Large deviations of the KPZ equation via the delta Bose gas

KPZ meets KPZ

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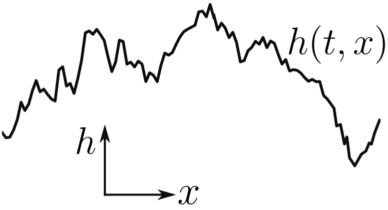
KPZ and SHE

Kardar–Parisi–Zhang (KPZ)

$$\partial_t h = \tfrac{1}{2}\partial_{xx} h + \tfrac{1}{2}(\partial_x h)^2 + \eta$$



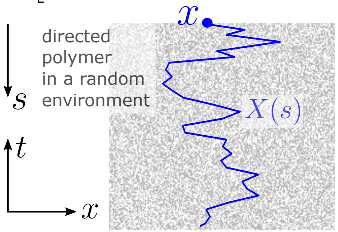
Stochastic Heat Equation (SHE)

$$\partial_t Z = \tfrac{1}{2}\partial_{xx} Z + \eta\, Z$$

Feynman–Kac:

$$Z(T,x) =$$

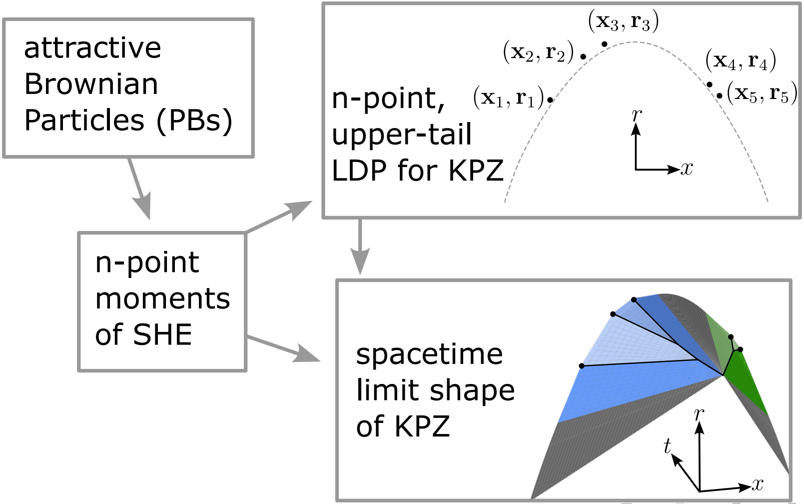
$$\mathbb{E}_{\mathrm{BM}}\Big[e^{\int_0^T \mathrm{d}s\ \eta(T-s, X(s))} Z(0, X(T))\Big]$$



$$e^{h(t,x)} = Z(t,x)$$

Plan for this talk

$Z(0,\cdot) = \delta_0$ for most of the talk



SHE moments ⇔ LDP of KPZ equation

$$\mathbf{E}\Big[\big(Z(T,0)e^{\frac{T}{24}}\big)^{\mathfrak{m}}\Big] = \mathbf{E}\Big[e^{\mathfrak{m}(h(T,0)+\frac{T}{24})}\Big] \approx e^{\frac{T}{24}\mathfrak{m}^3}, \quad \mathfrak{m} \in \mathbb{Z}_{>0}$$

- Proven by [Chen 15] by Feynman–Kac (for flat-like initial condition).
- Proven in [Corwin–Ghosal 20] by formulas.

[Das-T 21]

$$\mathbf{E}\Big[\big(Z(T,0)e^{\frac{T}{24}}\big)^{p}\Big] = \mathbf{E}\Big[e^{p(h(T,0)+\frac{T}{24})}\Big] = e^{\frac{T}{24}p^3}, \quad p \in \mathbb{R}_{>0}$$

which gives the upper tail LDP

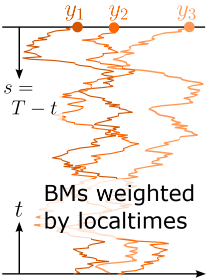
$$\mathbf{P}\Big[h(T,0) + \frac{T}{24} \approx \mathbf{r}\Big] \approx e^{-T\,\frac{4\sqrt{2}}{3}\mathbf{r}^{3/2}}, \quad \mathbf{r} \in \mathbb{R}_{>0}$$

SHE moments ⇔ attractive BPs

By Feynman–Kac,

$$\mathbf{E}\big[Z(T,y_1)\cdots Z(T,y_m)\big]$$

$$= \mathbb{E}_{\mathrm{BM}}\Big[e^{\int_0^T \mathrm{d}s\, \sum_{i<j}\delta_0(X_i - X_j)} \prod_{i=1}^{m} Z(0, X_i(T))\Big]$$



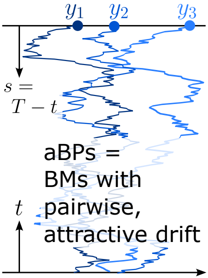
By Tanaka + Girsanov,

$$\mathbb{E}_{\mathrm{BM}}\Big[e^{\int_0^T \mathrm{d}s\, \sum_{i<j}\delta_0(X_i - X_j)}\,(\,\cdot\,)\Big]$$

$$= e^{\frac{T}{24}(m^3 - m)}$$

$$\mathbb{E}_{\mathrm{aBP}}\Big[e^{\sum_{i,j}\frac{1}{2}|X_i(0)-X_j(0)|}\,(\,\cdot\,)\,e^{-\sum_{i,j}\frac{1}{2}|X_i(T)-X_j(T)|}\Big]$$

$$\mathrm{d}X_i(s) = \sum_{j=1}^{m} \frac{1}{2}\mathrm{sgn}(X_j - X_i)\mathrm{d}s + \mathrm{d}B_i(s)$$



Another way to get the attractive BPs

Define $Q(t, y_1, \ldots, y_m) := \mathbf{E}\big[Z(t, y_1) \cdots Z(t, y_m)\big]$. By Itô,

$$\partial_t Q = \underbrace{\Big(\frac{1}{2} \sum_{i=1}^{m} \partial_{y_i}^2 + \sum_{i<j} \delta_0(y_i - y_j) \Big)}_{:= -H} \; Q$$

The Hamiltonian H has the ground state

$$\psi(y_1, \ldots, y_m) = \exp\Big(- \frac{1}{2} \sum_{i<j} |y_i - y_j| \Big)$$

Performing the ground-state transformation gives

$$\frac{1}{\psi}\big(- H\big)\psi = \frac{m^3 - m}{24} + \underbrace{\sum_{i \neq j} \frac{1}{2} \mathrm{sgn}(x_j - x_i) \partial_{x_i} + \frac{1}{2} \sum_{i=1}^{m} \partial_{x_i}^2}_{:= L, \text{ generator of attractive BPs}}$$

Multi-point moments

The goal of this talk is to

1. get

$$\mathbf{E}\Big[\prod_{\mathfrak{c}=1}^{n}\big(Z(T,T\mathbf{x}_{\mathfrak{c}})e^{\frac{T}{24}}\big)^{\mathfrak{m}_{\mathfrak{c}}}\Big]=\mathbf{E}\big[e^{\sum_{\mathfrak{c}=1}^{n}\mathfrak{m}_{\mathfrak{c}}(h(T,T\mathbf{x}_{\mathfrak{c}})+\frac{T}{24})}\big]\approx e^{T\,L_{\mathrm{SHE}}(\vec{\mathbf{x}},\vec{\mathfrak{m}})},$$

   for $\vec{\mathbf{x}}=(\mathbf{x}_1,\ldots,\mathbf{x}_n)\in\mathbb{R}^n, \vec{\mathfrak{m}}\in(\mathbb{Z}_{>0})^n$ and an explicit $L_{\mathrm{SHE}}(\vec{\mathbf{x}},\vec{\mathfrak{m}})$, and
2. then use the moments to get the LDP and limit shape (caveat to be explained).

[Lin 23]: Doing Step 1 by formulas.

High powers

We'll actually consider powers that grow to $+\infty$ with $T \to \infty$:

$$\mathbf{E}\Big[\prod_{\mathfrak{c}=1}^{n} \big(Z(T,0)e^{\frac{T}{24}}\big)^{\mathfrak{m}_{\mathfrak{c}} N}\Big] = \mathbf{E}\Big[e^{\sum_{\mathfrak{c}=1}^{n} \mathfrak{m} N (h(T,\mathbf{x}_{\mathfrak{c}})+\frac{T}{24})}\Big] \approx e^{TN^3 L_{\mathrm{SHE}}(\vec{\mathbf{x}},\vec{\mathfrak{m}})},$$

for $\vec{\mathbf{x}} \in \mathbb{R}^n$, $\vec{\mathfrak{m}} \in (\frac{1}{N}\mathbb{R}_{>0})^n$, and $N \to \infty$ at arbitrary rate relative to $T \to \infty$.

Why doing this?

$$\mathbf{E}\Big[\big(Z(T,0)e^{\frac{T}{24}}\big)^{\mathfrak{m}}\Big] = \mathbf{E}\Big[e^{\mathfrak{m}(h(T,0)+\frac{T}{24})}\Big] \approx e^{\frac{T}{24}\mathfrak{m}^3}, \quad \mathfrak{m} \in \mathbb{Z}_{>0}$$

High powers

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$$\mathbf{E}\Big[\prod_{\mathfrak{c}=1}^{n} \big(Z(T,0)e^{\frac{T}{24}}\big)^{\mathfrak{m}_\mathfrak{c} N}\Big] = \mathbf{E}\big[e^{\sum_{\mathfrak{c}=1}^{n} \mathfrak{m} N (h(T,\mathbf{x}_\mathfrak{c})+\frac{T}{24})}\big] \approx e^{TN^3 L_{\mathrm{SHE}}(\vec{\mathbf{x}},\vec{\mathfrak{m}})},$$

for $\vec{\mathbf{x}} \in \mathbb{R}^n, \vec{\mathfrak{m}} \in (\frac{1}{N}\mathbb{R}_{>0})^n$, and $N \to \infty$ at arbitrary rate relative to $T \to \infty$.

Why doing this?

$$\mathbf{E}\Big[\big(Z(T,0)e^{\frac{T}{24}}\big)^{N\mathfrak{m}}\Big] = \mathbf{E}\big[e^{N\mathfrak{m}(h(T,0)+\frac{T}{24})}\big] \approx e^{\frac{TN^3}{24}\mathfrak{m}^3}, \quad \mathfrak{m} \in \tfrac{1}{N}\mathbb{Z}_{>0}$$

- $\frac{1}{N}\mathbb{Z}_{>0} \to \mathbb{R}_{>0}$ which gives the full upper-tail LDP.
- But doing this changes the scale of deviations:

$$\mathbf{P}\big[h(T,NT\mathbf{x}_\mathfrak{c}) + \tfrac{T}{24} \approx TN^2\mathbf{r}_\mathfrak{c}, \mathfrak{c} = 1,\ldots,n\big] \approx \exp\big(-TN^3 I_{\mathrm{KPZ}}(\vec{\mathbf{x}},\vec{\mathbf{r}})\big)$$

LDP for the attractive BPs

$$\boldsymbol{\mu}_N(s) := \frac{1}{N}\sum_{i=1}^{N\mathfrak{m}} \delta_{X_i^N(s)}, \qquad \boldsymbol{\mu}_N \in \mathscr{C}\big([0,1],\, \mathfrak{m}\mathscr{P}(\mathbb{R})\big)$$

Take any $T = T_N$ with $N^2T = N^2T_N \to \infty$.

Theorem (T 23)

As $N \to \infty$, the empirical measure $\boldsymbol{\mu}_N$ satisfies an LDP on $\mathscr{C}([0,1], \mathfrak{m}\mathscr{P}(\mathbb{R}))$ with speed N^3T and an explicit rate function $\mathbb{I}$.

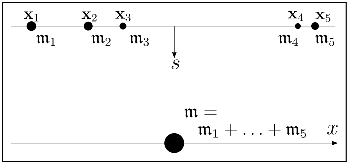
Remark

- Under the diffusive scaling, $N \to \infty$ and N^2T fixed, [Dembo–Shkolnikov–Varadhan–Zeitouni 16] proved the LDP for a general class of rank-based diffusions.
- The behavior under $N^2T \to \infty$ (considered here) is very different from that under the diffusion scaling (considered in [DSVZ 12]).

Back to moment Lyapunov exponents

Corollary

Under $Z(0,\bullet) = \delta_0$, $\quad \mathbf{E}\Big[\prod_{\mathfrak{c}=1}^{n} Z(T, NT\mathbf{x}_{\mathfrak{c}})^{N\mathfrak{m}_{\mathfrak{c}}}\Big] \approx e^{N^3 T \cdot L_{\mathrm{SHE}}(\vec{\mathfrak{m}})}$



$$L_{\mathrm{SHE}}(\vec{\mathfrak{m}}) = L_{\mathrm{SHE}}(\mathfrak{m}_1, \ldots, \mathfrak{m}_n)$$
$$:= \frac{\mathfrak{m}^3}{24} + \sum_{\mathfrak{c},\mathfrak{c}'=1}^{n} \frac{1}{2}\mathfrak{m}_{\mathfrak{c}}\mathfrak{m}_{\mathfrak{c}'}|\mathbf{x}_{\mathfrak{c}} - \mathbf{x}_{\mathfrak{c}'}| - \mathbb{I}_*$$

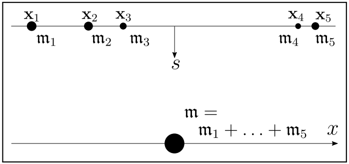
$$\mathbb{I}_* := \inf\Big\{\mathbb{I}(\mu) : \mu \in \mathscr{C}([0,1], \mathfrak{m}\mathscr{P}(\mathbb{R})), \mu(0) = \sum_{\mathfrak{c}=1}^{n} \mathfrak{m}_{\mathfrak{c}}\delta_{\mathbf{x}_{\mathfrak{c}}}, \mu(1) = \mathfrak{m}\delta_0\Big\}$$

Remark. The initial condition should actually be: $Z(0,\bullet) = \mathbf{1}_{[-\alpha,\alpha]}$, with $N \to \infty$ first and $\alpha \to 0$ later. A separate argument in [Lin–T 23] shows that this initial condition approximates the true delta initial condition.

Back to moment Lyapunov exponents

Corollary

Under $Z(0,\cdot)=\delta_0$, $\mathbf{E}\Big[\prod_{\mathfrak{c}=1}^{n} Z(T,NT\mathbf{x}_{\mathfrak{c}})^{N\mathfrak{m}_{\mathfrak{c}}}\Big] \approx e^{N^3T\cdot L_{\mathrm{SHE}}(\vec{\mathfrak{m}})}$



$$L_{\mathrm{SHE}}(\vec{\mathfrak{m}}) = L_{\mathrm{SHE}}(\mathfrak{m}_1,\ldots,\mathfrak{m}_n) := \frac{\mathfrak{m}^3}{24} + \sum_{\mathfrak{c},\mathfrak{c}'=1}^{n} \frac{1}{2}\mathfrak{m}_{\mathfrak{c}}\mathfrak{m}_{\mathfrak{c}'}|\mathbf{x}_{\mathfrak{c}}-\mathbf{x}_{\mathfrak{c}'}| - \mathbb{I}_*$$

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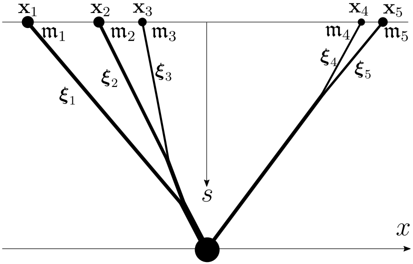
Theorem (T 23)

The infimum has a unique minimizer $\mu=\xi$, *the* ***optimal deviation****, which we describe next.*

Optimal deviation



Optimal clusters $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_n$ and optimal deviation $\boldsymbol{\xi}$

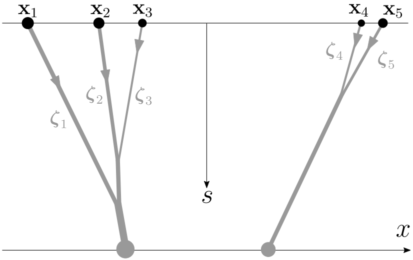
- $\boldsymbol{\xi}_{\mathfrak{c}}(s) :=$ $\qquad$ $\boldsymbol{\xi}(s) = \sum_{\mathfrak{c}=1}^{n} \mathfrak{m}_{\mathfrak{c}} \delta_{\boldsymbol{\xi}_{\mathfrak{c}}(s)}$

Optimal deviation

$$\mathrm{d}X_i^N = \frac{1}{N}\sum_{j=1}^{N\mathfrak{m}} \frac{1}{2}\mathrm{sgn}(X_j^N - X_i^N)\,\mathrm{d}s + \frac{1}{\sqrt{N^2 T}}\mathrm{d}B_i(s)$$

Inertia clusters, $\zeta_1, \ldots, \zeta_\mathfrak{c}$

- $\zeta_\mathfrak{c}$ has mass $\mathfrak{m}_\mathfrak{c}$.



Optimal clusters $\xi_1, \ldots, \xi_n$ and optimal deviation ξ

- $\xi_\mathfrak{c}(s) :=$

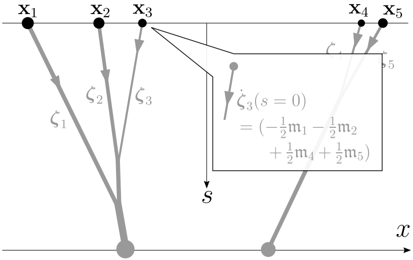
$\xi(s) = \sum_{\mathfrak{c}=1}^{n} \mathfrak{m}_\mathfrak{c}\delta_{\xi_\mathfrak{c}(s)}$

Optimal deviation

$$dX_i^N = \frac{1}{N}\sum_{j=1}^{N\mathfrak{m}} \frac{1}{2}\mathrm{sgn}(X_j^N - X_i^N)\,ds + \frac{1}{\sqrt{N^2T}}dB_i(s)$$

Inertia clusters, $\zeta_1, \ldots, \zeta_{\mathfrak{c}}$

- $\zeta_{\mathfrak{c}}$ has mass $\mathfrak{m}_{\mathfrak{c}}$.
- Start with velocity $(\ldots - \frac{1}{2}\mathfrak{m}_{\mathfrak{c}-1} + \frac{1}{2}\mathfrak{m}_{\mathfrak{c}+1} + \ldots)$.



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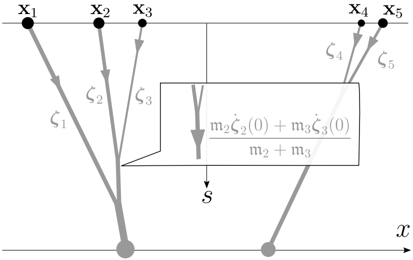
- Merge according to conservation of momentum.



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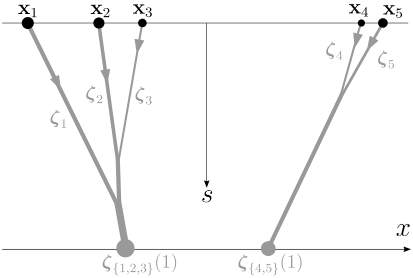

Inertia clusters, $\zeta_1, \ldots, \zeta_{\mathfrak{c}}$

- $\zeta_{\mathfrak{c}}$ has mass $\mathfrak{m}_{\mathfrak{c}}$.
- Start with velocity $(\ldots - \frac{1}{2}\mathfrak{m}_{\mathfrak{c}-1} + \frac{1}{2}\mathfrak{m}_{\mathfrak{c}+1} + \ldots)$.
- Merge according to conservation of momentum.

Branches, $\mathfrak{b}$: $\mathfrak{c}, \mathfrak{c}' \in \mathfrak{b}$ if and only if $\zeta_{\mathfrak{c}}(1) = \zeta_{\mathfrak{c}'}(1)$

Optimal clusters $\xi_1, \ldots, \xi_n$ and optimal deviation ξ

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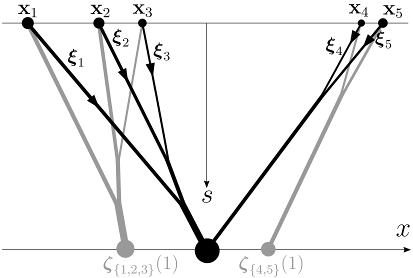
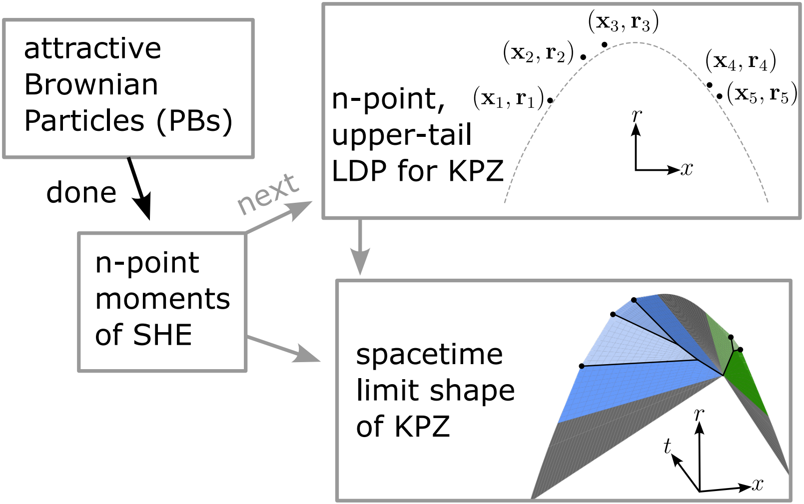


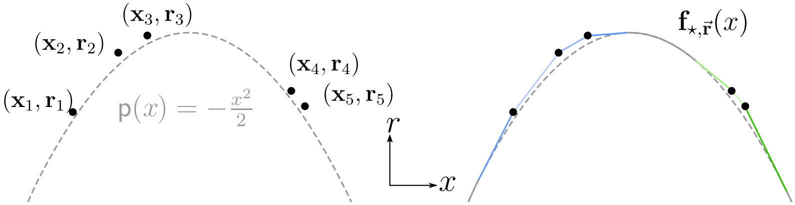
Branches, $\mathfrak{b}$: $\mathfrak{c}, \mathfrak{c}' \in \mathfrak{b}$ if and only if $\zeta_{\mathfrak{c}}(1) = \zeta_{\mathfrak{c}'}(1)$

Optimal clusters $\xi_1, \ldots, \xi_n$ and optimal deviation ξ

- $\xi_{\mathfrak{c}}(s) := \zeta_{\mathfrak{c}}(s) + (-\zeta_{\mathfrak{b}}(1))\,s, \quad \mathfrak{c} \in \mathfrak{b}$ $\qquad \xi(s) = \sum_{\mathfrak{c}=1}^{n} \mathfrak{m}_{\mathfrak{c}}\delta_{\xi_{\mathfrak{c}}(s)}$

So far and what's next



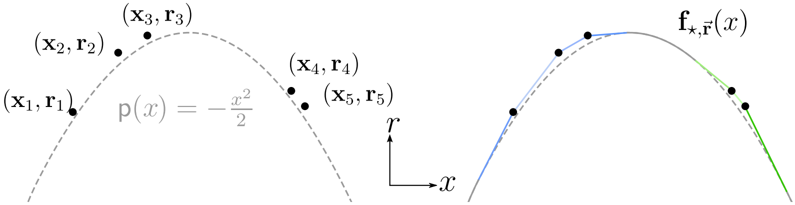


n-point, upper-tail rate function

$$I_{\mathrm{KPZ}}(\vec{\mathbf{r}}) = I_{\mathrm{KPZ}}(\mathbf{r}_1, \ldots, \mathbf{r}_n) := \int_{\mathbb{R}} \mathrm{d}x \, \big(\tfrac{1}{2}(\partial_x \mathsf{f}_{\star,\vec{\mathbf{r}}})^2 - \tfrac{1}{2}(\partial_x \mathsf{p})^2 \big)$$

Gibbs line ensembles [Corwin–Hammond 14, 16] and [Ganguly–Hegde 22].

Moments of SHE → LDP for KPZ: Legendre transform



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Theorem (T 23)

Let $\mathscr{R}_{conc} := \{\vec{\mathbf{r}} : \mathsf{f}_{\star,\vec{\mathbf{r}}} \geq \mathsf{p},\ \mathsf{f}_{\star,\vec{\mathbf{r}}}$ *is concave*$\}$. *The functions*

$$L_{\text{SHE}}(\vec{\mathfrak{m}}) : [0,\infty)^n \to [0,\infty) \qquad I_{\text{KPZ}}(\vec{\mathbf{r}}) : \mathscr{R}_{conc} \to [0,\infty)$$

are strictly convex and the Legendre transform of each other.

n-point, upper-tail LDP for the KPZ equation

$$h_N(t,x) := \tfrac{1}{N^2T}\big(h(Tt,NTx) - \log\sqrt{T}\big)$$
$$\mathcal{E}_{N,\delta}(\vec{\mathbf{r}}) := \{|h_N(1,\mathbf{x}_{\mathfrak{c}}) - \mathbf{r}_{\mathfrak{c}}| \leq \delta, \mathfrak{c} = 1,\ldots,n\}$$

Corollary (T 23 & Lin–T 23)

Under delta initial condition $Z(0,\cdot) = \delta_0$, for any $\vec{\mathbf{r}} \in \mathscr{R}^{\circ}_{\text{conc}}$,

$$\mathbf{P}\big[\mathcal{E}_{N,\delta}(\vec{\mathbf{r}})\big] \approx e^{-N^3T\cdot I_{\text{KPZ}}(\vec{\mathbf{r}})}$$

$N \to \infty$ *and* $N^2T = N^2T_N \to \infty$ *first;* $\delta \to 0$ *later.*

Related results

First, when $n = 1$ and $\mathbf{x}_1 = 0$, we recover $I_{\text{KPZ}}(\mathbf{r}) = \frac{4\sqrt{2}}{3}\mathbf{r}^{3/2}$.

One point, upper-tail LDPs

- Hyperbolic scaling regime

$$\mathbf{P}\big[\tfrac{1}{T}h(Tt,0) \approx -\tfrac{1}{24} + \mathbf{r}\big] \approx e^{-T\,\frac{4\sqrt{2}}{3}\mathbf{r}^{3/2}}, \quad T \to \infty, \mathbf{r} > 0$$

  - Predicted in [Le Doussal–Majumdar–Schehr 16]; proven in [Das–T 21].
- Other scaling regimes and/or other initial conditions
  - Physics: Asida, Hartman, Janas, Kolokolov, Korshunov, Katzav, Krajenbrink, Le Doussal, Majumdar, Livne, Meerson, Prolhac, Rosso, Sasorov, Schmidt, Smith, Vilenkin, ...
  - Math rigorous: Corwin, Das, Gaudreau Lamarre, Ghosal, Lin, Tsai, ...

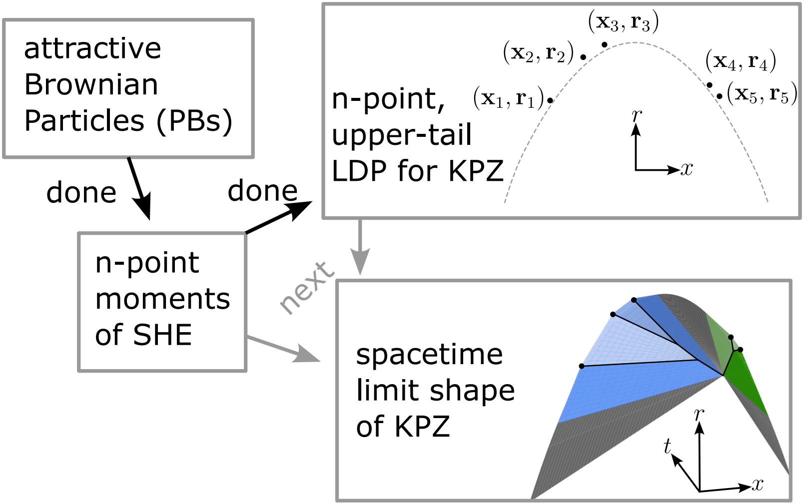
***n*-point upper tails and terminal-time limit shape**

Related results

n-point upper tails and terminal-time limit shape

- [Ganguly–Hegde 22]
  - Detailed and optimal n-point bounds that hold for all $t > t_0$.
  - When specialized onto the hyperbolic scaling regime: the n-point LDP and the terminal-time limit shape $\mathsf{f}_{\star,\mathbf{r}}$.

So far and what's next



Spacetime limit shape

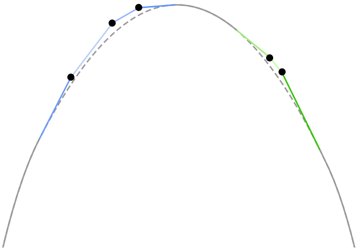
Theorem (Lin–T 23)

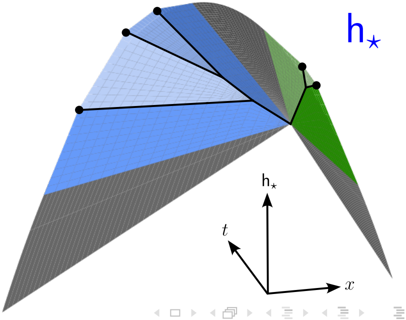
Under $Z(0,\cdot) = \delta_0$, *for any* $\vec{\mathbf{r}} \in \mathscr{R}^{\circ}_{\mathrm{conc}}$ *and* $R < \infty$,

$$\mathbf{P}\Big[\|h_N - \mathsf{h}_\star\|_{\mathscr{L}^\infty([\frac{1}{R},1]\times[-R,R])} < \tfrac{1}{R} \,\Big|\, \mathcal{E}_{N,\delta}(\vec{\mathbf{r}}) \Big] \longrightarrow 1$$

$N \to \infty$ *and* $N^2 T = N^2 T_N \to \infty$ *first;* $\delta \to 0$ *later.*

$$\mathsf{f}_{\star,\vec{\mathbf{r}}}(x) = \mathsf{h}_\star(1,x)$$

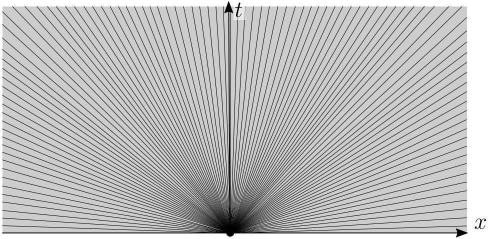


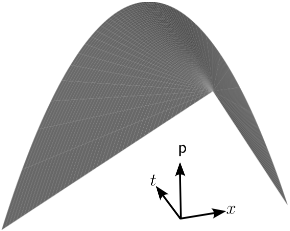


Limit shape

Hydrodynamic limit (without conditioning)

- [Janjigian–Rassoul-Agha–Seppäläinen 22] The hydrodynamic limit h_0 is the entropy solution of $\partial_t \mathsf{h}_0 = \frac{1}{2}(\partial_x \mathsf{h}_0)^2$.
- [Amir–Corwin–Quastel 11] Here $\mathsf{h}_0(t,x) = \mathsf{p}(t,x) := -x^2/(2t)$.





Limit shape (with conditioning)

- $\mathsf{h}_\star(t,x)$ also solves $\partial_t \mathsf{h}_\star = \frac{1}{2}(\partial_x \mathsf{h}_\star)^2$, but is a *non-entropy* solution.

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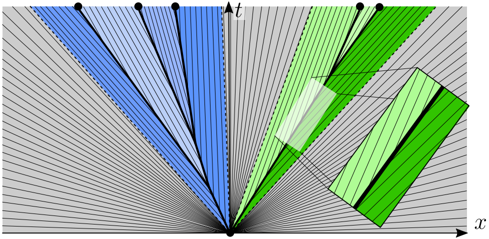
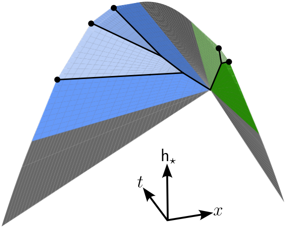


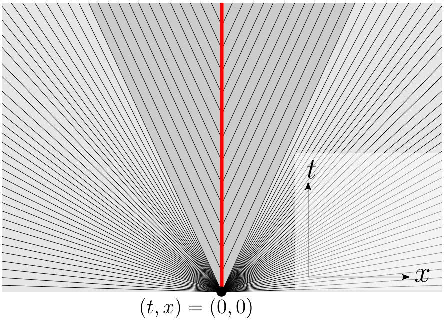


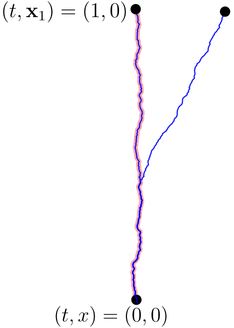
- How to describe $\mathsf{h}_\star$?
  $\mathsf{h}_\star(1-s,x)$ is the entropy solution of the *backward* equation
  $-\partial_s \mathsf{h}_\star = \frac{1}{2}(\partial_x \mathsf{h}_\star)^2$.
  Consistent with [Jensen 00] [Varadhan 04]

Mechanism of the deviations

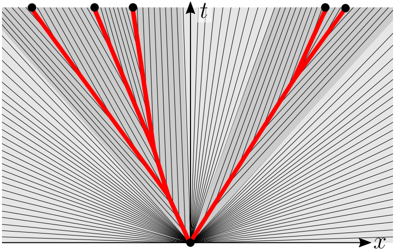
$$e^{h(T,NTx)} = Z(T,NTx) = \mathbb{E}_{\mathrm{BM}}\big[e^{\int_0^T \mathrm{d}s\, \eta(T-s,X(s))}\delta_0(X(T))\big]$$

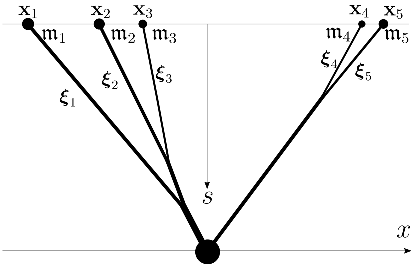
Consider $n = 1$ and $\mathbf{x}_1 = 0$.





Noise corridors = optimal clusters in aBPs





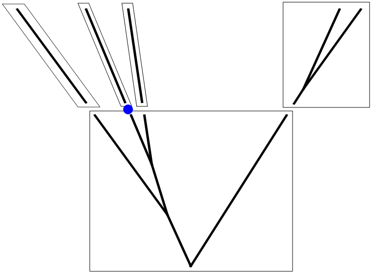
Proposition (T 23)

(Noise corridors in KPZ) = (optimal clusters in attractive BPs)

Elements of the proof

Given $\mathcal{E}_{N,\delta}(\vec{\mathbf{r}})$, we want to argue $h_N \approx \mathsf{h}_\star$.

- Use a tree structure to show that $h_N(t,x) \approx \mathsf{h}_\star(t,x)$ at any point (t,x) on the noise corridors / shocks.



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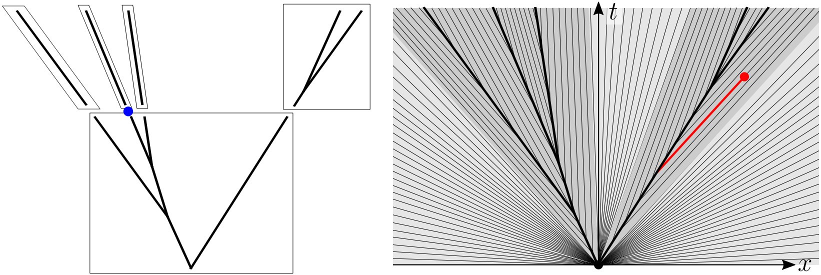
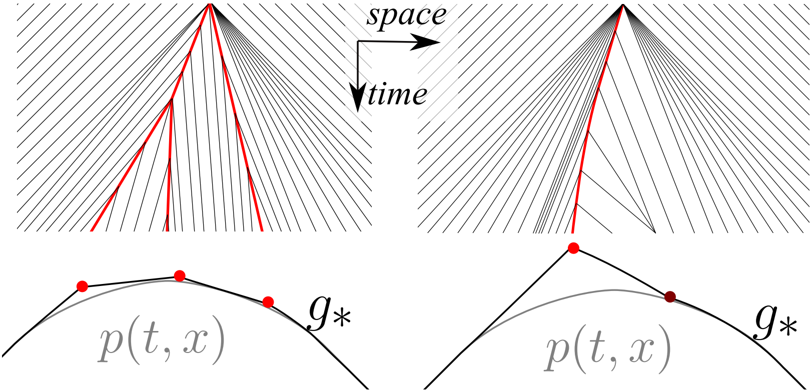
- Use a tree structure to show that $h_N(t,x) \approx \mathsf{h}_\star(t,x)$ at any point (t,x) on the noise corridors / shocks.
- Once $h_N \approx \mathsf{h}_\star$ holds along the noise corridors / shocks, the rest can be obtained by analyze the increments of h_N along characteristics.



Food for thought?



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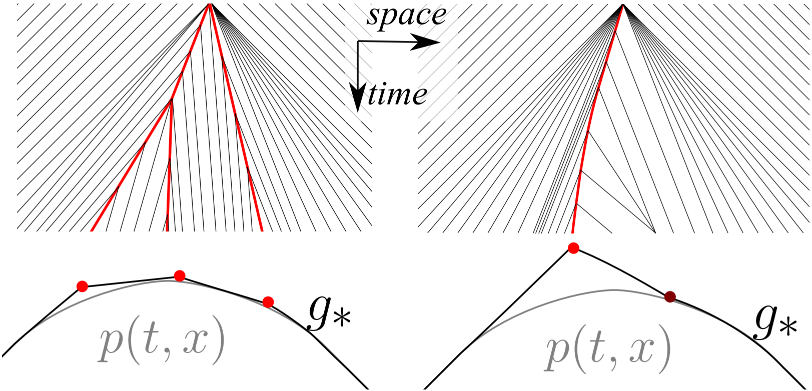

Thank you for the attention and thanks to the organizers!