# The Semi-Simple Theory of Acylindricity in Higher Rank


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"A semi-simple object is one that can be decomposed into a sum of simple objects."

Semi-simplicity Wikipedia

## Outline:

I. Gromov product, $\delta$-hyp.

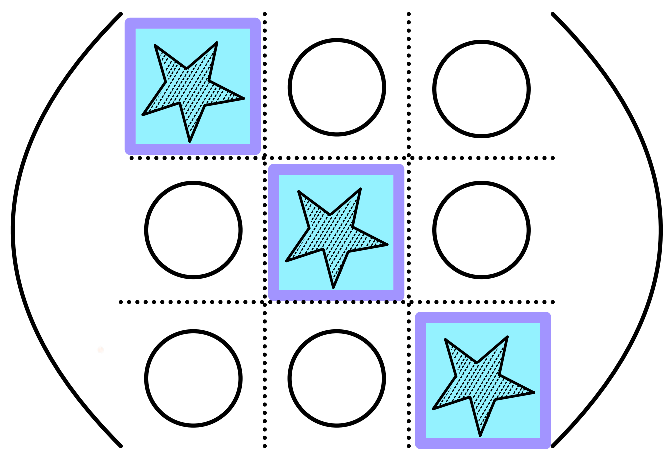
II. $\mathrm{Aut}(X)$, $X = \prod_{i=1}^{D} X_i$

III. Acylindricity & examples

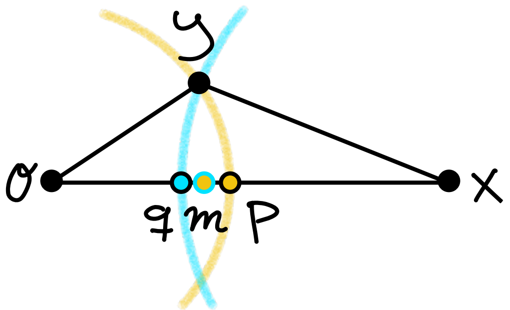
IV. Tits' Alternative

V. Product decompositions

VI. $\mathrm{Out}(\Gamma)$ (partial resolution to Sela's Conjecture 2023)



A geometric construction in geodesic space:




$d(o,y) < d(y,x) < d(x,o)$

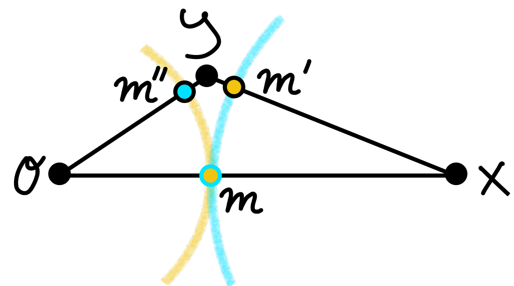
1) Find $p \in [o,x]$ st.
$$d(o,p) = d(o,y)$$

2) Find $q \in [o,x]$ st.
$$d(x,q) = d(x,y)$$

3) Find $m$ midpoint of the segment $[p,q]$.

Def: Gromov product:
$$\langle x,y\rangle_o = \tfrac{1}{2}\left[d(x,o) + d(o,y) - d(x,y)\right] \geq 0$$

* Measure of how far $o$ is from being on $[x,y]$




* In this example,

$$\langle x,y\rangle_o = d(o,m)$$

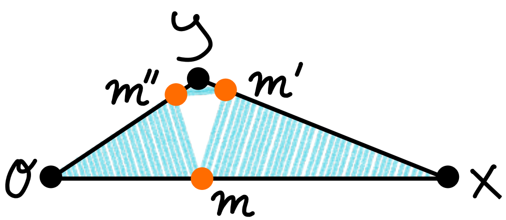
$$\langle y,o\rangle_x = d(x,m)$$

$$\langle o,x\rangle_y = d(y,m') = d(y,m'')$$

**Def:** $X$ geodesic metric space is $\delta$-hyperbolic if

$\forall\, o, x, y \in X$

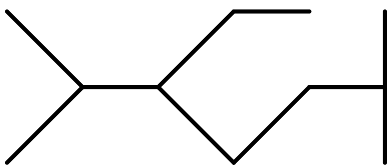
$\forall\, [o,x], [x,y], [y,o]$ geodesics




all distances $\triangle \leq \delta$.

**Example:** Trees ; $\delta = 0$

$T$ :



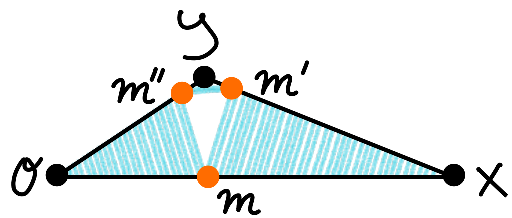
$\ast$ Class not closed under direct products

**Def:** $X$ geodesic metric space is CAT(0) if

nonpositively curved

$\forall\, o, x, y \in X$

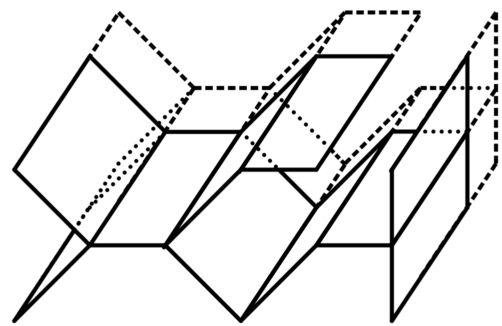
$\forall\, [o,x], [x,y], [y,o]$ geodesics




all distances $\triangle \leq$ Euclidean

**Example:** Products of trees

$T \times \mathbb{R}$



$\ast$ Class is closed under direct products

**Def:** $\Gamma$ group is $\delta$-hyp/CAT(0)

if $\Gamma$ admits a geometric action on $X$ is $\delta$-hyp/CAT(0)

i.e $\exists\ \Gamma \to \mathrm{Isom}\, X$ s.t.

**Proper:** $\forall x \in X\ \forall r > 0$

$\#\{\gamma \in \Gamma : d(\gamma x, x) \le r\} < \infty$

**Co-compact:** $\Gamma \backslash X$ compact.

⚠ proper co-compact makes $\Gamma$ a uniform lattice

✶ These spaces and groups are

$\boxed{\text{nonpositively curved}}$

**Objective:** Expand universe of nonpositively curved spaces & nonpositively groups:

$X_1, \ldots, X_D$ are $\delta$-hyperbolic

$$\mathbb{X} := \prod_{i=1}^{D} X_i$$

$$\mathrm{Aut}\,\mathbb{X} := \underbrace{\mathrm{Sym}_X{}^D}_{} \ltimes \prod_{i=1}^{D} \mathrm{Isom}\, X_i$$

(permutes isometric factors)

$$\cong \mathrm{Isom}(\mathbb{X}, \ell^1\text{-product})$$

Provided $X_i \underset{QI}{\not\sim} \mathbb{R}$.

$\boxed{\text{TH:}}$ (Bowditch, Eskin-Farb, Kapovich-Kleiner-Leeb)

If $\varphi : \mathbb{X} \to \mathbb{X}$ is a quasi-isom.

then $\exists\, \sigma \in \mathrm{Sym}\,\mathbb{X}$, $\varphi_i \in QI(X_i)$

$\varphi$ is uniformly close to $\sigma \circ (\varphi_1, \ldots, \varphi_D)$.

**Def:** $\Gamma \to \mathrm{Aut}(X)$ is

- **uniformly proper** if
  $\forall \epsilon > 0 \ \exists N > 0 \ \forall x \in X$ st
  $\#\{\gamma : d(\gamma x, x) \leq \epsilon\} < N$

- **proper** if
  $\forall \epsilon > 0 \ \exists \cancel{N > 0} \ \forall x \in X$ st
  $\#\{\gamma : d(\gamma x, x) \leq \epsilon\} < \cancel{N} \ \infty$
  ojo

- **nonuniformly proper** if
  proper & not uniformly proper.

**Def:** $\Gamma \to \mathrm{Aut}(X)$ is

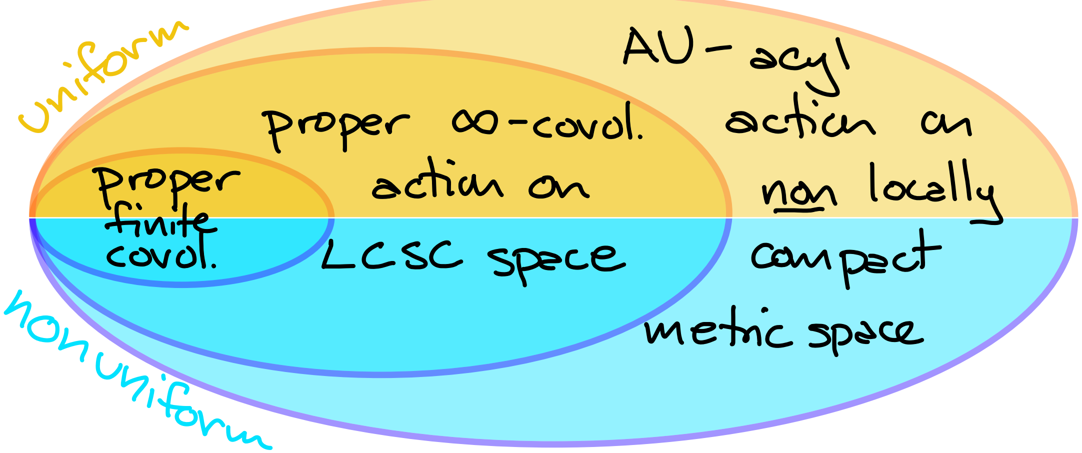
- **acylindrical** if $\forall \epsilon > 0 \ \exists R, N > 0$ st
  $d(x,y) \geq R \Rightarrow \#\{\gamma : d(\gamma x, x), d(\gamma y, x) \leq \epsilon\} < N$

- **AU-**(Ambiguous Uniformity) **acylindrical** if $\forall \epsilon > 0 \ \exists R, \cancel{N} > 0$ st
  $d(x,y) \geq R \Rightarrow \#\{\gamma : d(\gamma x, x), d(\gamma y, x) \leq \epsilon\} < \cancel{N} \ \infty$
  ojo

- **nonuniformly acylindrical** if
  AU-acylindrical & not acylindrical.

## Philosophy:

AU-acylindricity generalizes notion of a lattice, or proper finite covolume actions



# Examples of groups admit unbounded AU-acyl. action on non-positively curved space

- $\Gamma < SL_n\overline{\mathbb{Q}}$ finitely gen'ed

  $n=2 \rightsquigarrow$ action on $X$

- Corollary to Dowba:

  $\Gamma < SU_2 \times SU_2$

  finitely generated.

  (action on $X$)

- $D=1$ : Acylindrically hyperbolic
  - Mapping Class Groups
  - Relatively hyperbolic groups

    $G * \mathbb{Z}$, $G$ any group.

- Petyt-Spriano:

  $\Gamma$ a Higherarchically Hyperbolic Group (HHG)

  - Many (all?) Cubulated groups
  - Many Acyl. hyp. groups
  - Margolis:

    Groups QI to $X$

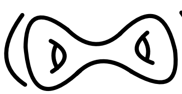
- Bestvina Brady Kernels:

  $K \xhookrightarrow[\text{isom}]{} F_2 \times F_2$

  (not finitely presented)

- Groups with prop(QT)

  $\Rightarrow$ No "best" representation

Ex: $\Gamma = MCG($$)$ $D=1$ ✓

& $\exists$ $\Gamma \longrightarrow \prod_{i=1}^{D} \mathrm{Isom}\, T_i$ → Quasi-trees

s.t. orbits are QI emb $\Rightarrow$ AU acyl!

Class is closed under:

- Finite extensions
- direct products
- subgroups

ojo class of $\Gamma < F_2$ is countable
class of $\Gamma < F_2 \times F_2$ is uncountable

Which groups do not admit AU-acylindrical action on $X$?

- Grigorchuk groups
- Burnside groups
- Tarski Monsters
- Thompson Groups

Groups with hereditary (NL)

- Groups with "hereditary NL"

only parabolic & elliptic actions even up to finite index.

Question: What can be said about such a large class of groups?

Answer: Not much without some restrictions on action:

- The trivial action $\Gamma \to \mathrm{Isom}\{*\}$ is acylindrical.
- Every (countable) group admits a proper parabolic action via the following "horoball" construction.

(& proper $\Rightarrow$ AU-acyl.)

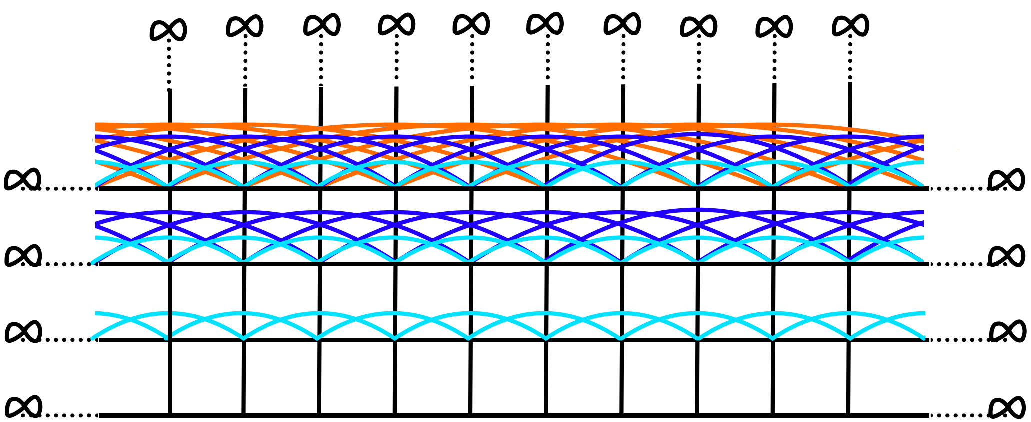
$\Gamma$ countable $\Rightarrow$ $\exists$ graph $Y_0$ on which $\Gamma$ acts properly.

Inductively, let $Y_n$ be the graph obtained by connecting points at distance $2^n$.

Define:

$$X := \bigcup_{n\in\mathbb{N}} \{n\} \times Y_n$$

Example $\Gamma = Y_0 = \mathbb{Z}$




"upper half plane model"

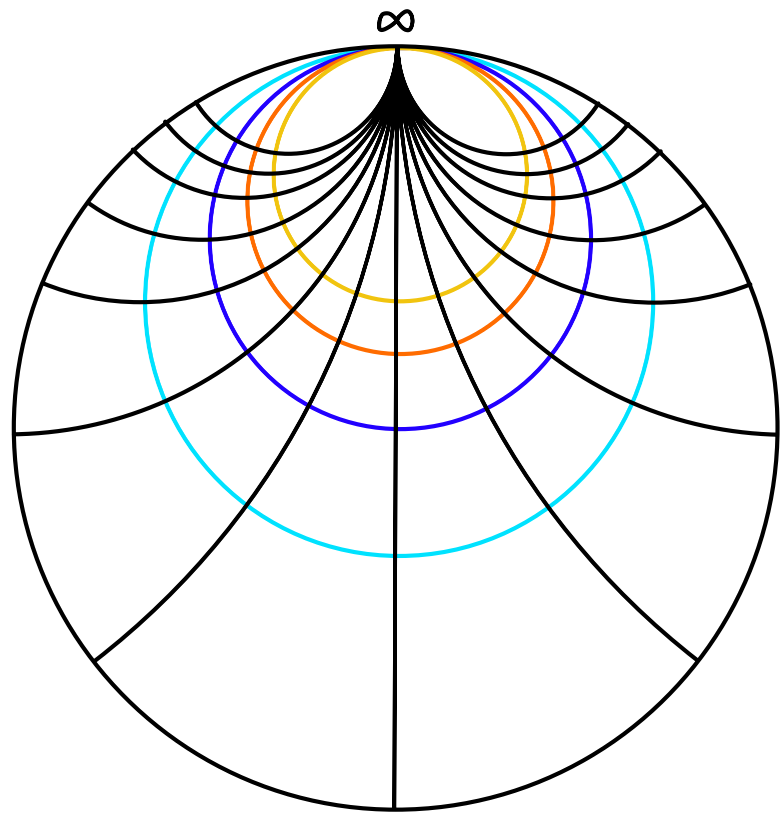
Rescaling:

length$( \;\; ) = \frac{1}{2}$

length$( \;\; ) = \frac{1}{2^2}$

length$( \;\; ) = \frac{1}{2^3}$

obtain "disK/ball model"



$\boxed{\text{TH:}}$ Tits' Alternative for linear groups (1972)

If $\Gamma < SL_n\mathbb{C}$ then either $\Gamma > F_2$ or $\exists\, \Gamma_0 \underset{\text{fin.}}{<} \Gamma$ s.t. up to conjugation $\Gamma_0$ is upper triangular:

$$(n=4) \quad \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & * \end{pmatrix}$$

($\Rightarrow \Gamma_0$ is solvable.)

- Tits' alternative (& its proof) is a powerful tool in the study of linear groups.
- There are many versions for different classes of groups.
- Open problem:

  Tits' Alternative for subgroups of CAT(0) groups.

$\boxed{\text{TH:}}$ (Tits' Alternative)

$\Gamma \to \mathrm{Aut}(X)$ acylindrical & not elliptic. Then either $\Gamma \geq F_2$ or $\Gamma \underset{\text{virt.}}{\simeq} \mathbb{Z}^K$, $0 \leq K \leq D$.

$\boxed{\text{TH:}}$ (Tits' Alternative)

$\Gamma \to \mathrm{Aut}(X)$ $\boxed{\text{acylindrical}}$

& not elliptic. Then either

$\Gamma \geq F_2$ or $\Gamma \underset{\text{virt.}}{\sim} \mathbb{Z}^K$, $0 \leq K \leq D$.

Proof scheme:

Step 1: Apply classification of actions on $\delta$-hyp. spaces to factors & show that elliptic & parabolic actions don't "contribute" to acyl. quasi-parabolic factor in step 2 becomes lineal.

Step 2: If $\Gamma \not\geq F_2$ then the factor actions lineal & may assume $X_i \underset{QI}{\sim} \mathbb{R}$ for $i = 1, \ldots, D$.

Step 3: Show $\Gamma$ is amenable (every finitely gen'ed subgroup has polynomial growth)

Step 4: Replace $X_i$ with $\mathbb{R}$ via Busemann (amenability) ~~pseudo~~ character & show acylindricity preserved.

Step 5: $\mathbb{R}^D$ locally compact so acyl. $\Rightarrow$ proper $\Rightarrow$ $\Gamma \underset{\text{virt.}}{\sim} \mathbb{Z}^K$

$0 \leq K \leq D$

**Def:** An internal product decomp.

$$\Gamma = \Gamma_1 \cdots \Gamma_K$$

is cannonical if whenever

$\Gamma = \Lambda_1 \cdots \Lambda_L$ then $L = K$ &

up to permutation $\Lambda_i = \Gamma_i$.

is strongly cannonical if

whenever $\Gamma' < \Gamma$ finite index

& $\Gamma' = \Gamma_1' \cdots \Gamma_L'$ then $L = K$

& up to perm. $\Gamma_i' = \Gamma_i \cap \Gamma'$.

Non-example: $\mathbb{Z}^2 \cong \mathbb{Z} \times \mathbb{Z}$

factors are cannonical but

$\Gamma_1 = \{(n,0) : n \in \mathbb{Z}\}, \Gamma_2 = \{(n,n) : n \in \mathbb{Z}\}$

$$\mathbb{Z}^2 = \Gamma_1 \cdot \Gamma_2$$

**Def:** $\mathrm{Aut}(\Gamma)$ is the group of isomorphisms $\Gamma \to \Gamma$.

$H < \Gamma$ is characteristic if $\varphi(H) = H \;\; \forall \varphi \in \mathrm{Aut}(\Gamma)$

$\boxed{\text{TH:}}$ Let $\Gamma$ be fin. gen'ed & $\Gamma \to \mathrm{Aut}(X)$ AU acyl, with general type factors.

$\exists$ characteristic subgroups

$$A \triangleleft \Gamma_1 \triangleleft \Gamma$$

$$|A| < \infty \;,\; [\Gamma : \Gamma_1] < \infty$$

st. $\Gamma_1 / A$ admits a strongly cannonical product decomp.

$\boxed{\text{TH:}}$ Let $\Gamma$ be fin. gen'ed & $\Gamma \to \mathrm{Aut}(X)$ AU acyl, with general type factors.

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$$A \lhd \Gamma_1 \lhd \Gamma$$

$$|A| < \infty \quad , [\Gamma : \Gamma_1] < \infty$$

st. $\Gamma_1 / A$ admits a strongly cannonical product decomp.

Proof Scheme:

Step 1: Must make characteristic, ±:

$$A = \mathrm{Ker}(\Gamma \to \mathrm{Aut}\, X)$$

$$\Gamma_0 = \mathrm{Ker}(\Gamma \to \mathrm{Sym}_X D)$$

Step 2: If $\Gamma_0' = \Gamma_0 / A$ not strongly irreducible then virtually isomorphic to $\Gamma_0' = H_1 \cdot H_2$.

⚠ $D = \#$ of factors gives bound on complexity

Step 3: Using this bound $\exists$ characteristic $\Gamma_1 \lhd \Gamma_0$ with strongly cannonical product decomp.

**Def:** $\Gamma$ is virtually isomorphic to $\Lambda$

$$\exists A \triangleleft \Gamma_0 \triangleleft \Gamma,\ B \triangleleft \Lambda_0 \triangleleft \Lambda$$

$$|A|, |B|, [\Gamma : \Gamma_0], [\Lambda : \Lambda_0] < \infty$$

$$\Gamma_0/A \cong \Lambda_0/B$$

**Def:** The inner automorphism group of $\Gamma$ is the image of $\Gamma \to \mathrm{Aut}(\Gamma)$, where

$$\gamma \mapsto C_\gamma, \quad C_\gamma(x) := \gamma x \gamma^{-1}$$

The outer automorphism group of $\Gamma$ is

$$\mathrm{Out}(\Gamma) = \mathrm{Aut}(\Gamma)/\mathrm{Inn}(\Gamma)$$

$\boxed{\text{TH:}}$ Let $\Gamma$ be fin. gen'ed & $\Gamma \to \mathrm{Aut}(X)$ AU acyl, with general type factors,

Up to virtual isomorphism

$$\mathrm{Out}(\Gamma) \underset{\substack{\uparrow \\ \text{finite index}}}{\gtrsim} \prod_{i=1}^{k} \mathrm{Out}(\Gamma_i)$$

As a corollary we (partially) resolve Sela's Conjecture (2023):
If $\Gamma$ is a "nice" HHG then $\mathrm{Out}(\Gamma)$ has a similar structure to $\mathrm{Out}(\Lambda)$, $\Lambda$ is 1-ended hyperbolic.

Answer: True if & only if true for irreducible factors & ...

# Open problems:

- What can you say about (AU-) acylindrical actions on non-positively curved spaces? CAT(0) cube cpx:

  Chatterji - Martin 2016
  Genevois 2019

  rank 1

  & work in progress
  Balasubramanya - Fernós
  higher rank.

- Which groups admit an "interesting" action on a nonpositively curved space but do not admit AU-acyl on a nonpositively curved space?

- Are your favorite results true for products & subgroups?

- Are there interesting properties extending acyl?

- If $\Sigma, \Sigma'$ are sufficiently complex surfaces then (it seems) $MCG(\Sigma) \times MCG(\Sigma') \leq_{fin} MCG(\Sigma \times \Sigma')$

  What can you say about the subgroups?

- If $\Gamma$ not acyl. hyperbolic & $\Gamma \to Aut(X_1 \times X_2)$ AU-acyl. can $Out(\Gamma)$ be $\infty$?

- Assume $\Gamma \to Aut(X_1 \times X_2)$ is AU-acyl. & $\exists \mu \in Prob(\Gamma)$ s.t. Furstenberg-Poisson bound. $(B, \nu) \cong (\partial X_1 \times \partial X_2, \nu_1 \otimes \nu_2)$.

  Does $\Gamma$ satisfy normal subgroup theorem? Maybe with additional hypotheses?

- ...

# THANKS!