Grothendieck spaces: topology and applications

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March 11, 2024 version

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More recently, Hamel and I have shown that these results hold for a much larger class of logics, highlighting the importance of conditions that assert the exchangeability of certain double limits. See our arXiv preprint: *On the undefinability of pathological Banach spaces.*

C_p-Theory

Definition

Let X be a topological space. We will assume X is completely regular unless specified otherwise. Then $C_p(X)$ is the collection of real-valued continuous functions on X, given the topology of pointwise convergence, which means the topology $C_p(X)$ inherits from $\mathbb{R}^X$.

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C_p-theory studies the interrelations between the topology of X and the topology of $C_p(X)$. Good references are Arhangel'skiĭ's *Topological Function Spaces* and V. V. Tkachuk's 4-volume *A C_p-theory Problem Book*.

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Theorem (Arhangel'skiĭ-Pytkeev)

All finite powers of X are Lindelöf if and only if $C_p(X)$ is countably tight.

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Corollary

If X is compact, $C_p(X)$ is countably tight.

Grothendieck Spaces

Definition

A subspace Y of X is

- countably compact in X if every infinite subset of Y has a limit point in X. We equivalently say Y is relatively countably compact.
- relatively compact if $\overline{Y}$ is compact.

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Definition

A space X is weakly Grothendieck if $C_p(X)$ is a g-space.
A space X is Grothendieck if $C_p(X)$ is a hereditary g-space. (A hereditary g-space is also called angelic.)

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Theorem

Every countable union of countably compact spaces is Grothendieck.

Definition

A Lindelöf Σ-space is a continuous image of a perfect pre-image of a separable metrizable space. Equivalently, a member of the smallest class of spaces containing all compact spaces, all separable metrizable spaces, and that is closed under finite products, closed subspaces and continuous images.

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Every countably tight space is weakly Grothendieck. Every k-space is weakly Grothendieck.

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CH $\implies$ *there is such an example.*

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See my Top. Appl. paper: *Countable tightness and the Grothendieck property in C_p-theory.*

Double Limit Conditions

Definition

Let X be a topological space and $A \subseteq C_p(X, [0,1])$. We write $\mathrm{DLC}(A, X)$ if: for every X-sequence $\{x_n\}_{n<\omega}$ and A-sequence $\{f_m\}_{m<\omega}$ the following double limits agree whenever they both exist: $\lim_{n\to\infty} \lim_{m\to\infty} f_m(x_n) = \lim_{m\to\infty} \lim_{n\to\infty} f_m(x_n)$.

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We write $\mathrm{DLC}(X)$ if $\mathrm{DLC}(A,X)$ holds for all $A \subseteq C_p(X,[0,1])$. We say that X satisfies the double-limit condition if for each $A \subseteq C_p(X,[0,1])$ we have: $\mathrm{DLC}(A,X) \iff A$ is relatively countably compact.

Theorem (Pták)

Let X be compact and $A \subseteq C_p(X)$ pointwise bounded. Then A is relatively compact in $C_p(X)$ if and only if $\mathrm{DLC}(A, X)$ holds.

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Theorem (Iovino)

A space X is compact if and only if every ultralimit in X exists.

Definition

Let X be a topological space and $A \subseteq C_p(X,[0,1])$. We write $\mathrm{DULC}(A,X)$ if: for every pair of sequences $\{x_n\}_{n<\omega} \subseteq X$ and $\{f_m\}_{m<\omega} \subseteq A$, and ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on ω, the double limits below agree whenever they both exist:

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$\mathrm{DLC}(X) \implies \mathrm{DULC}(X)$.

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Definition

c_0 is the Banach space of sequences of real numbers converging to 0.
ℓ^p is the Banach space of sequences $\{x_n\}_{n<\omega}$ of real numbers such that $\Sigma_{n<\omega} |x_n|^p < \infty$.

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We proved (**I**) $\iff$ (**IV**) for X countably tight, i.e., countably tight spaces satisfy the double ultralimit condition.

We trivially have (**I**) $\iff$ (**II**) if X is weakly Grothendieck.

An immediate consequence of the ultralimit definitions:

Lemma

If X is weakly Grothendieck and satisfies the double ultralimit condition, then for every $A \subseteq C_p(X, [0,1])$ we have $\mathrm{DULC}(A, X) \implies A$ is relatively compact.

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The $\implies$ can be strengthened to $\iff$ for countably tight spaces.

Theorem

Let X be countably tight. Then $A \subseteq C_p(X, [0,1])$ is relatively compact $\iff \mathrm{DULC}(A, X)$.

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This greatly improves the undefinability results in our paper *C_p-theory for model theorists*, which appeared in Iovino's book *Beyond First-order Model Theory, II*. This new paper, *On the undefinability of pathological Banach spaces*, is available on arXiv and will hopefully enable us to generalize the results of Iovino et al on Deep Learning.

Conclusion

In the many applications of double limit exchange, much weaker assumptions than compactness suffice. In particular, the following classes of spaces are all weakly Grothendieck and satisfy the double ultralimit condition: countably compact spaces, countably tight spaces, k-spaces, separable spaces. This observation enables us to extend Iovino et al's undefinability results from compact logics to countably compact logics, which they claimed in their first preprint but later withdrew.

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We will therefore give the proof of our result. Arhangel'skiĭ's proof can be found in his book. It is straightforward general topology.

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Proof. Let $\overline{A}$ denote the closure of A in $[0,1]^X$. Suppose $\overline{A} \cap C_p(X)$ is not compact in $C_p(X)$. Then $\overline{A} \cap C_p(X)$ is closed but it is not countably compact in $C_p(X)$, since X is weakly Grothendieck.

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Let $\{f_n\}_{n<\omega}$ be a subset of A with closure disjoint from $\overline{A} \cap C_p(X)$. Since $\overline{A}$ is a compact subset of $[0,1]^X$, each ultralimit of the sequence $\{f_n\}_{n<\omega}$ exists, and is discontinuous. Take a nonprincipal ultrafilter $\mathcal{U}$ over ω and let $\lim_{n\to\mathcal{U}} f_n = g$, where g is discontinuous by assumption.

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g is a real-valued function so there is some $g(y) \in \mathbb{R}$ and an open interval about it such that its inverse under g is not open, so the complement of that inverse is not closed.

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g is a real-valued function so there is some $g(y) \in \mathbb{R}$ and an open interval about it such that its inverse under g is not open, so the complement of that inverse is not closed.

Then there are $\varepsilon > 0$ and $y \in X$ such that $y \in \overline{Y}$, where

$$Y = X \setminus g^{-1}(g(y) - \varepsilon, g(y) + \varepsilon).$$

Theorem (HT)

Let X be countably tight. A subset A of $C_p(X,[0,1])$ is relatively compact in $C_p(X,[0,1])$ if and only if $\mathrm{DULC}(A,X)$.

Proof. Suppose $\overline{A} \cap C_p(X)$ is not compact in $C_p(X)$.

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Since $t(X) = \aleph_0$, there is some $Z \subseteq Y$ with $|Z| = \aleph_0$ and $y \in \overline{Z}$. Suppose $Z = \{x_m\}_{m<\omega}$ and for each open neighbourhood U of y, let $M_U := \{m < \omega : x_m \in U\}$.

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Suppose $Z = \{x_m\}_{m<\omega}$ and for each open neighbourhood U of y, let $M_U := \{m < \omega : x_m \in U\}$. Clearly, the family of all M_U is centred (i.e., all finite subfamilies have non-empty intersections) and so it can be extended to an ultrafilter $\mathcal{V}$ on ω so that

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since each f_n is continuous.

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On the other hand, $\lim_{m\to\mathcal{V}} \lim_{n\to\mathcal{U}} f_n(x_m) = \lim_{m\to\mathcal{V}} g(x_m)$ exists by the compactness of $[0,1]$.

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However, by the choice of each x_m, we have $|g(y) - g(x_m)| \geq \varepsilon$ and so the ultralimits exist but are different, a contradiction. □

The application of these topological results to Model Theory and then to the undefinability of Banach spaces is rather technical so we won't do it here. It can be found in the paper of Hamel and myself recently posted on arXiv.

Grothendieck's Theorem

For a topological space X, $C_p(X)$ is the set of continuous real-valued functions on X, given the pointwise topology inherited from $\mathbb{R}^X$. The classic theorem of Grothendieck states:

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Let X be countably compact and let $A \subseteq C_p(X)$ be such that every infinite subset of A has a limit point in $C_p(X)$. Then the closure of A in $C_p(X)$ is compact.

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[Gro52] A. Grothendieck. Critéres de compacité dans les espaces fonctionnels généraux. *Amer. J. Math.*, **74**:168–186, 1952.

Countably Tight & Grothendieck

Definition ([Arh98])

Recall that $A \subseteq X$ is relatively countably compact if every infinite subset of A has a limit point in X.
X is a g-space if each $A \subseteq X$ which is countably compact in X has compact closure.
X is a Grothendieck space (resp. weakly Grothendieck space) if $C_p(X)$ is a hereditary g-space (resp. a g-space).

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Problem

If countably compact subspaces of $C_p(X)$ are compact, is X Grothendieck?

Free Sequences

Definition

$\{x_\alpha : \alpha < \kappa\}$ is free if for every $\eta < \kappa$, $\{x_\alpha : \alpha < \eta\}$ and $\{x_\alpha : \eta < \alpha < \kappa\}$ have disjoint closures.

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PFA $\implies$ *If X does not include an uncountable free sequence, then every countably compact subspace of $C_p(X)$ is compact.*

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Can that conclusion be strengthened to X being Grothendieck?

Countable Tightness and the Grothendieck Property in C_p Theory

The Grothendieck property has become important in research on the definability of pathological Banach spaces [CI18], [HT20], [HT22].

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The proof of Grothendieck's Theorem involves interchanging double limits as one often does in Analysis, e.g. under suitable conditions,

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Jose Iovino noticed a connection between interchanging double limits and definability in model theory. He and P. Casazza used this to prove the undefinability in first order (continuous) logic of a famous pathological Banach space: Tsirelson's space. I saw that their results could be greatly generalized using C_p-theory, but today I'm just talking about topology rather than model theory.

Countable Tightness and the Grothendieck Property in C_p Theory

The Grothendieck property has become important in research on the definability of pathological Banach spaces [CI18], [HT20], [HT22].

[CI18] P. Casazza and J. Iovino. On the undefinability of Tsirelson's space and its descendants. ArXiv: 1812.02840, 2018.

[HT20] C. Hamel and F. D. Tall. Model theory for C_p-theorists. Top. Appl., paper 107197, 2020.

[HT23] C. Hamel and F. D. Tall, C_p-theory for model theorists, in J. Iovino, ed., *Beyond first order model theory, II*, CRC Press, Boca Raton, 2023.

[HT24] C. Hamel and F. D. Tall, On the undefinability of pathological Banach spaces, submitted.

We here answer a question of Arhangel'skiĭ by proving it undecidable whether countably tight spaces with Lindelöf finite powers are Grothendieck.

We here answer a question of Arhangel'skiĭ by proving it undecidable whether countably tight spaces with Lindelöf finite powers are Grothendieck.

We answer another of his questions by proving that PFA implies Lindelöf countably tight spaces are Grothendieck.

Strengthening Arhangel'skiĭ's Result

In [Arh98], Arhangel'skiĭ proved:

Proposition 2

$\mathrm{MA} + \neg\mathrm{CH}$ *implies that if* X *is countably tight and* X^n *is Lindelöf for all* $n < \omega$*, then* X *is Grothendieck.*

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Example

Assuming $\diamondsuit$ plus Kurepa's Hypothesis, Ivanov [Iva78] constructed a compact space Y of cardinality $2^{\mathfrak{c}}$ such that Y^n is hereditarily separable for all $n < \omega$. $C_p(Y)$ is the required counterexample.

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To see this, we require several results from the literature.

Arhangel'skiĭ's Result and ZFC

Lemma ([Arh92])

X^n is Lindelöf for every $n < \omega$ if and only if $C_p(X)$ is countably tight.

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Definition

A space X is Fréchet-Urysohn if whenever x is a limit point of $Z \subseteq X$, there is a sequence in Z converging to x.

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A space X is Fréchet-Urysohn if whenever x is a limit point of $Z \subseteq X$, there is a sequence in Z converging to x.

Arhangel'skiĭ later proved:

Lemma ([Arh98])

X is Grothendieck if and only if it is weakly Grothendieck and compact subspaces of $C_p(X)$ are Fréchet-Urysohn.

Arhangel'skiĭ's Result and ZFC

Lemma ([Arh92])

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He also proved:

Lemma ([Arh92])

X embeds into $C_p(C_p(X))$.

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Lemma ([Arh92])

X embeds into $C_p(C_p(X))$.

Clearly, separable Fréchet-Urysohn spaces have cardinality $\leq \mathfrak{c}$. Ivanov's space Y is too big to be Fréchet-Urysohn, yet it embeds in $C_p(C_p(Y))$, so $C_p(Y)$ cannot be Grothendieck, although it is weakly Grothendieck.

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$(C_p(Y))^n$ is, however, (hereditarily) Lindelöf for all $n < \omega$ by the Velichko-Zenor theorem:

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Lemma ([Vel81], [Zen80])

If X^n is hereditarily separable for all $n < \omega$, then $(C_p(X))^n$ is hereditarily Lindelöf for all $n < \omega$.

[Arh92] A. V. Arhangel'skiĭ. Topological Function Spaces, vol. 78 of *Mathematics and its Applications (Soviet Series).* Kluwer Academic Publishers Group, Dordrecht, 1992.

[Arh98] A. V. Arhangel'skiĭ. Embedding in C_p-spaces. *Topology Appl.*, **85**:9–33, 1998.

[Iva78] A. V. Ivanov On bicompacta all finite powers of which are hereditarily separable. *Doklady Akademii Nauk SSSR*, **243**(5):1109–1112, 1978.

[Vel81] N. V. Velichko. Weak topology of spaces of continuous functions. *Mathematical Notes of the Academy of Sciences of the USSR*, **30**:849–854, 1981.

[Zen80] P. Zenor. Hereditary m-separability and the hereditary m-Lindelöf property in product spaces and function spaces. *Fund. Math.*, **106**(3):175–180, 1980.

Finally, $C_p(Y)$ is countably tight since Y is compact, so $C_p(Y)$ is Lindelöf, countably tight, but not Grothendieck.

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A dramatic strengthening of the conclusion of Arhangel'skiĭ's Proposition is

Theorem

PFA *implies Lindelöf countably tight spaces are Grothendieck.*

Finally, $C_p(Y)$ is countably tight since Y is compact, so $C_p(Y)$ is Lindelöf, countably tight, but not Grothendieck.

A dramatic strengthening of the conclusion of Arhangel'skiĭ's Proposition is

Theorem

PFA *implies Lindelöf countably tight spaces are Grothendieck.*

The proof actually follows easily from known results.

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A space is surlindelöf if it is a subspace of $C_p(X)$ for some Lindelöf X.

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Arhangel'skiĭ proved:

Lemma ([Arh92])

PFA *implies that every surlindelöf compact space is countably tight.*

Definition

A space is surlindelöf if it is a subspace of $C_p(X)$ for some Lindelöf X.

Lemma ([Arh92])

PFA *implies that every surlindelöf compact space is countably tight.*

Okunev and Reznichenko proved:

Lemma ([OR07])

MA_{ω_1} *implies that every separable surlindelöf compact countably tight space is metrizable.*

The Proof: Fréchet-Urysohn

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PFA *implies that every surlindelöf compact space is Fréchet-Urysohn.*

Proof. Metrizable spaces are clearly Fréchet-Urysohn. By countable tightness, if K is compact and $L \subseteq K$ and $p \in \overline{L}$, then there is a countable $M \subseteq L$ such that $p \in \overline{M}$. But $\overline{M}$ is separable compact and so metrizable. □

This proves the Theorem.

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PFA *implies Lindelöf countably tight spaces are Grothendieck.*

In fact, as often happens, we have:

Theorem

If ZFC *is consistent, so is* ZFC *plus "every Lindelöf countably tight space is Grothendieck".*

[Arh92] A. V. Arhangel'skiĭ. Topological Function Spaces, vol. 78 of *Mathematics and its Applications (Soviet Series).* Kluwer Academic Publishers Group, Dordrecht, 1992.

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Example

A space X and a $Y \subseteq X$ such that Y is countably compact in X but $\overline{Y}$ is not countably compact. In Ψ-space, the closure of ω includes an uncountable closed discrete set.

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A space X and a $Y \subseteq X$ such that Y is countably compact in X but $\overline{Y}$ is not countably compact. In Ψ-space, the closure of ω includes an uncountable closed discrete set.

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Countably tight spaces are weakly Grothendieck.

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A space is realcompact if it can be embedded as a closed subspace in a product of copies of the real line.

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Lemma

If X is countably tight, then $C_p(X)$ is realcompact.

Definition

A space is *wD* if whenever $\{d_n : n < \omega\}$ is closed discrete, there is an infinite $S \subseteq \omega$ and a discrete collection of open sets $\{U_n : n \in S\}$ with $d_n \in U_n$ for all $n \in S$.

Definition

A space is *wD* if whenever $\{d_n : n < \omega\}$ is closed discrete, there is an infinite $S \subseteq \omega$ and a discrete collection of open sets $\{U_n : n \in S\}$ with $d_n \in U_n$ for all $n \in S$.

Lemma

Every realcompact space is wD.

Lemma

Let X be wD. Let Y be countably compact in X. Then $\overline{Y}$ is countably compact.

Lemma

Let X be wD. Let Y be countably compact in X. Then $\overline{Y}$ is countably compact.

Proof. Suppose not. Let $\{d_n : n < \omega\}$ be a closed discrete subspace of $\overline{Y}$. Let $\{U_n : n < \omega\}$ be a discrete collection of open subsets of X, with $d_n \in U_n$ for every $n \in S$, where $S \subseteq \omega$ is infinite. Pick $e_n \in U_n \cap Y$. Then $\{e_n : n \in S\}$ is a closed discrete subspace of Y, contradiction. □

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Proof. Let X be countably tight. Then $C_p(X)$ is realcompact and hence wD. Let Y be countably compact in $C_p(X)$, then $\overline{Y}$ is countably compact. But $\overline{Y}$ is realcompact so $\overline{Y}$ is compact. □