The Conformal Dimension and Minimality of Stochastic Objects

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Quasisymmetric mappings and conformal dimension

A homeomorphism $f\colon X \to Y$ is η-quasisymmetric, where $\eta : [0, \infty) \to [0, \infty)$ is a homeomorphism, if

$$\frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \leq \eta\left(\frac{d_X(x, y)}{d_X(x, z)}\right)$$

for all $x, y, z \in X$ with $x \neq z$. The map f is called quasisymmetric if it is η-quasisymmetric for some distortion function η.

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X is minimal if $\dim_C(X) = \dim(X)$.

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- It may be applied to Universality theory. Given a discrete model that converges to a random objects as the scaling limit, people may wonder whether the same convergence can be obtained when we "distorted" the discrete model.
- Sullivan dictionary in probability. There are similarities between metric spaces that raised in geometric group theory, complex dynamics and probability. The community are searching the third line(probability) of Sullivan dictionary.

Examples and results

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- $M^n, \dim(M^n) = \dim_C(M^n) = n.$
- $SG, \dim(SG) = \frac{\log 3}{\log 2}, \dim_C(SG) = 1.$(Tyson, Wu)

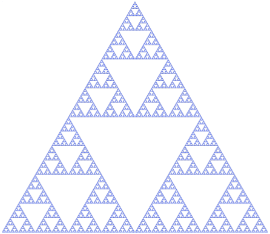
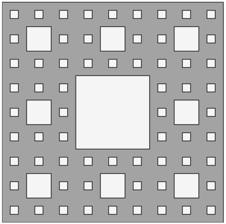


Figure: A Sierpiński Gasket

Examples and results

- $S_3, \dim(S_3) = \frac{\log 8}{\log 3}, 1 + \frac{\log 2}{\log 3} \leq \dim_C(S_3) < \frac{\log 8}{\log 3}$. Open



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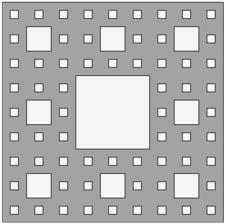

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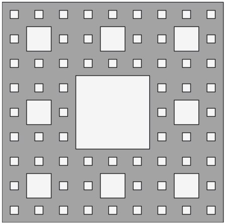

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S_p is quasisymmetric to S_q iff $p = q$. (Bonk, Merenkov)

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- $\exists X \subset [0,1]$ s.t. $\dim(X) = \dim_C(X) = 1$ and $\mathcal{H}^1(X) = 0$.(Hakobyan)
- For any $\alpha \geq 1$, there exists a X such that $\dim(X) = \dim_C(X) = \alpha$.(Bishop, Tyson)

Construction of minimal spaces:
Let X be an Q-Ahlfors regular space, say, a cantor set($Q < 1$) or a snowflake($Q \geq 1$), then $X \times [0,1]$ is minimal.

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- The lower bound: Much harder.

The main obstacle of computing the conformal dimension is to estimate a lower bound.

Modulus estimation

One criterion for conformal dimension lower bounds starts from the idea of "sufficiently rich curve family", i.e., the existence of a family of curves with some positive modulus.

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Theorem 1 (Bishop,Tyson)

Let (X, d, μ) *be a compact, doubling metric measure space satisfying:*

- $\mu(B(x,r)) \leq C \cdot r^q$ *for all balls* $B(x,r) \subset X$ *and some* $C > 0$.
- $\mathrm{Mod}_p(\Gamma) > 0$ *for some* $1 < p \leq q$ *and some curve family* $\Gamma \subset X$.

Then $\dim_C(X) \geq q$.

Fuglede modulus

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Let (X, d, μ) be a metric measure space and $p \geq 1$. The p-modulus of a family of measures $\mathbf{E}$ on X is defined as

$$\mathrm{Mod}_p(\mathbf{E}) = \inf \int_X \rho^p \, d\mu$$

where the infimum is taken over all Borel functions $\rho : X \to [0, \infty)$ such that $\int \rho \; d\lambda \geq 1, \;\; \forall \;\; \lambda \in \mathbf{E}$.

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Fuglede modulus is an outer measure of measures on X.

Fuglede modulus estimation

For this to work one has to assume that X contains a family $\mathcal{E} = \{E_i\}_{i \in I}$ of subsets and a family of measures $\mathbf{E} = \{\lambda_i\}_{i \in I}$ so that λ_i is supported on E_i which are essentially 1-dimensional, and such that the modulus of $\mathbf{E}$ is positive.

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Theorem 2 (Hakobyan)

Let (X,μ) be a compact, doubling metric measure space satisfying

- $\mu(B(x,r)) \le C\cdot r^q$ *for every ball* $B(x,r)\subset X$ *for some constant* $C>0$.
- $\dim_C E \ge 1,\ \forall\ E\in\mathcal{E}$ *for some collection subsets* $\mathcal{E}$ *of* X.
- *there exists a collection of measures* $\mathbf{E} = \{\lambda_E\}_{E\in\mathcal{E}}$ *supported on* E *such that for* $\forall\ E\in\mathcal{E}$, $x\in E$ *and some* $C>0$
$$\lambda_E(B(x,r)\cap E)\ge C\cdot r.$$
- $\mathrm{Mod}_p(\mathbf{E})>0$ *for some* $1\le p\le q$

Then $\dim_C(X)\ge q$.

Cantor set conformal dimension

Theorem 3 (Binder, Hakobyan and L.)

Let $E = \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{2^i} E_{i,j}$ *be a metric cantor space. If*

1. *there exists a constant* $L \geq 1$ *such that for any* $E_{i,j}, E'_{i,j}$ *we have*

$$\frac{1}{L} \leq \frac{\mathrm{diam}(E_{i,j})}{\mathrm{diam}(E'_{i,j})} \leq L,$$

2. $\lim_{i \to \infty} \Delta(E_{i,j}, E'_{i,j}) \to 0,$

then $\dim_C(E) \geq 1$.

A minimal space

Let $I_0 = [0,1] \times [0,1]$ be the unit square in $\mathbb{R}^2$. We denote by

$$\mathcal{Q}_n := \left\{ \left[\frac{i}{4^n}, \frac{i+1}{4^n} \right] \times \left[\frac{j}{2^n}, \frac{j+1}{2^n} \right] \right\}_{i,j}$$

and call any element in $\mathcal{Q}_n$ a *n-block*.

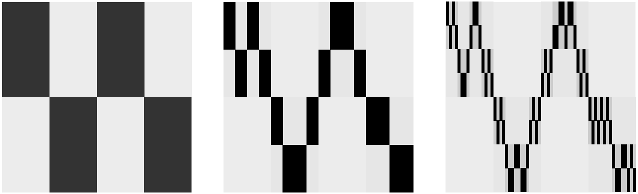
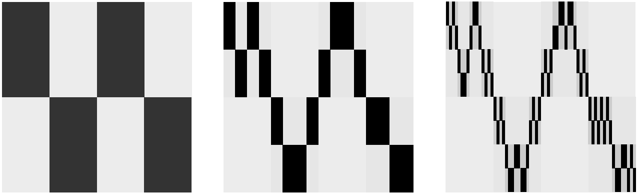


Figure: A minimal graph with 3 generations.

A minimal space

- We choose two 1-blocks from the upper half and two 1-blocks from the lower half of I_0, and call their union I_1.
- Suppose that I_n is already constructed and $I_n = \bigcup_i Q_i$ where $Q_i \in \mathcal{Q}_n$. Similarly, we choose two $(n+1)$-blocks from the upper half and two $(n+1)$-blocks from the lower half of each Q_i.
- Then I_{n+1} is the union of all the chosen $(n+1)$-blocks.
- Finally, we define $I := \bigcap_{n=0}^{\infty} I_n$.



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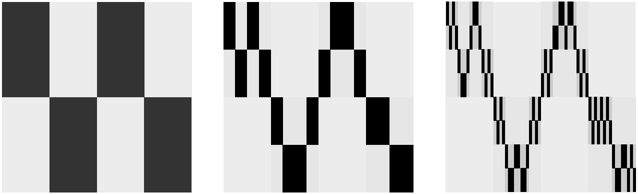
Remark: It is possible to construct I as a graph by specifically choosing I_n in each step.

A minimal space

Some Facts: $dim(I_a) = \frac{1}{2}$ for every $a \in [0,1]$, and $\dim(I) = \frac{3}{2}$.



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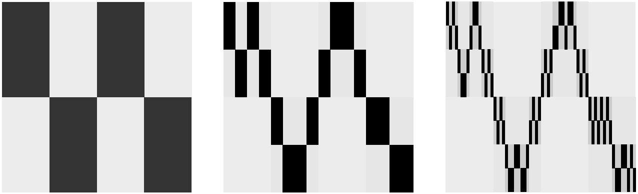

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Lemma 4 (Mass distribution principle)

$\mu(U) \le C \cdot (\mathrm{diam}(U))^{\alpha}, \mu(X) > 0 \Longrightarrow \dim(X) \ge \alpha.$

It is clear that the Minkowski dimension of them are $\frac{1}{2}$ and $\frac{3}{2}$, respectively. Letting μ and μ_x be probability measures on I and I_x that distributed uniformly, then, by Mass distribution principle, we finish the proof.

A minimal space

Recall that *Marstrand Slicing Theorem* asserts that for any $A \subset \mathbb{R}^2$ with $\dim(A) \geq 1$ and any $A_x = \{x : (x, y) \in A\}$, we have $\dim(A_x) \leq \dim(A) - 1$ for almost every x. This implies some product-like structure in the random fractal I. The rigidity illustrates the following theorem.

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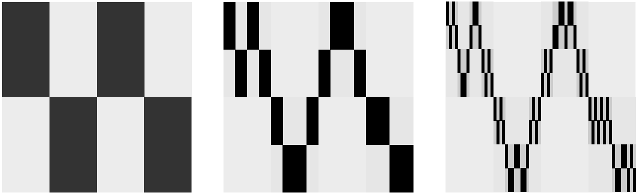
Theorem 5 (Binder, Hakobyan and L.)

I is minimal and $\dim_C(I) = \frac{3}{2}$.



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- $\dim(\Gamma_B) = \frac{3}{2}$ a.s..

- $\dim T(x) = \frac{1}{2}$ for every x a.s. for $T(x) = \{t : B(t) = x\}$.

Brownian motion and local time

A few results of 1-dimensional Brownian motion.

- we define $L^a(t)$ to be the local time of the standard Brownian motion $B(t)$ at level a, i.e.,

$$L^a(t) = \lim_{n \to \infty} 2^{-n+1} D^n(a, t)$$

where $D^n(a, t)$ is the number of downcrossings before time t of the n^{th}-dyadic interval containing a. Notice that $L^a(t)$ is a random field.

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- For any fixed a, there exists a constant $C > 0$ independent of a such that a.s. for all $t > 0$, the local time

$$L^a(t) = \mathcal{H}^{\varphi}\left(T(0) \cap [0,t]\right)$$

for the gauge function $\varphi(r) = C\sqrt{r \log(\log(1/r))}$.

Brownian motion and local time

The graph of 1-dimension Brownian motion "looks like" the example in Theorem 5. We explore the graph from this point of view to construct the product-like structure provided by the local time. This observation is in the heart of the proof of the following theorem.

Brownian motion and local time

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Theorem 6 (Binder, Hakobyan and L.)

Let $B(t)$ be a 1-dimension Brownian motion, then the graph of B is minimal a.s., i.e., the conformal dimension of Γ_B is $\frac{3}{2}$ a.s..

Structures on the graph of $B(t)$

- We define

$$\mu(A) := \int_{-\infty}^{\infty} l^a(A \cap Z_a)\, da.$$

- There exists $C > 0$ such that

$$\mu(B(x, r) \cap \Gamma(B)) \leq C \cdot r^{\frac{3}{2}-\epsilon}.$$

for any $\epsilon > 0$.

We construct a Cantor set E in the following way:

- Pick subset of $\Gamma(B)$ that lies between two adjacent hitting times of $Z_0, Z_{1/2}$ and another subset that lies between two adjacent hitting times of $Z_{1/2}, Z_1$.
- These two elements are the first generation of E.
- Suppose a n^{th}-generation element is given, then we pick two subsets that lies between two adjacent hitting times of two adjacent dyadic levels, receptively.
- All these 2^{n+1} elements forms the $(n+1)^{\text{th}}$-generation of E.
- Finally, $E = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{2^n} E^{n,m}$.
- We collect all the Cantor sets constructed by the above method and denote them by $\mathcal{E}$.

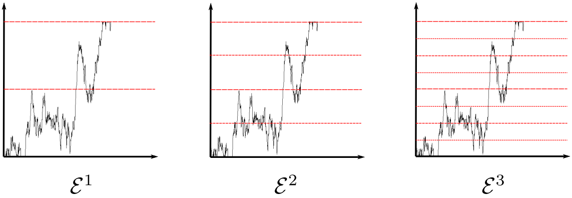


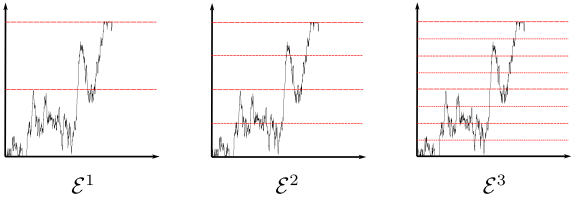
Figure: A standard linear Brownian motion and the collection of $E_j^{n,m}$ for 3 generations.

Proof of Theorem 6

Proof.

- There exists a measure μ s.t. $\mu(B(x,r)\cap\Gamma(B)) \lesssim r^{\frac{3}{2}-\epsilon}$ for any $\epsilon > 0$.
- For any $E \in \mathcal{E}, \dim_C E \geq 1$ by Theorem 3.
- Let λ_E be the project measure from y-axis to E thus it satisfies $\lambda_E(B(x,r)\cap E) \geq C \cdot r$ for some C naturally.

It follows from Theorem 2 that the only thing left is to prove that $\mathrm{Mod}_1(\mathbf{E}) > 0$.



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- We will show that $\mathrm{Mod}_1(\mathbf{E}) > 0$.
- Let ρ be admissible for $\mathbf{E}$. We replace ρ by an alternative admissible ϱ with smaller mass.
- Let ϱ_1 be a function on E_1 such that the integral of ρ_1 on every 1^{th}-generation element that achieves the minimal among all the 1^{th}-generations of the same level.
- Similarly, define ϱ_n iteratively in the same way to cover all the n^{th}-generation.
- Finally, we define $\varrho := \liminf_{n\to\infty} \varrho_n$.

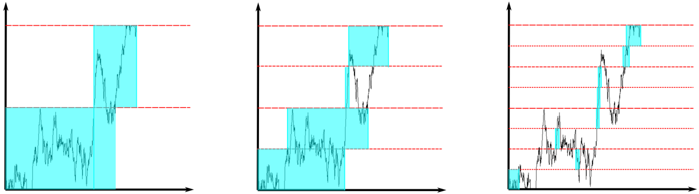
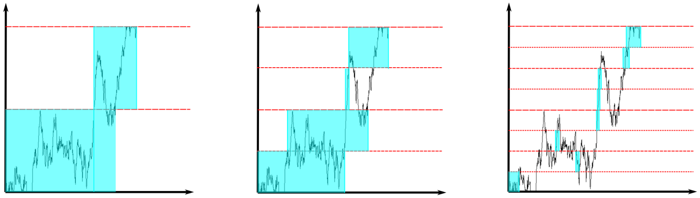


Figure: Constructing ϱ_n.

□

Proof of Theorem 6

- For any $E \in \mathcal{E}$,
$$\int_E \varrho d\lambda_E \geq \int_F \rho \lambda_F \geq 1.$$
Thus ϱ is admissible for $\mathcal{E}$.
- $\int_X \varrho d\mu \leq \int_X \rho d\mu$ where X is the space that $\mathcal{E}$ covers.
- It is sufficient to prove that $\int_X \varrho d\mu > 0$.
- Recall that $\mu(B(x,r)) = \int_0^1 l_a(B(x,r) \cap Z_a) da$.
-
$$\int_X \varrho d\mu \geq \left(\int_E \varrho d\lambda_E \right) \inf_{a \in [0,1]} L^a(Z_a cap X) \geq \inf_{a \in [0,1]} L^a(Z_a \cap X) > \delta.$$
for some $\delta > 0$.



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Further developments

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Higher dimensional Brownian motion? We guess it is not minimal a.s..

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The studies of the intersection of the SLE trace with $\mathbb{R}$ or semi-circles show that the SLE trace has a product-like structure.

We claim that the SLE_κ trace is minimal almost surely for $\kappa > 4$, i.e., the conformal dimension of the SLE_κ curve, $\kappa > 4$, is $1 + \frac{\kappa}{8}$ almost surely.

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Cannon Conjecture: Let G be a Gromov hyperbolic group whose boundary at infinity $\partial_\infty G$ is homeomorphic to $\mathbb{S}^2$. Then $\partial_\infty G$ is quasisymmetrically equivalent to $\mathbb{S}^2$.

Thurston Characterization Theorem: A Thurston map $f\colon S^2 \to S^2$ with hyperbolic orbifold is topological conjugate to a rational map if and only if it has no Thurston obstruction.

Further developments

Interesting ideas from geometric group theory and complex dynamics?

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Question: Is the Brownian sphere S quasisymmetric to $\mathbb{S}^2$?

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A dynamical Theorem[Bonk,L.,Li]: Suppose $f\colon S^2 \to S^2$ is an expanding Thurston map without periodic critical points, and $\mathcal{C} \subseteq S^2$ is an f-invariant Jordan curve with $\operatorname{post}(f) \subseteq \mathcal{C}$. Let F be a subsystem of f with respect to C and Ω be its tile maximal invariant set. If Ω is homeomorphic to the standard Sierpiński carpet, then the following conditions are equivalent:

1. There exists a postcritically-finite rational map $g\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with no periodic critical points and a g-invariant set $\Theta \subset \widehat{\mathbb{C}}$ such that $f|_\Omega$ is topologically conjugate to $g|_\Theta$.
2. The set Ω is quasisymmetrically equivalent to a round carpet in $\hat{\mathbb{C}}$.
3. There is no Thurston obstruction for F.

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Question: How is a corresponding Brownian carpet? Is the CLE_κ carpet quasisymmetrically equivalent to a round carpet?

Thank you!