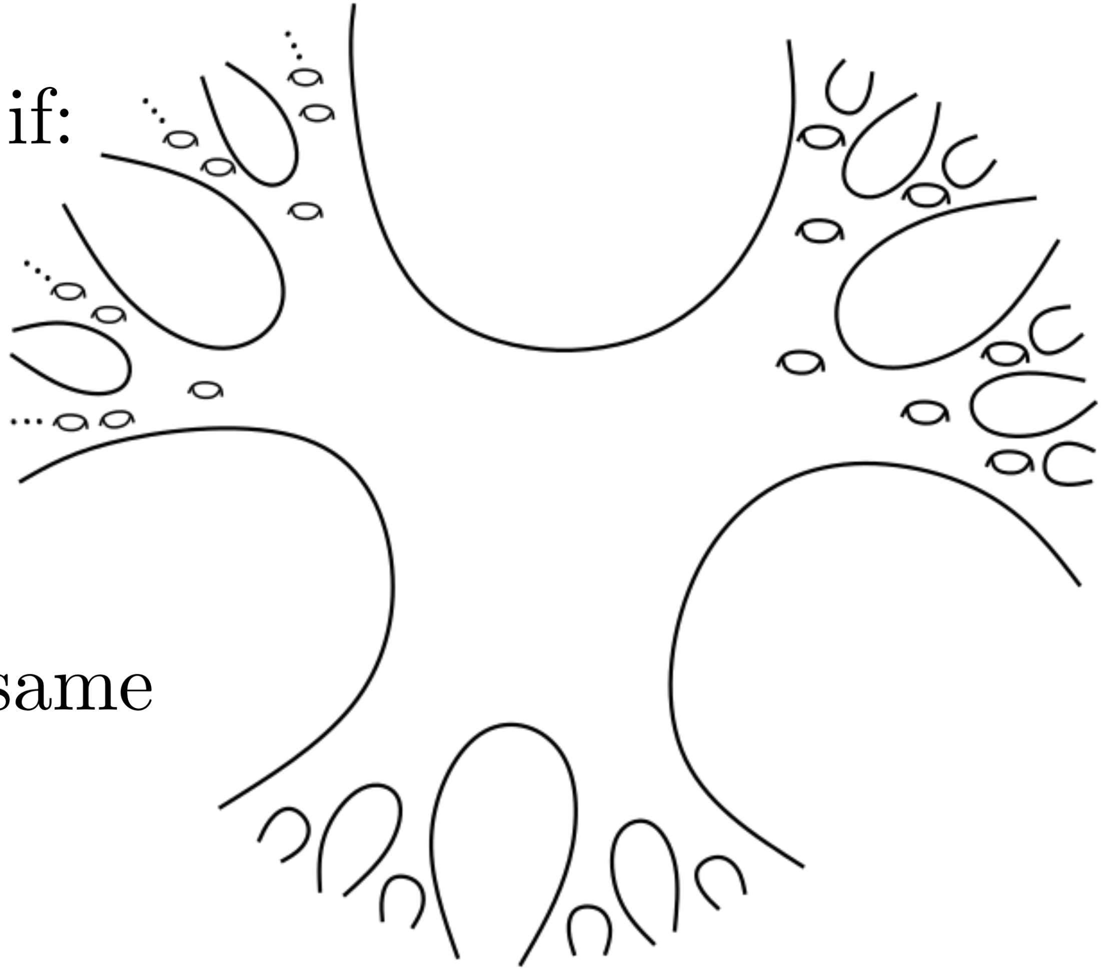

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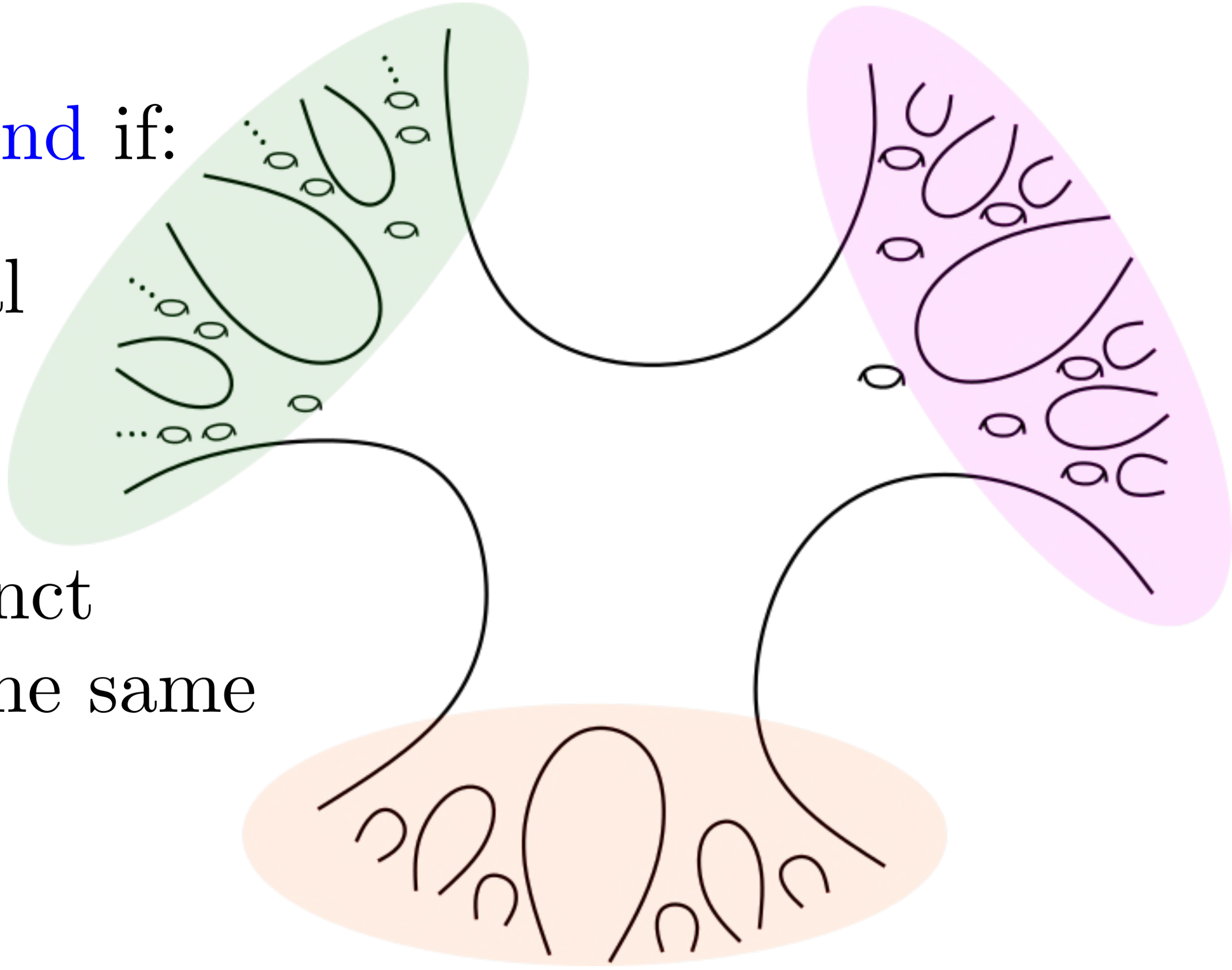

Grand Arcs

An arc α converges to an end e if for any neighborhood U of e, α eventually never leaves this neighborhood.

An arc with endpoints e_1, e_2 is grand if:

1. e_1 and e_2 are both maximal
2. $e_1 \neq e_2$
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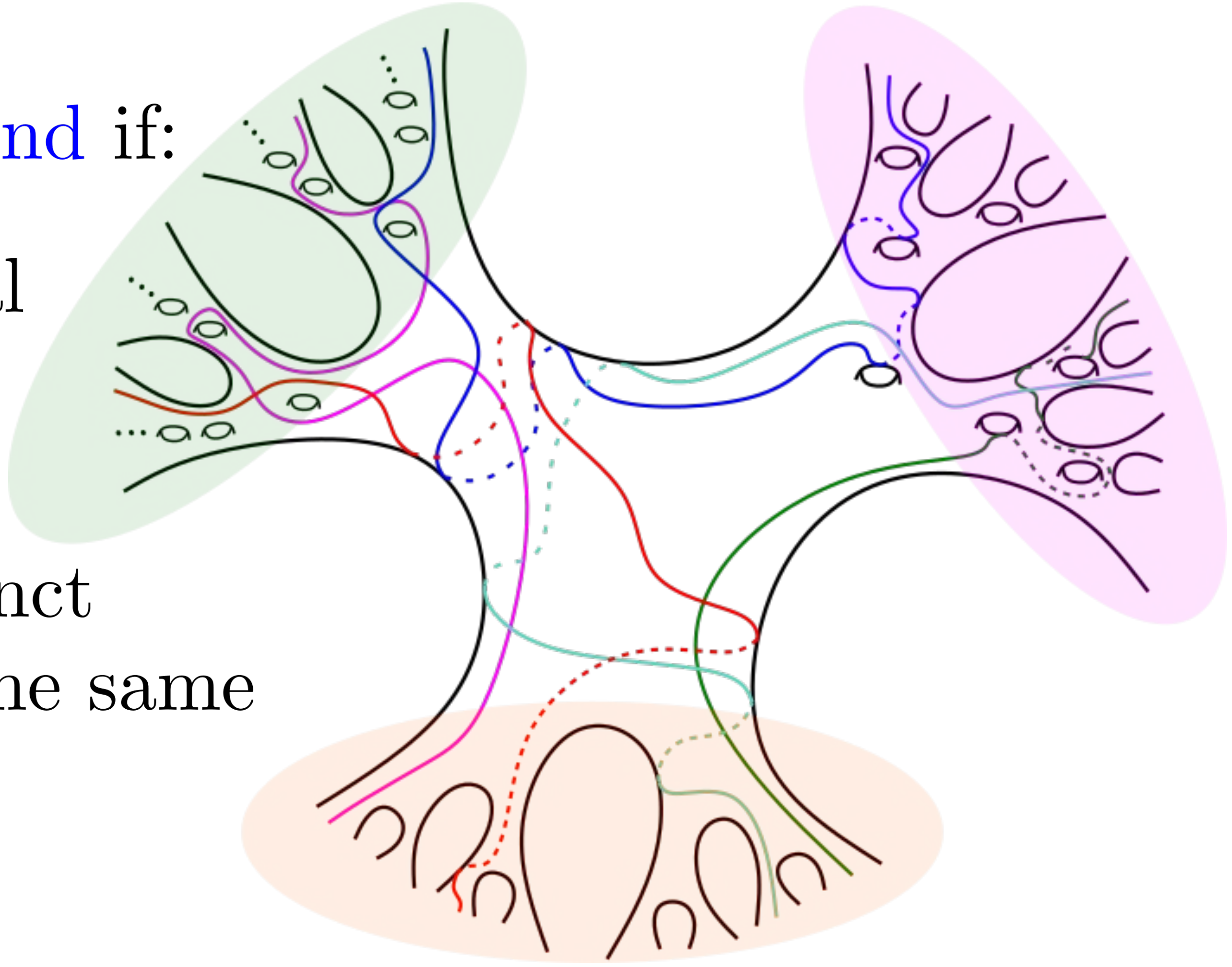
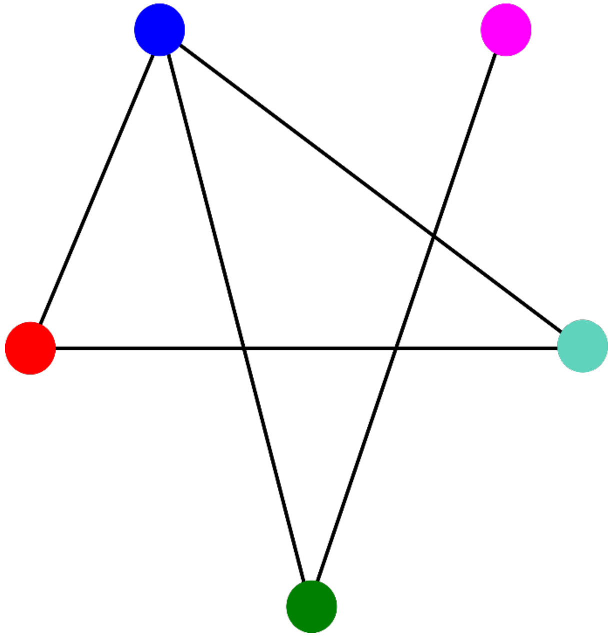
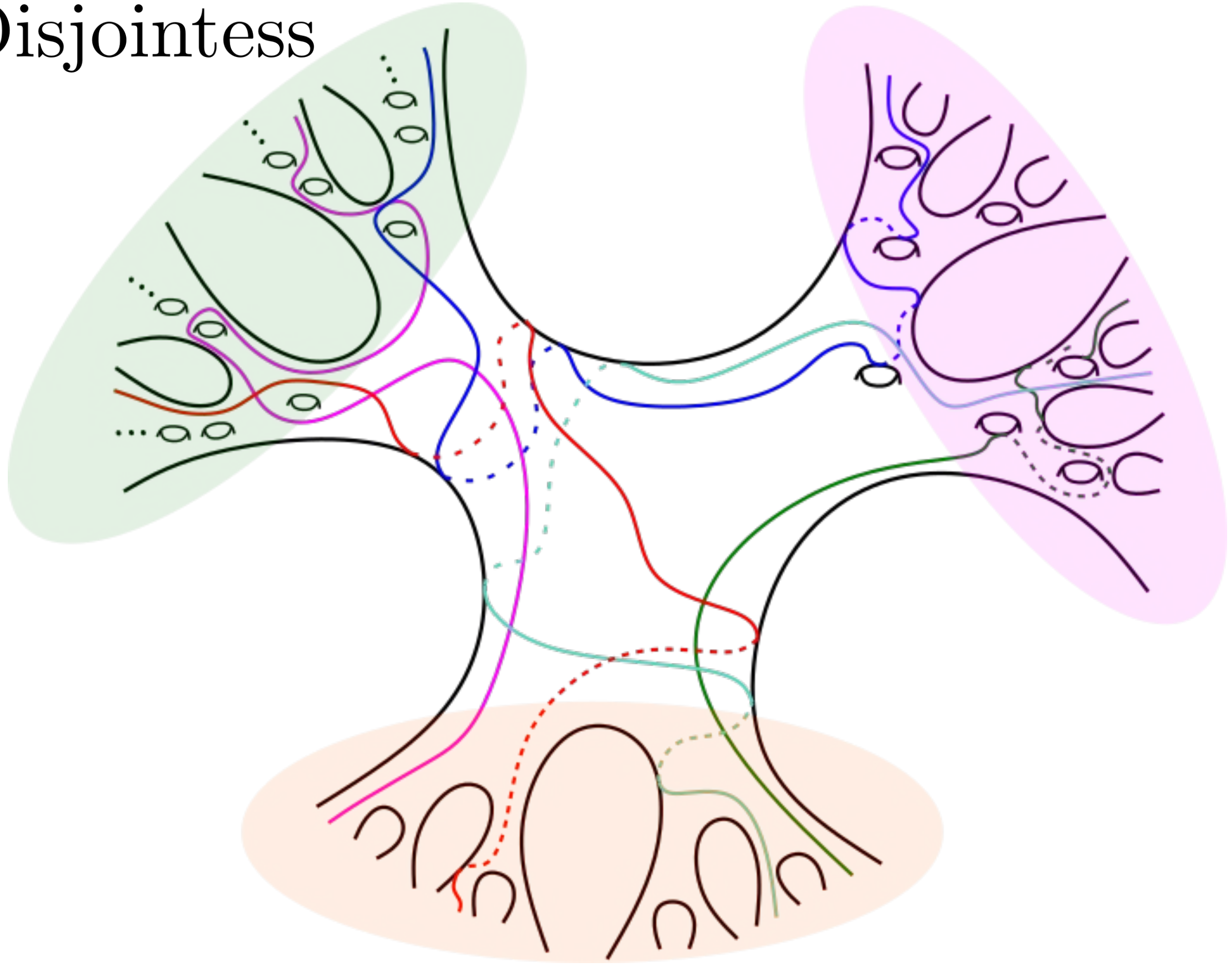


Grand Arc Graph

Grand Arc Graph, $\mathcal{G}(\Sigma)$ Vertices: isotopy classes of grand arcs

Edges: Disjointess





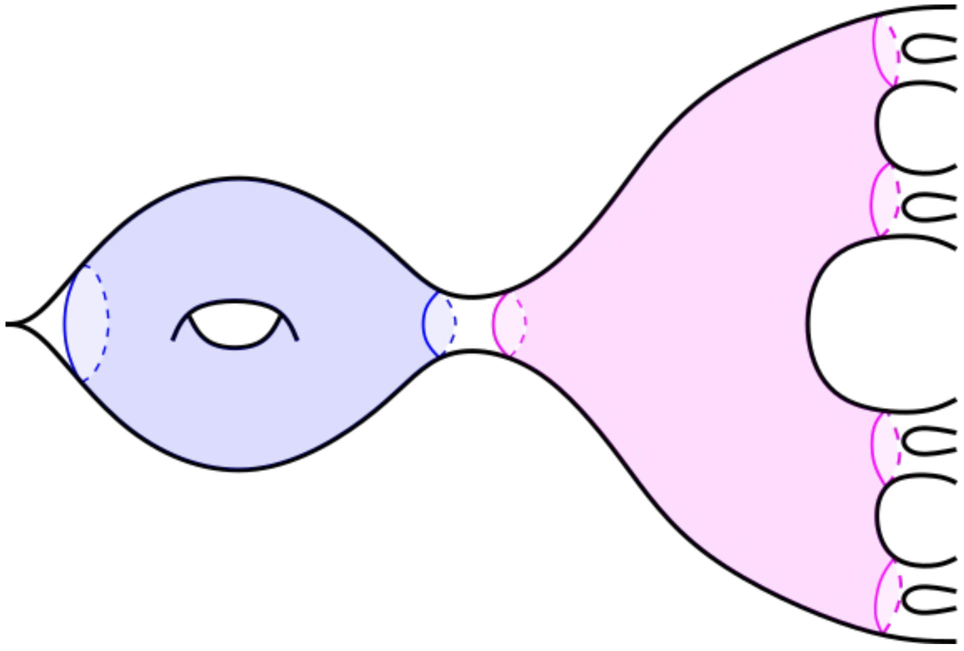
Main Theorem

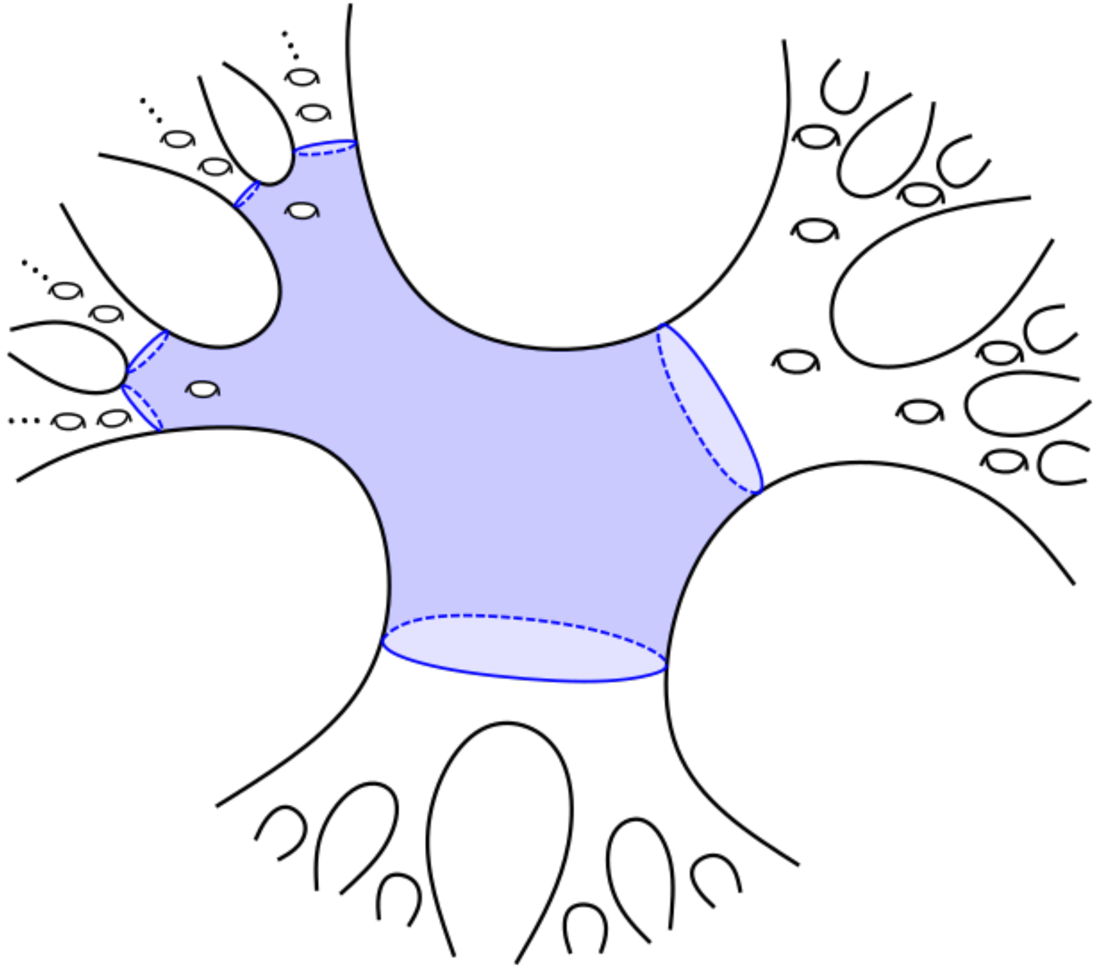
Theorem (Bar-Natan – V.): For a large class of surfaces, the grand arc graph is connected, hyperbolic, has infinite diameter, and MCG(Σ) acts continuously on visible boundary.

Proof Sketch

Witnesses

A witness, $W \subseteq \Sigma$ is a subsurface which intersects every grand arc.





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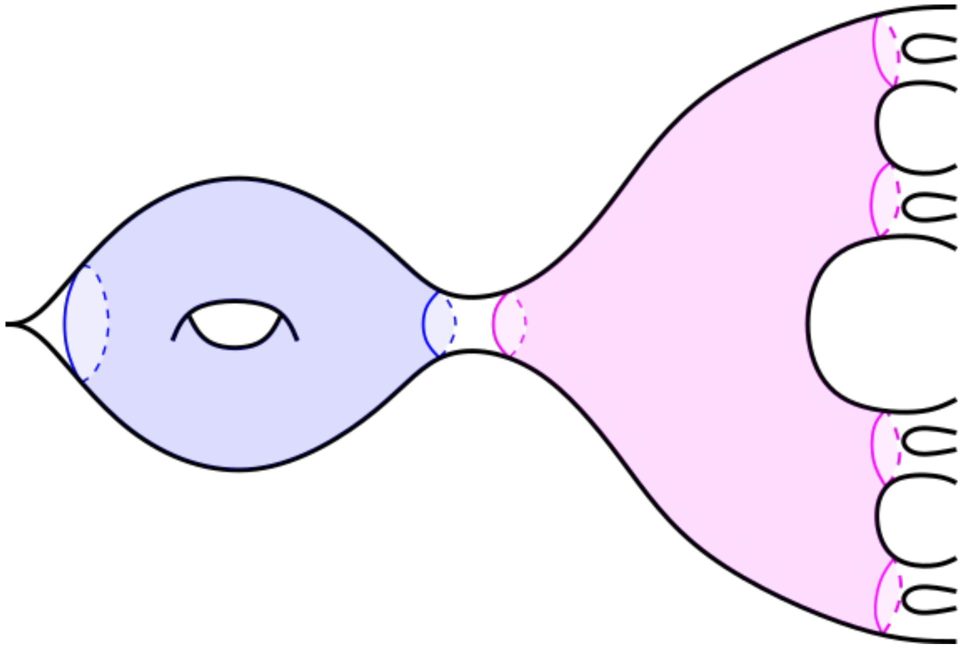
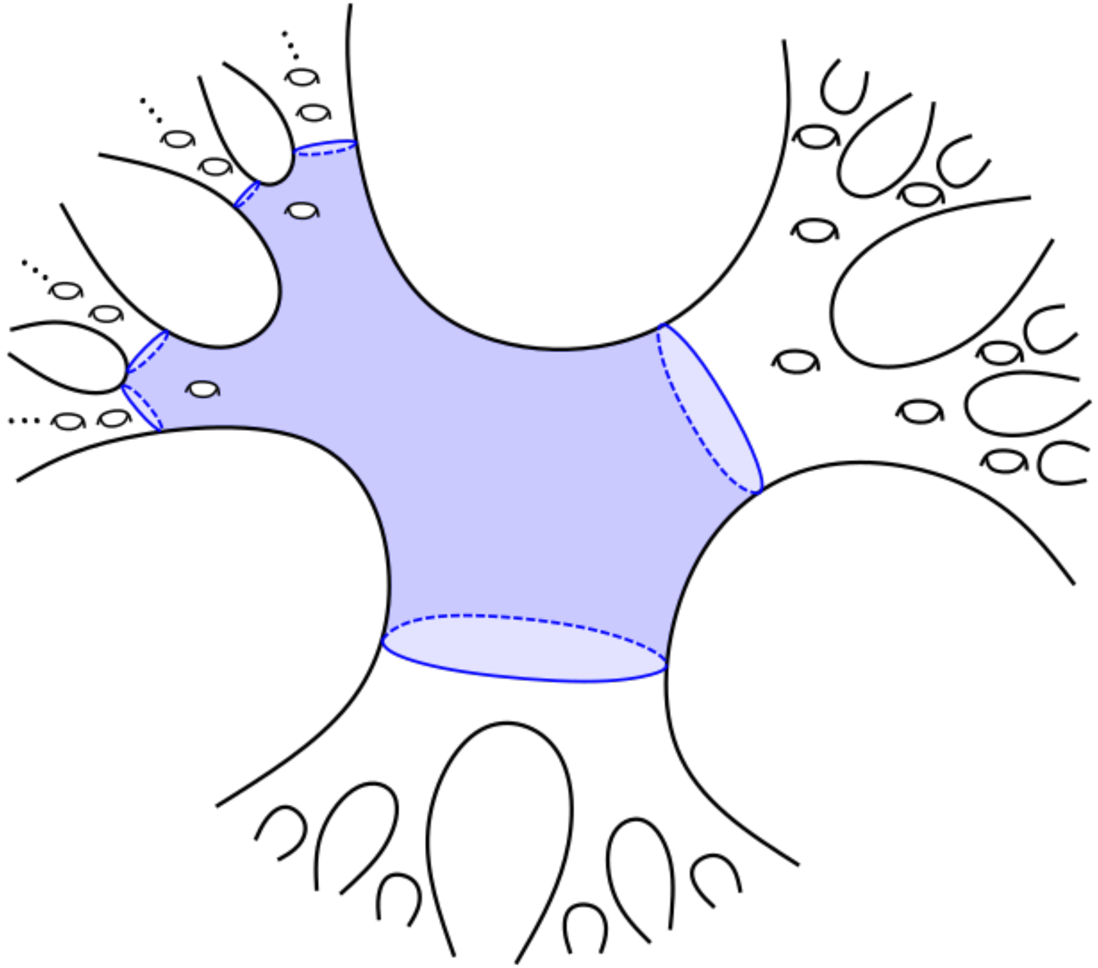
Theorem (Bar-Natan – V.) If a surface has greater than 2 self-similar equivalences classes of maximal ends, then all witnesses intersect.





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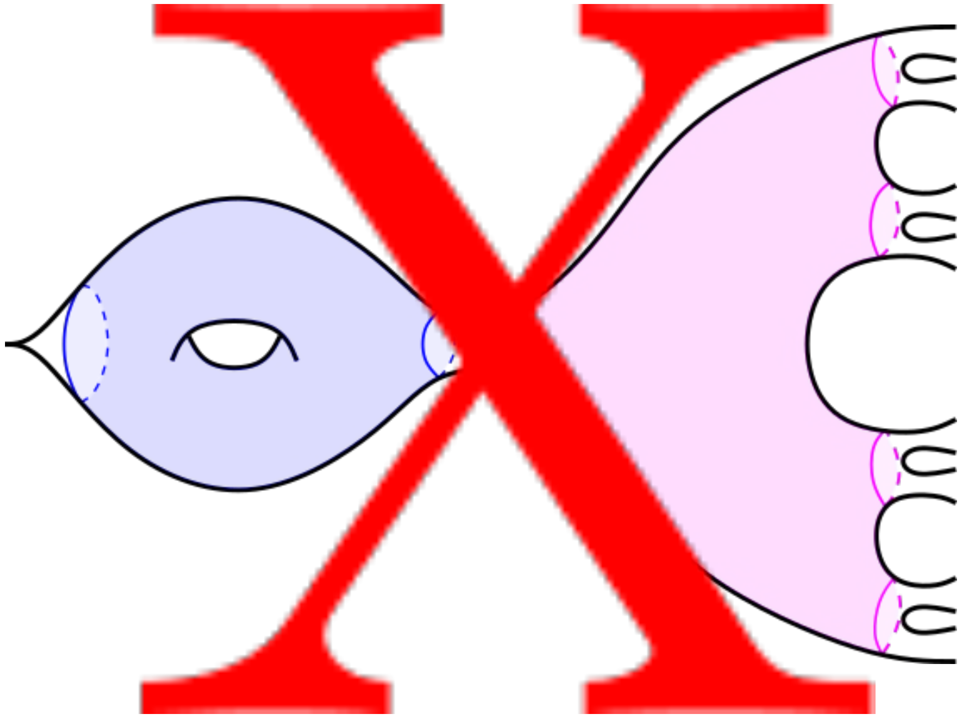

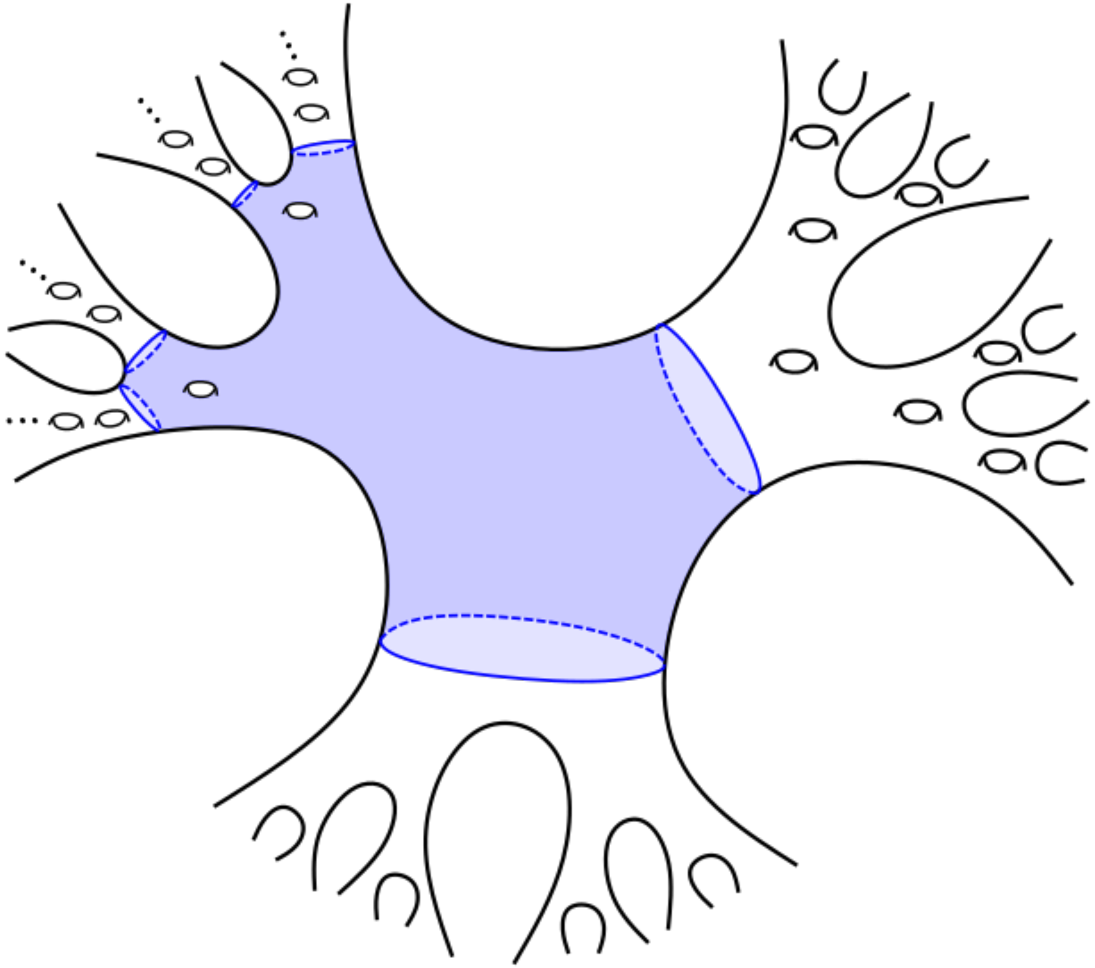

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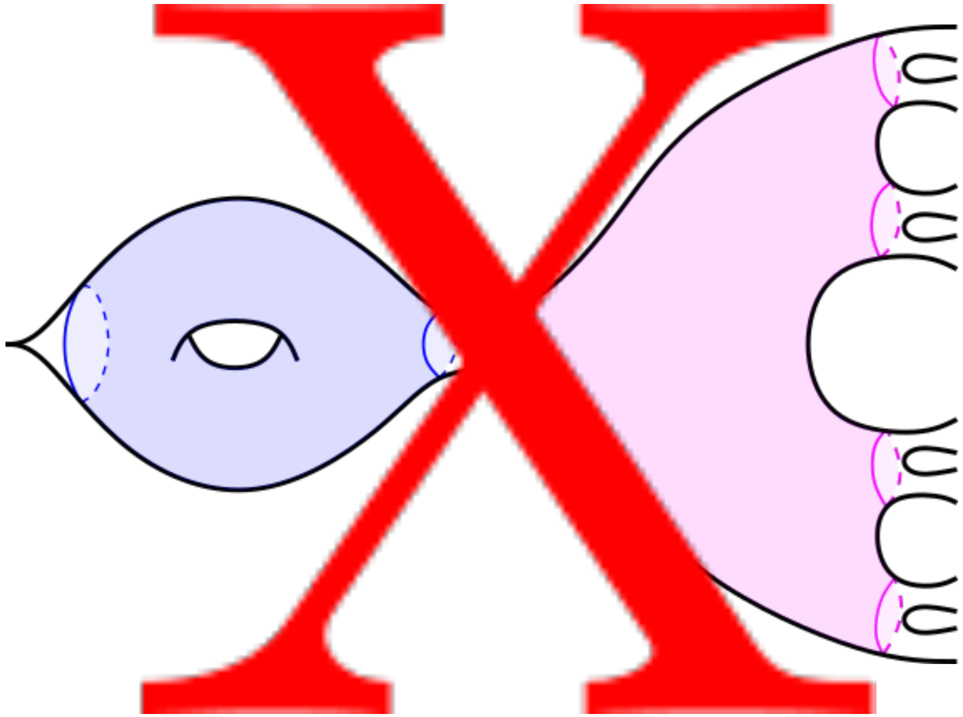
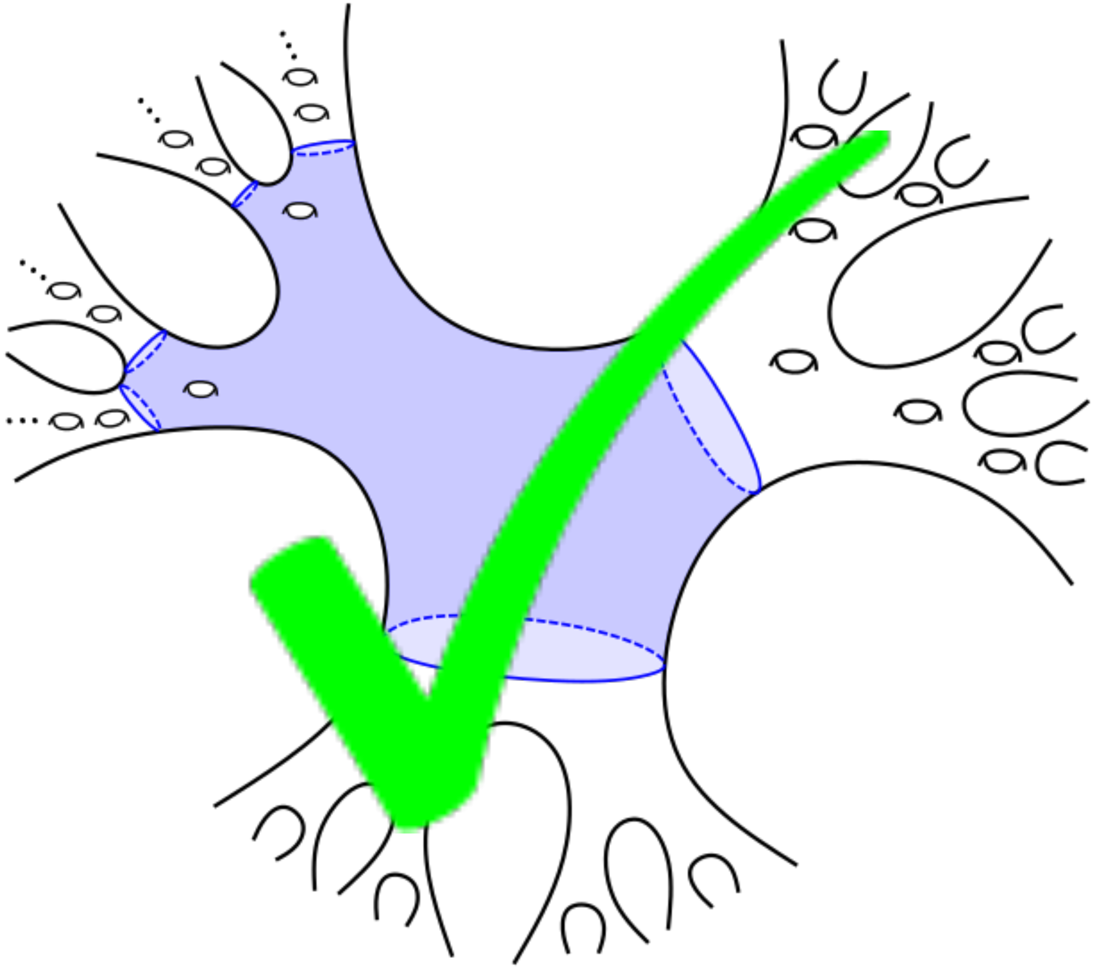
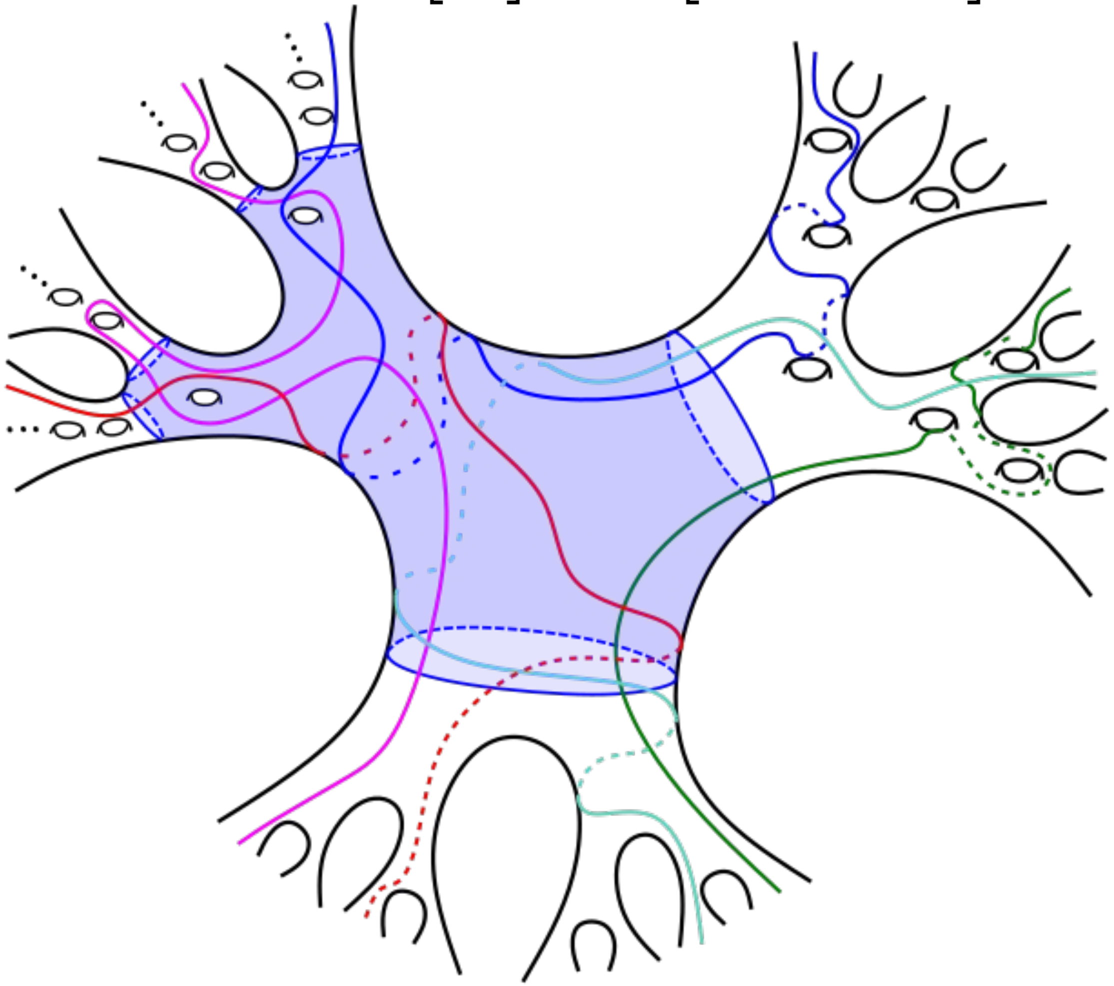




Witness Projection Program

$$\pi_W : \mathcal{G}(\Sigma) \to \mathcal{A}(W)$$

$$[\alpha] \mapsto [\alpha \cap W]$$



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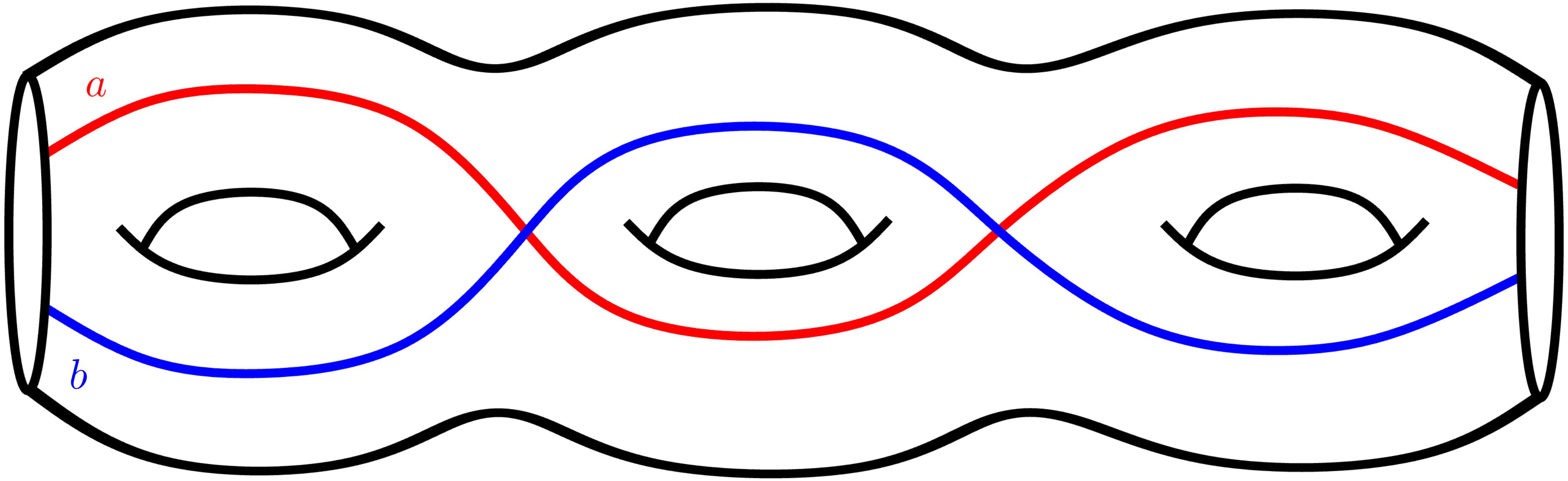
$\implies$ infinite diameter

Proof Sketch

Unicorn paths (Hensel–Przytycki–Webb [2013]):



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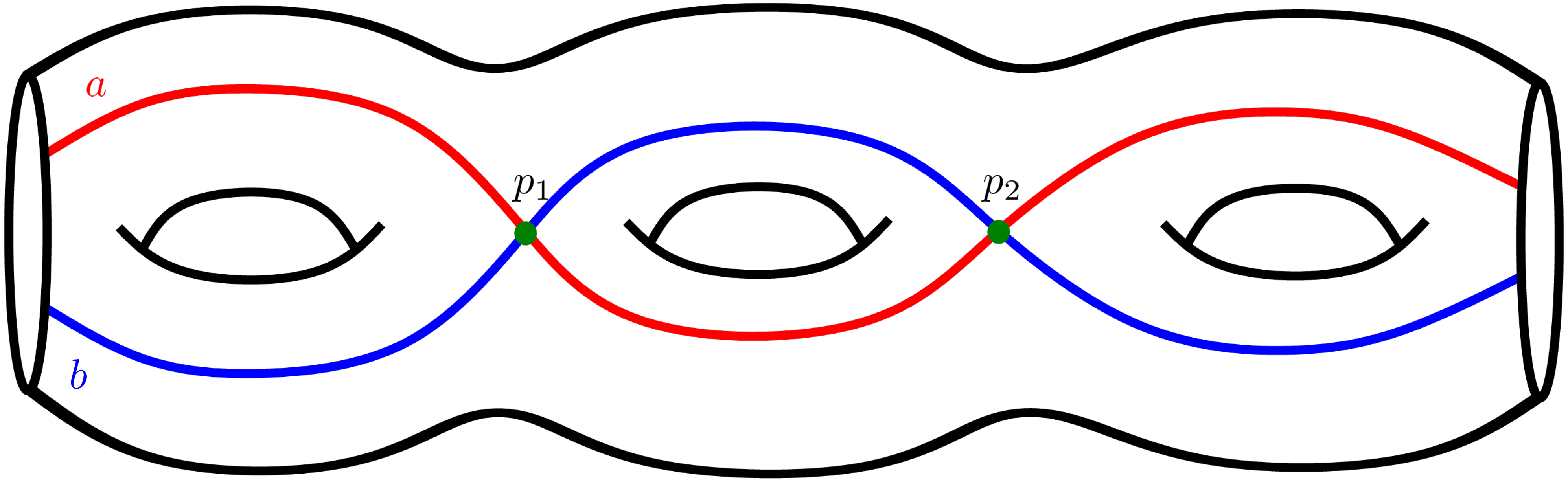

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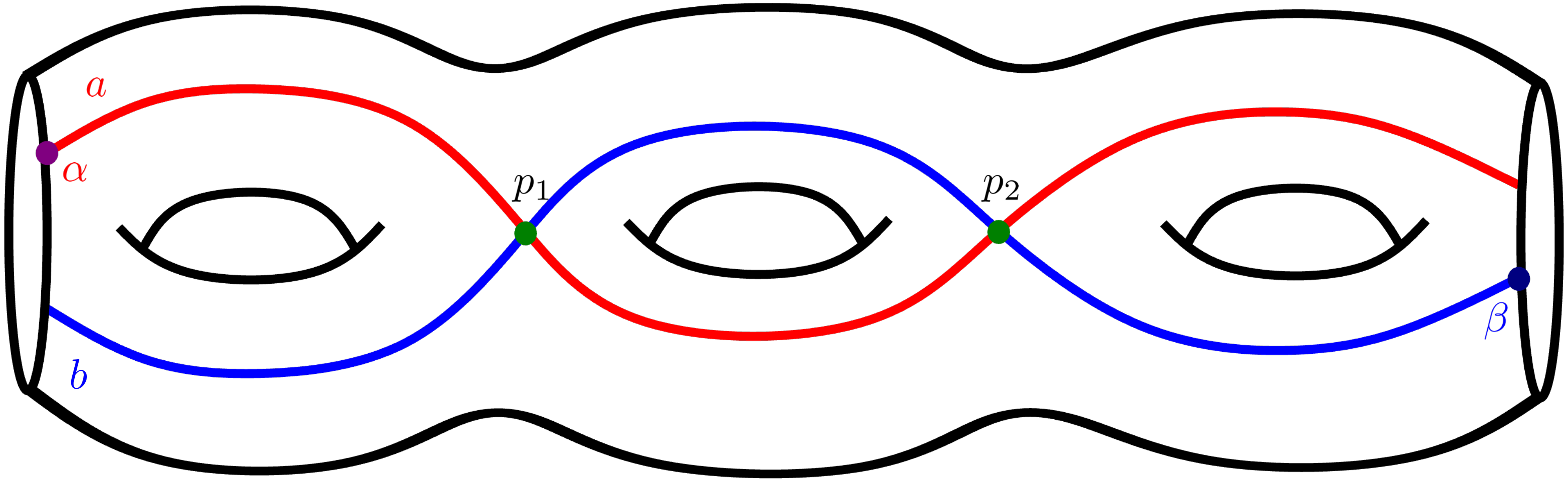

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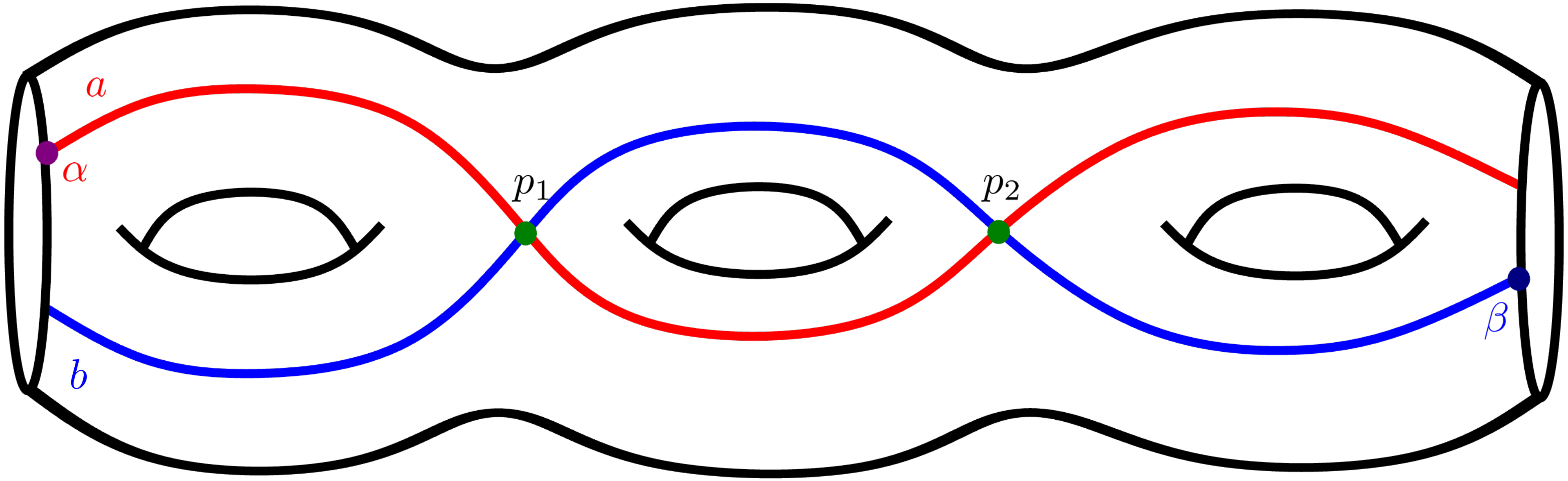
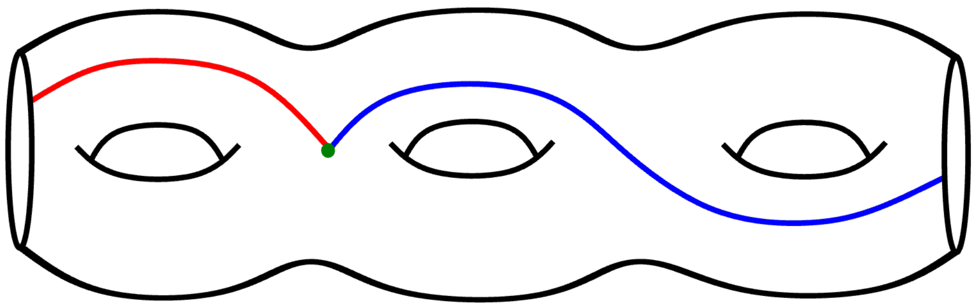


First unicorn arc:



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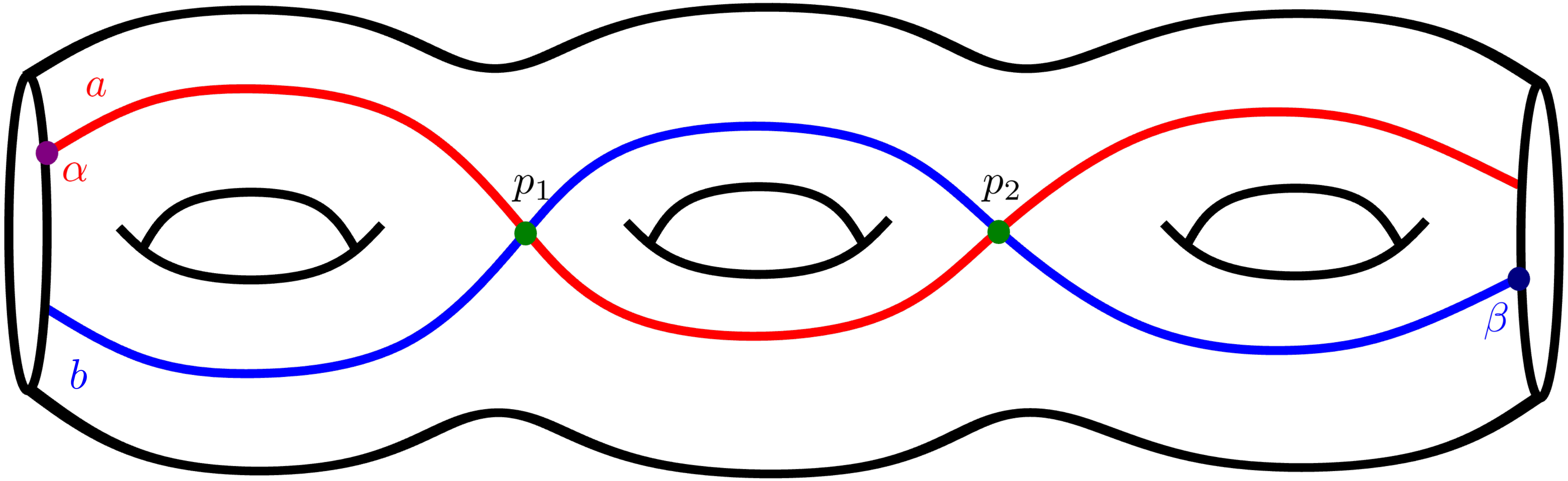

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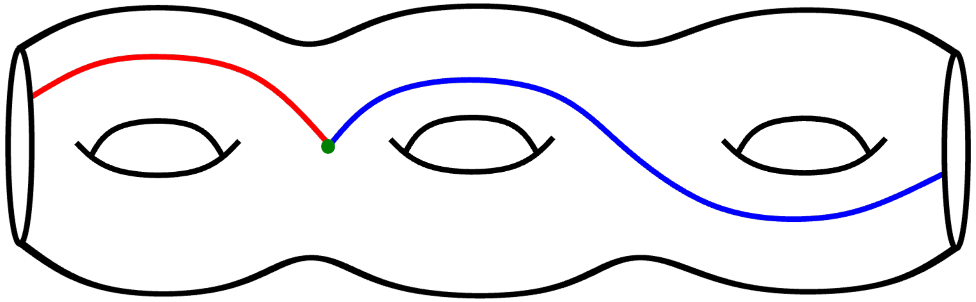
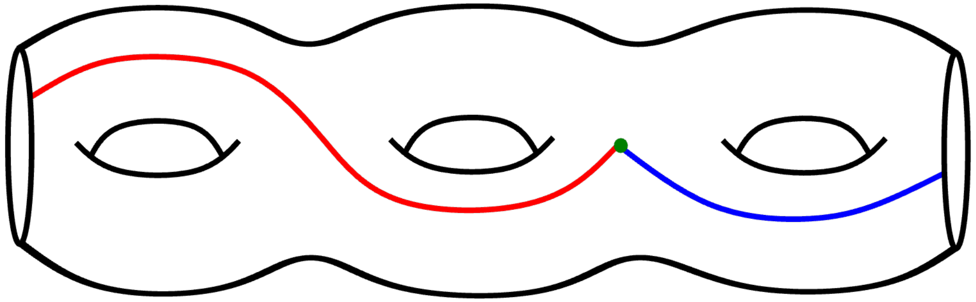


Second unicorn arc:



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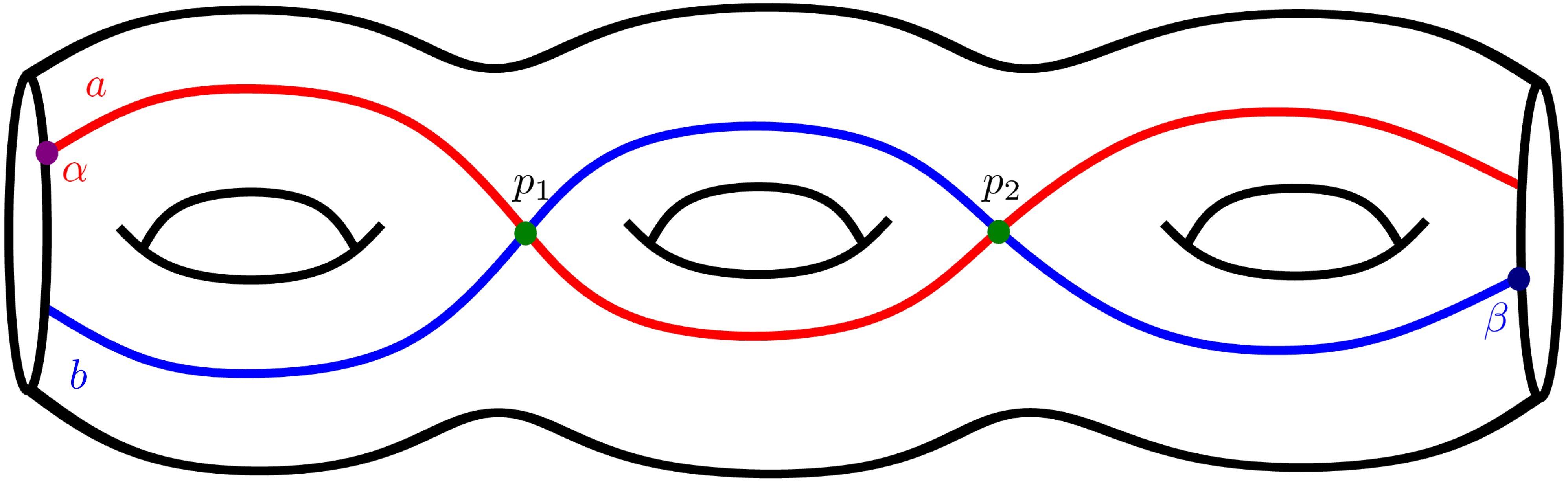

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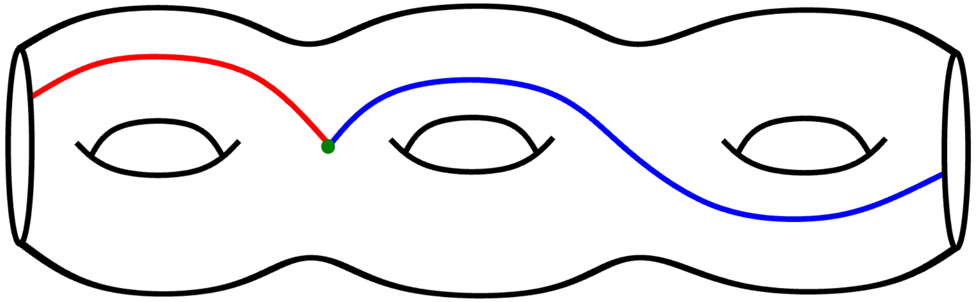

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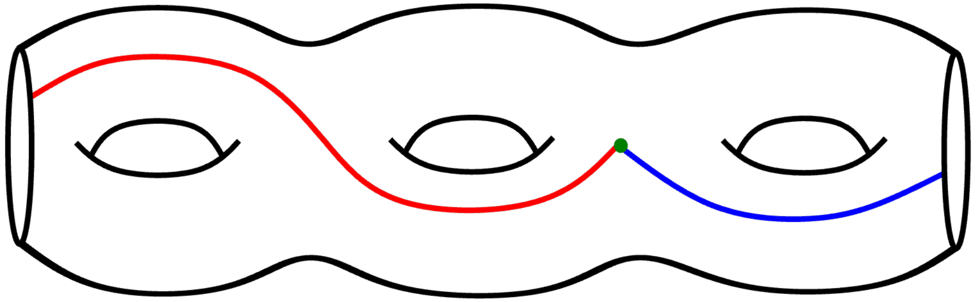

Consecutive unicorn arcs are disjoint.

Unicorn paths (Hensel–Przytycki–Webb):

Fanoni–Ghaswala–McLeay: Generalized unicorn paths for infinite-type surfaces.

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Unicorn paths allow us to show $\mathcal{G}(\Sigma)$ is:

- Connected
- Hyperbolic

Notice!

Two disjoint witnesses $\implies$ not δ-hyperbolic!

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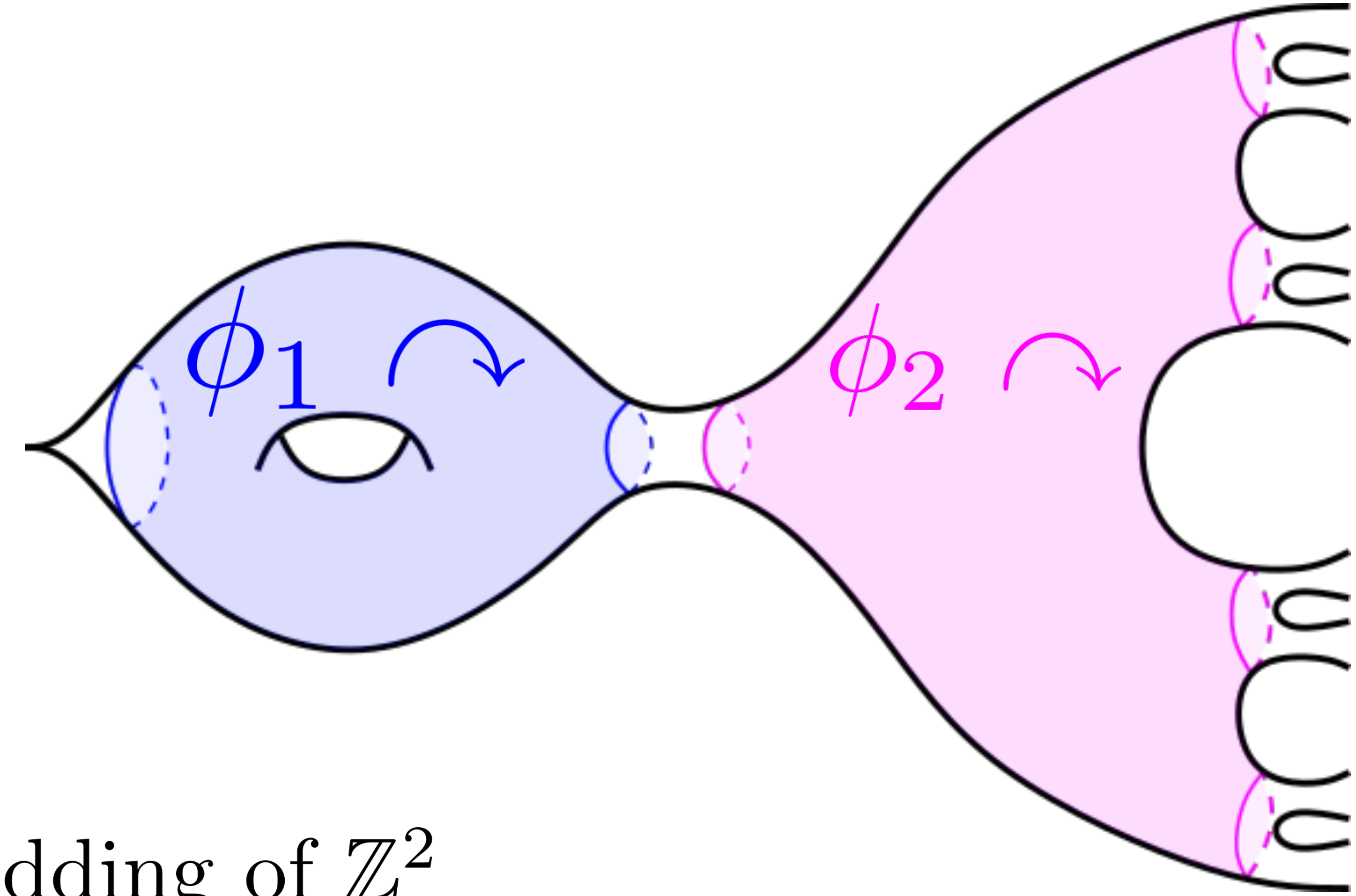

Q.I embedding of $\mathbb{Z}^2$

Mapping Class Group Action

$\mathrm{MCG}(\Sigma) \curvearrowright \mathcal{G}(\Sigma)$ by isometries.

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$$\implies \text{continuous action on } \partial\mathcal{G}(\Sigma)$$

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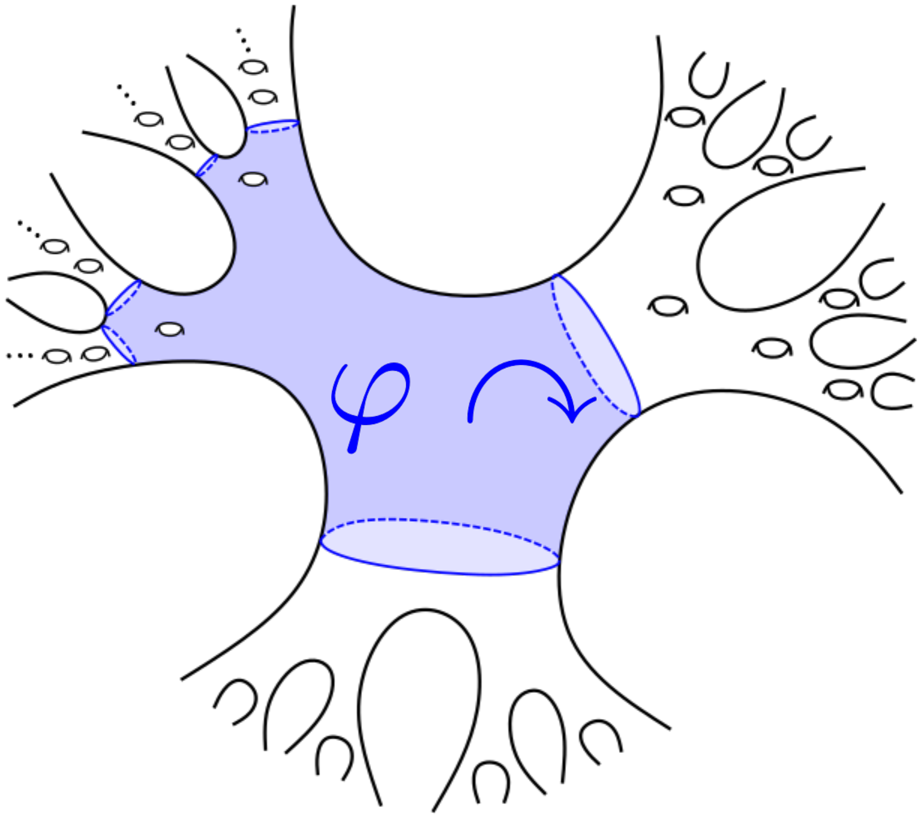
Theorem (Bar-Natan – V.): Let φ be a pseudo-Anosov mapping class that fixes the boundary of W. Let $\bar{\varphi} \in \mathrm{MCG}(\Sigma)$ be the homeomorphism fixing W^c and acting as φ on W. Then $\bar{\varphi}$ acts loxodromically on $\mathcal{G}(\Sigma)$.

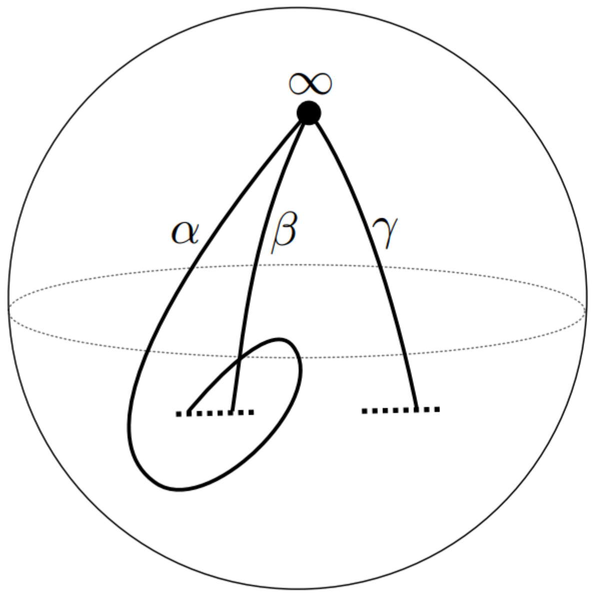


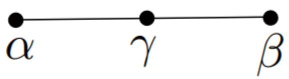
Relationship to Other Graphs

Ray Graph (Calegari)

Vertices: Isotopy classes of proper rays, with interior in the complement of K, from a point in K to infinity

Edges: Disjointness





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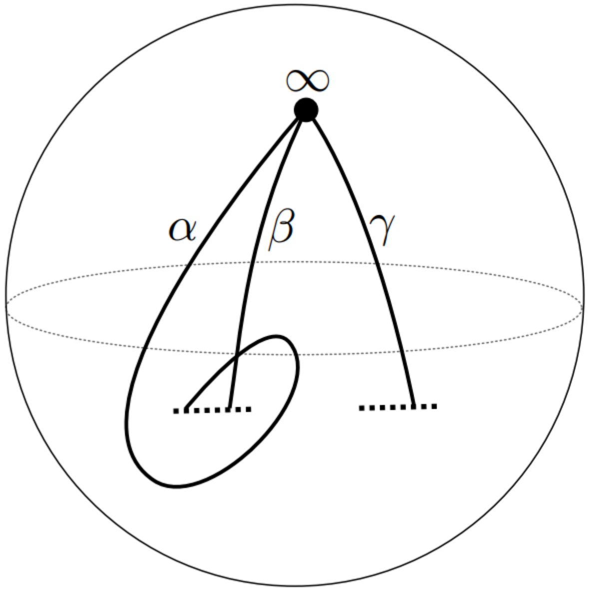

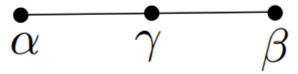

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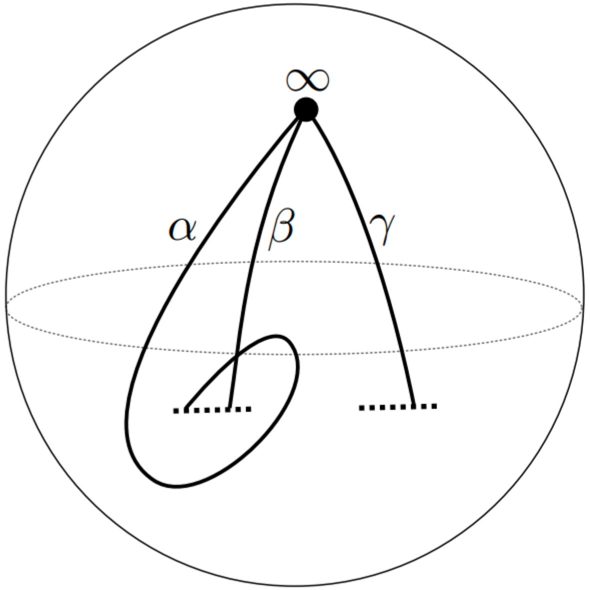

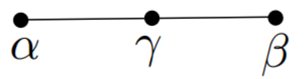

$\implies$ ray graph and $\mathcal{G}(\mathbb{R}^2 \setminus K)$ are the same!

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A clopen neighbourhood U of e is stable if for every clopen neighbourhood $V \subset U$ there is a clopen neighbourhood $U' \subset V$ such that $(U, U \cap \mathrm{Ends}_g(\Sigma)) \cong (U', U' \cap \mathrm{Ends}_g(\Sigma))$. We say that an end is stable if it admits a stable neighbourhood, and unstable otherwise. A surface is stable if all its ends are stable.

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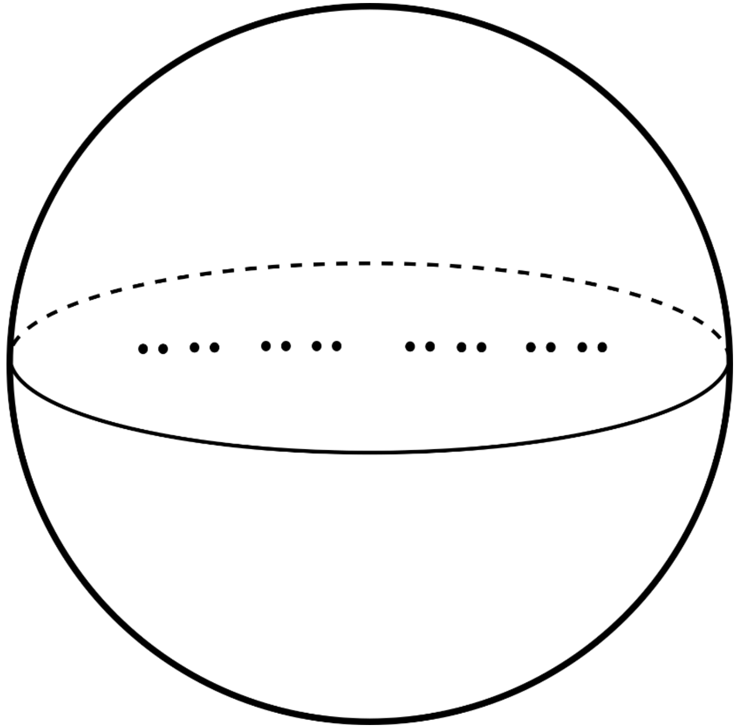

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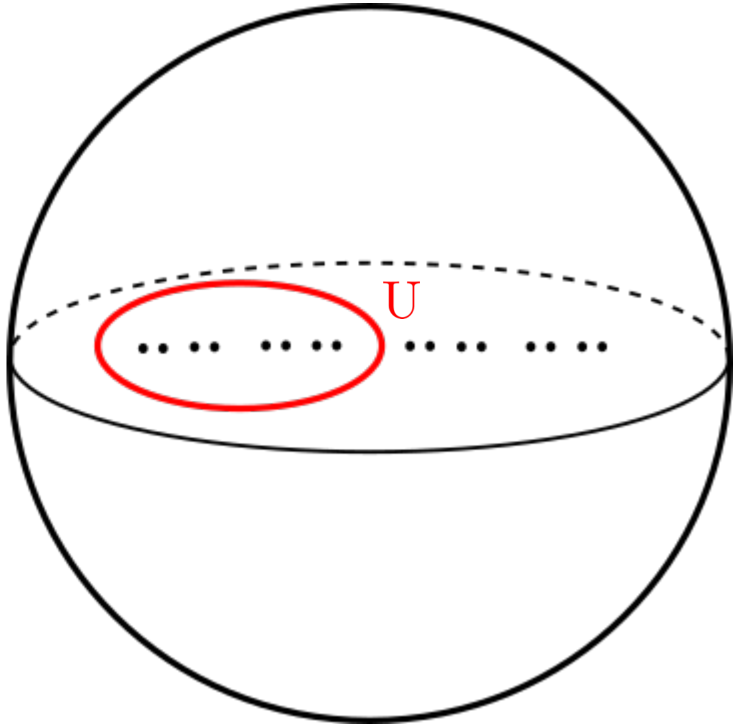

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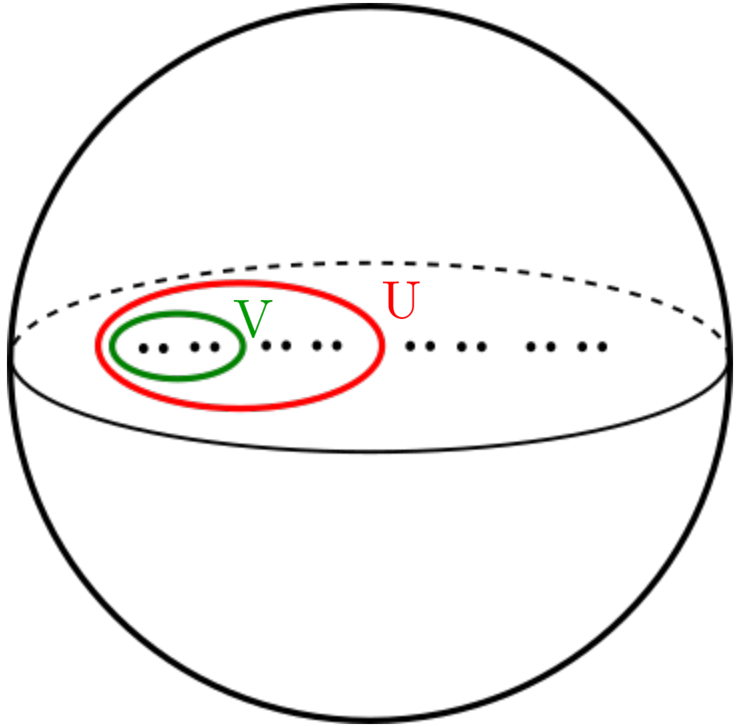

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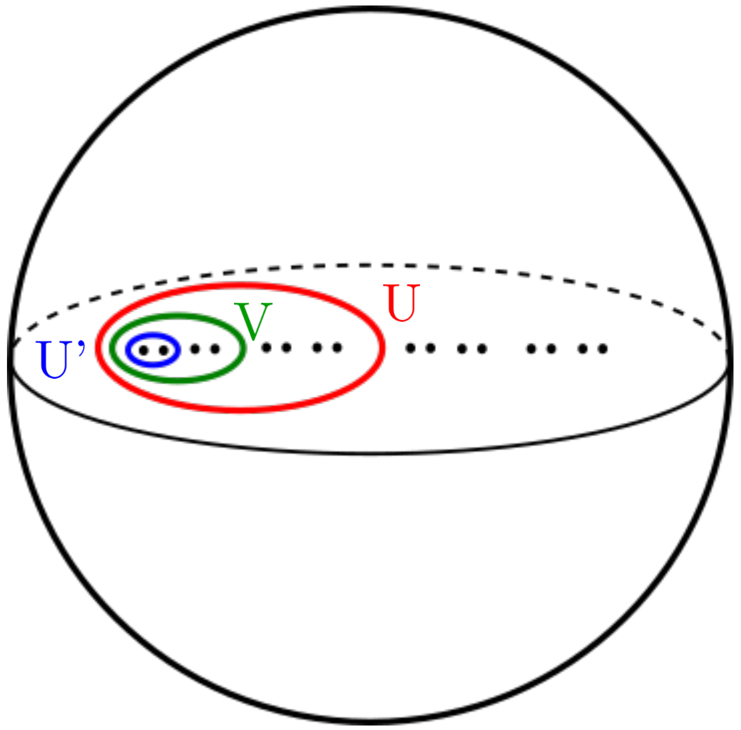

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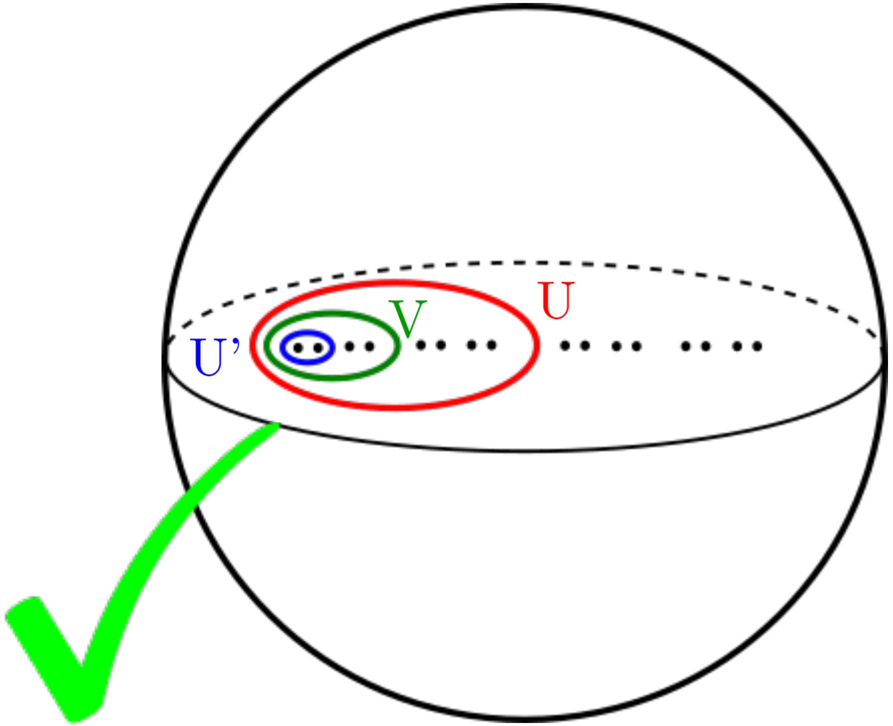

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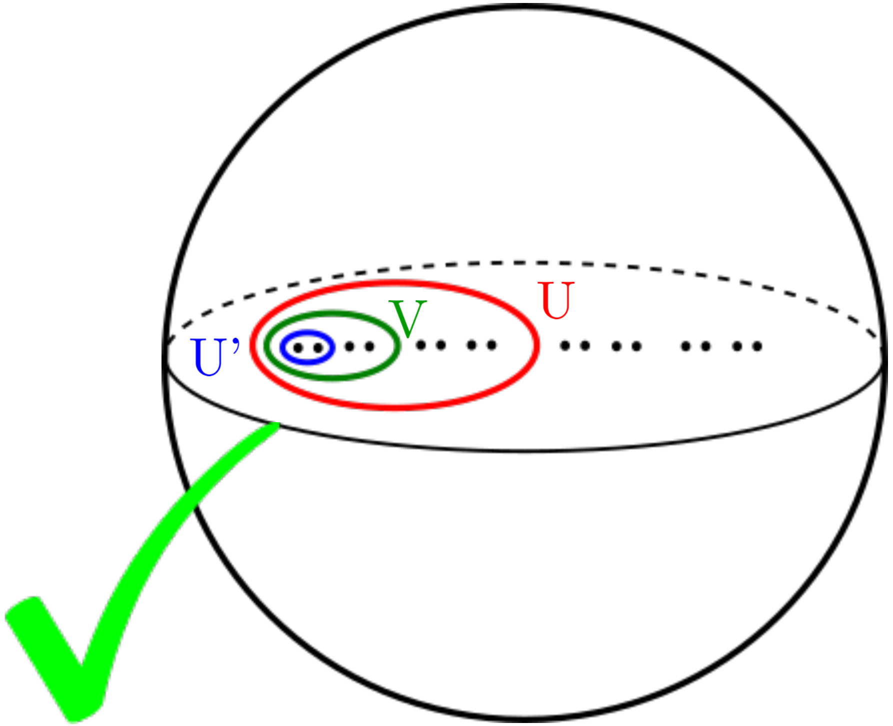

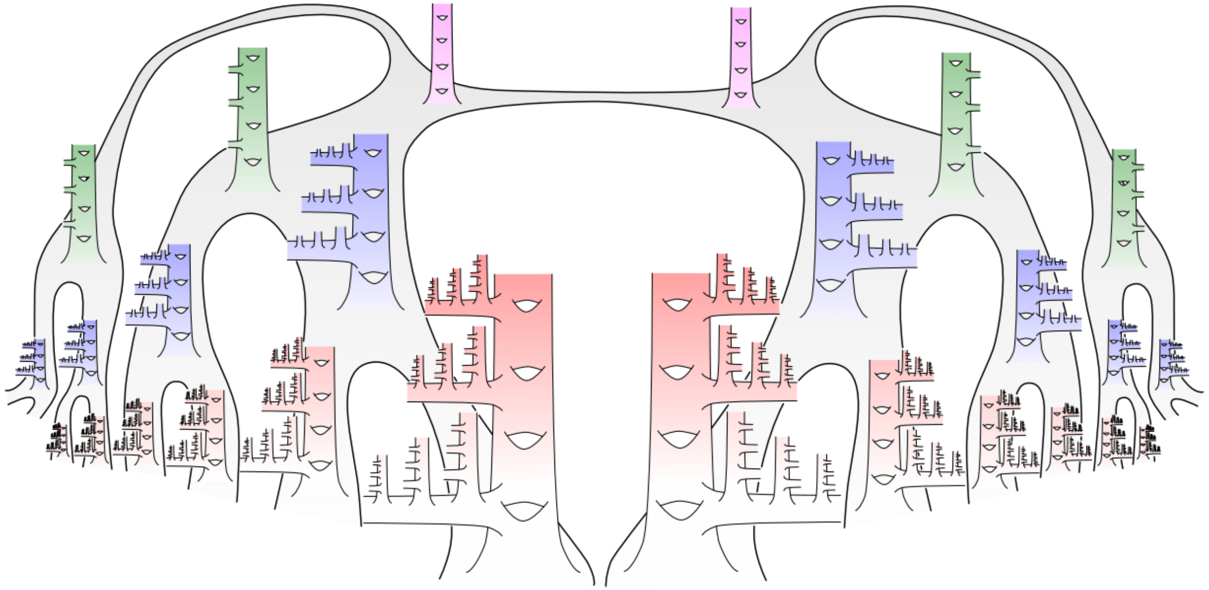

Image by Fanoni–Ghaswala–McLeay

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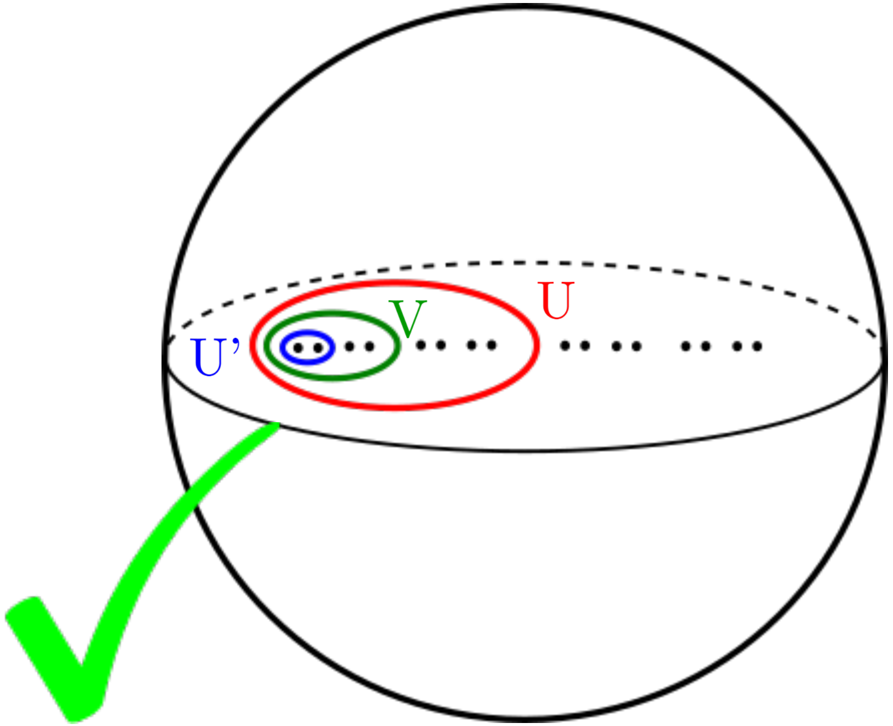

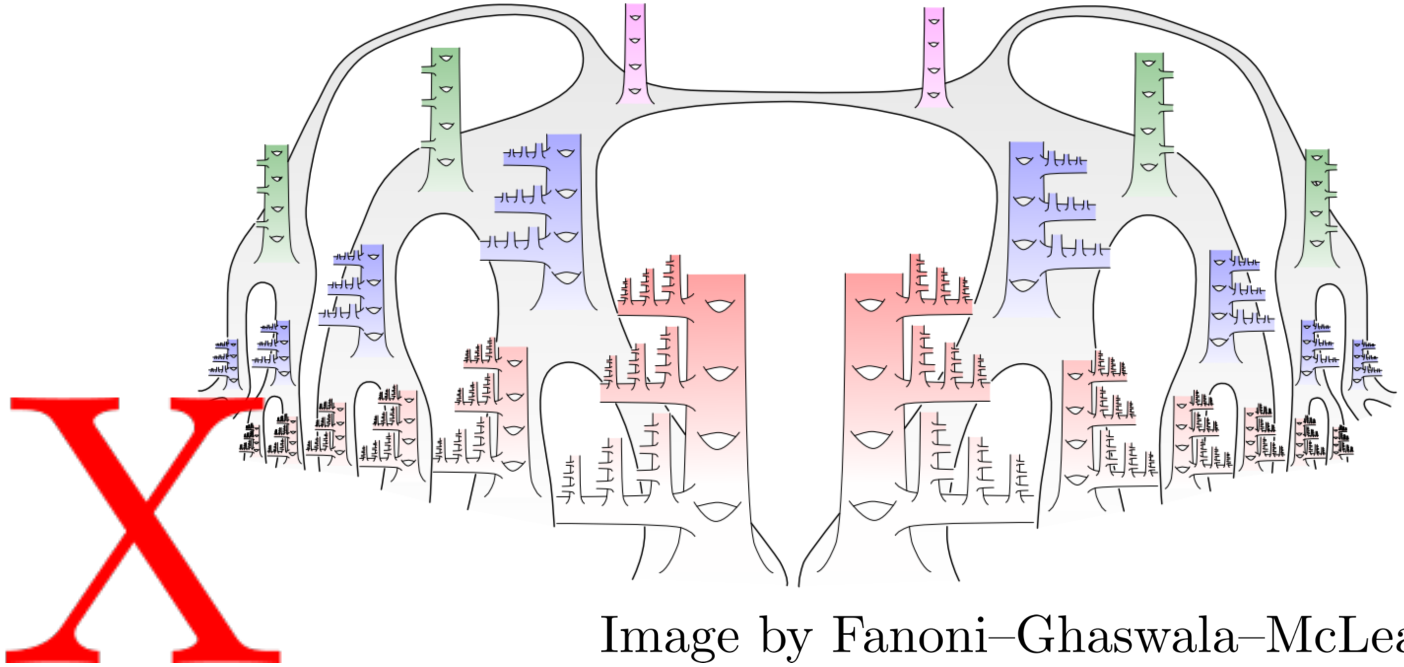

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Fact: Grand arcs aren't necessarily omnipresent

Overview

Theorem (Bar-Natan – V.): For a large class of surfaces, the grand arc graph is connected, hyperbolic, has infinite diameter, and $MCG(\Sigma)$ acts continuously on visible boundary.

+ loxodromic actions

+ nice connections to previous graphs