An Undecidable Extension of Morley's Theorem on the Number of Countable Models

Christopher J. Eagle, Clovis Hamel, Sandra Müller, and Franklin D. Tall

March 11, 2024 version

Morley's Theorem

Conjecture (Vaught, 1961)

Every countable first-order theory has either at most countably many or continuum many non-isomorphic countable models.

Morley's Theorem

Conjecture (Vaught, 1961)

Every countable first-order theory has either at most countably many or continuum many non-isomorphic countable models.

Theorem (Morley, 1970)

Every countable first-order theory has either at most $\aleph_1$ or continuum many non-isomorphic countable models.

Morley's Theorem

Conjecture (Vaught, 1961)

Every countable first-order theory has either at most countably many or continuum many non-isomorphic countable models.

Theorem (Morley, 1970)

Every countable first-order theory has either at most $\aleph_1$ or continuum many non-isomorphic countable models.

To avoid a trivial application of CH implying VC, we can identify countable models with elements of the Cantor set to obtain:

Morley's Theorem

To avoid a trivial application of CH implying VC, we can identify countable models with elements of the Cantor set to obtain:

Conjecture (Absolute Vaught)

Every countable first-order theory has either at most countably many or perfectly many non-isomorphic countable models.

Morley's Theorem

To avoid a trivial application of CH implying VC, we can identify countable models with elements of the Cantor set to obtain:

Conjecture (Absolute Vaught)

Every countable first-order theory has either at most countably many or perfectly many non-isomorphic countable models.

Theorem (Absolute Morley)

Every countable first-order theory has either at most $\aleph_1$ or perfectly many non-isomorphic countable models.

Coding countable structures as reals

Let $S = \{R_i\}_{i \in I}$ be a signature, where each R_i is a relation symbol with arity n_i, and I is countable.

Coding countable structures as reals

Let $S = \{R_i\}_{i \in I}$ be a signature, where each R_i is a relation symbol with arity n_i, and I is countable.

Suppose that $\mathcal{M}$ is a countable S-structure. Up to isomorphism, we may assume that the underlying set of $\mathcal{M}$ is ω.

Coding countable structures as reals

Let $S = \{R_i\}_{i \in I}$ be a signature, where each R_i is a relation symbol with arity n_i, and I is countable.

Suppose that $\mathcal{M}$ is a countable S-structure. Up to isomorphism, we may assume that the underlying set of $\mathcal{M}$ is ω.

For each $i \in I$ the interpretation $R_i^{\mathcal{M}}$ of R_i is a subset of ω^{n_i}, and so we identify R_i with an element of $\omega^{\omega^{n_i}}$.

Coding countable structures as reals

Let $S = \{R_i\}_{i \in I}$ be a signature, where each R_i is a relation symbol with arity n_i, and I is countable.

Suppose that $\mathcal{M}$ is a countable S-structure. Up to isomorphism, we may assume that the underlying set of $\mathcal{M}$ is ω.

For each $i \in I$ the interpretation $R_i^{\mathcal{M}}$ of R_i is a subset of ω^{n_i}, and so we identify R_i with an element of $\omega^{\omega^{n_i}}$.

As the structure $\mathcal{M}$ is completely determined by the interpretations of each of the relation symbols, we may identify $\mathcal{M}$ with an element of $\prod_{i \in I} 2^{\omega^{n_i}}$.

Coding countable structures as reals

Let $S = \{R_i\}_{i \in I}$ be a signature, where each R_i is a relation symbol with arity n_i, and I is countable.

Suppose that $\mathcal{M}$ is a countable S-structure. Up to isomorphism, we may assume that the underlying set of $\mathcal{M}$ is ω.

For each $i \in I$ the interpretation $R_i^{\mathcal{M}}$ of R_i is a subset of ω^{n_i}, and so we identify R_i with an element of $\omega^{\omega^{n_i}}$.

As the structure $\mathcal{M}$ is completely determined by the interpretations of each of the relation symbols, we may identify $\mathcal{M}$ with an element of $\prod_{i \in I} 2^{\omega^{n_i}}$.

This identification provides a bijective map from the collection of S-structures with universe ω to $\prod_{i \in I} 2^{\omega^{n_i}}$. We thus view the Cantor space $\prod_{i \in I} 2^{\omega^{n_i}}$ as being the space of countable S-structures, and define $\mathrm{Mod}_S = \prod_{i \in I} 2^{\omega^{n_i}}$.

Second-order Logic

Syntax

Semantics

Second-order Logic

Syntax Variables to represent individual elements of structures; for each n, variables to represent sets of n-tuples of elements (these are the n-ary relation variables).

Semantics

Second-order Logic

Syntax Variables to represent individual elements of structures; for each n, variables to represent sets of n-tuples of elements (these are the n-ary relation variables).

Formulas are then defined recursively in a straightforward manner.

Semantics

Second-order Logic

Syntax Variables to represent individual elements of structures; for each n, variables to represent sets of n-tuples of elements (these are the n-ary relation variables).

Formulas are then defined recursively in a straightforward manner.

Semantics As for first-order, with the following additions, where $\phi(P)$ is a second-order formula:

Second-order Logic

Syntax Variables to represent individual elements of structures; for each n, variables to represent sets of n-tuples of elements (these are the n-ary relation variables).

Formulas are then defined recursively in a straightforward manner.

Semantics As for first-order, with the following additions, where $\phi(P)$ is a second-order formula:

- If P is an n-ary relation variable and $A \subseteq M^n$, then $M \models \phi(P)[A]$ if and only if $(\mathcal{M}, A) \models \phi$, where $(\mathcal{M}, A)$ is the expanded structure obtained by interpreting P as A.

Second-order Logic

Syntax Variables to represent individual elements of structures; for each n, variables to represent sets of n-tuples of elements (these are the n-ary relation variables).

Formulas are then defined recursively in a straightforward manner.

Semantics As for first-order, with the following additions, where $\phi(P)$ is a second-order formula:

- If P is an n-ary relation variable and $A \subseteq M^n$, then $M \models \phi(P)[A]$ if and only if $(\mathcal{M}, A) \models \phi$, where $(\mathcal{M}, A)$ is the expanded structure obtained by interpreting P as A.
- $\mathcal{M} \models (\exists P)\phi(P)$ if and only if there is some $A \subseteq M^n$ such that $\mathcal{M} \models \phi(A)$. The definition for second-order universal quantifier is analogous.

Second-order Logic

Example

$\neg$ CH $\implies$ 2nd order VC fails: one can express in second-order logic that a linear order is a well-order and hence there is a second-order theory whose countable models are (up to isomorphism) exactly countable ordinals.

Second-order Logic

Example

$\neg$CH $\implies$ 2nd order VC fails: one can express in second-order logic that a linear order is a well-order and hence there is a second-order theory whose countable models are (up to isomorphism) exactly countable ordinals.

Just say "$<$ is a linear order"—that's the first order; then say every subset ordered by $<$ has a least element.

Theorem (A)

There is a forcing such that in the resulting universe of set theory, there is a second-order theory T in a countable signature such that the number of non-isomorphic countable models of T is exactly $\aleph_2$, while $2^{\aleph_0} = \aleph_3$. Add to L $\aleph_2$ Cohen reals and then $\aleph_3$ random reals.*

** Signature = collection of relation symbols, each with specified arity. Note we can code function symbols and constants as relations.*

Theorem (B (Foreman-Magidor))

Beginning with a supercompact cardinal, carry out the standard forcing iteration for producing a model of the Proper Forcing Axiom. Then 2nd order Absolute Morley holds. In fact, every equivalence relation on $\mathbb{R}$ that is in $L(\mathbb{R})$ either has $\leq \aleph_1$ equivalence classes or a perfect set of equivalence classes.

Theorem (B (Foreman-Magidor))

Beginning with a supercompact cardinal, carry out the standard forcing iteration for producing a model of the Proper Forcing Axiom. Then 2nd order Absolute Morley holds. In fact, every equivalence relation on $\mathbb{R}$ that is in $L(\mathbb{R})$ either has $\leq \aleph_1$ equivalence classes or a perfect set of equivalence classes.

Definition

A cardinal κ is supercompact if for every cardinal $\lambda \geq \kappa$ there exists an elementary embedding j_λ of V into an inner model M (i.e. a proper class model of ZFC included in V) with critical point κ (least ordinal j_λ moves) and $\lambda < j_\lambda(\kappa)$, such that M^λ is included in M.

Theorem (C)

If there are infinitely many Woodin cardinals, then there is a model of set theory in which the Absolute Morley Theorem holds for second-order theories in countable signatures. In fact, every σ-projective equivalence relation on $\mathbb{R}$ has $\leq \aleph_1$ or perfectly many equivalence classes.

Theorem (C)

If there are infinitely many Woodin cardinals, then there is a model of set theory in which the Absolute Morley Theorem holds for second-order theories in countable signatures. In fact, every σ-projective equivalence relation on $\mathbb{R}$ has $\leq \aleph_1$ or perfectly many equivalence classes.

Problem

Are large cardinals necessary to prove the conclusion of Theorem C? If so, how large?

Problem

Are large cardinals necessary to prove the conclusion of Theorem C? If so, how large?

Recently solved by J. Zhang. Just add at least $\aleph_2$ Cohen reals.
[**TZ**] just appeared in *Arch. Math Logic*.

Descriptive complexity of 2nd-order theories

Definition ($\mathrm{Mod}_S(\sigma)$)

For σ a sentence of some logic with signature S, define $\mathrm{Mod}_S(\sigma) = \{\mathcal{M} \in \mathrm{Mod}_S : \mathcal{M} \models \sigma\}$ where Mod_S is the collection of countable S-models and $\mathcal{M}$ is a model with universe ω. (If S is clear from context we may omit it.)

Descriptive complexity of 2nd-order theories

Definition ($\mathrm{Mod}_S(\sigma)$)

For σ a sentence of some logic with signature S, define $\mathrm{Mod}_S(\sigma) = \{\mathcal{M} \in \mathrm{Mod}_S : \mathcal{M} \models \sigma\}$ where Mod_S is the collection of countable S-models and $\mathcal{M}$ is a model with universe ω. (If S is clear from context we may omit it.)

Definition ($\cong_T$)

Let S be a countable signature, and let T be an S-theory of some logic. The equivalence relation of isomorphism of models of T is the equivalence relation $\cong_T$ on Mod_S defined by declaring that $\mathcal{M} \cong_T \mathcal{N}$ if and only if either: neither of the two structures is a model of T, or $\mathcal{M} \cong \mathcal{N}$.

Descriptive complexity of 2nd-order theories

Definition ($\mathrm{Mod}_S(\sigma)$)

For σ a sentence of some logic with signature S, define $\mathrm{Mod}_S(\sigma) = \{\mathcal{M} \in \mathrm{Mod}_S : \mathcal{M} \models \sigma\}$ where Mod_S is the collection of countable S-models and $\mathcal{M}$ is a model with universe ω. (If S is clear from context we may omit it.)

Definition ($\cong_T$)

Let S be a countable signature, and let T be an S-theory of some logic. The equivalence relation of isomorphism of models of T is the equivalence relation $\cong_T$ on Mod_S defined by declaring that $\mathcal{M} \cong_T \mathcal{N}$ if and only if either: neither of the two structures is a model of T, or $\mathcal{M} \cong \mathcal{N}$.

The equivalence classes of $\cong_T$ are thus one class for each isomorphism class of countable models of T, together with one additional class containing all elements of $\mathrm{Mod}_S \setminus \mathrm{Mod}_S(T)$.

Descriptive complexity of 2nd-order theories

Definition ($\cong_T$)

Let S be a countable signature, and let T be an S-theory of some logic. The equivalence relation of isomorphism of models of T is the equivalence relation $\cong_T$ on Mod_S defined by declaring that $\mathcal{M} \cong_T \mathcal{N}$ if and only if either: neither of the two structures is a model of T, or $\mathcal{M} \cong \mathcal{N}$.

The projective sets are obtained from the Borel sets by iterating the operations of projection (continuous real-valued images) and complementation, forming a hierarchy of length ω. Closing under countable unions and extending that hierarchy up through the countable ordinals yields the *σ-projective sets*.

Descriptive complexity of 2nd-order theories

Definition ($\mathrm{Mod}_S(\sigma)$)

For σ a sentence of some logic with signature S, define $\mathrm{Mod}_S(\sigma) = \{\mathcal{M} \in \mathrm{Mod}_S : \mathcal{M} \models \sigma\}$ where Mod_S is the collection of countable S-models and $\mathcal{M}$ is a model with universe ω. (If S is clear from context we may omit it.)

The projective sets are obtained from the Borel sets by iterating the operations of projection (continuous real-valued images) and complementation, forming a hierarchy of length ω. Closing under countable unions and extending that hierarchy up through the countable ordinals yields the *σ-projective sets*.

One can also characterize the collection of σ-projective sets as $L_{\omega_1}(\mathbb{R}) \cap \mathscr{P}(\mathbb{R})$.

Descriptive complexity of 2nd-order theories

The projective sets are obtained from the Borel sets by iterating the operations of projection (continuous real-valued images) and complementation, forming a hierarchy of length ω. Closing under countable unions and extending that hierarchy up through the countable ordinals yields the *σ-projective sets*.

One can also characterize the collection of σ-projective sets as $L_{\omega_1}(\mathbb{R}) \cap \mathscr{P}(\mathbb{R})$.

Lemma

Let S be a countable signature, and let σ be a second-order S-sentence. Then $\mathrm{Mod}_S(\sigma)$ is projective.

Descriptive complexity of 2nd-order theories

Definition ($\cong_T$)

Let S be a countable signature, and let T be an S-theory of some logic. The equivalence relation of isomorphism of models of T is the equivalence relation $\cong_T$ on Mod_S defined by declaring that $\mathcal{M} \cong_T \mathcal{N}$ if and only if either: neither of the two structures is a model of T, or $\mathcal{M} \cong \mathcal{N}$.

Lemma

Let S be a countable signature, and let σ be a second-order S-sentence. Then $\mathrm{Mod}_S(\sigma)$ is projective.

Proof.

By induction on the complexity of σ. □

Descriptive complexity of 2nd-order theories

Lemma

Let S be a countable signature, and let T be a second-order S-theory. Then $\mathrm{Mod}_S(T)$ is σ-projective; it is in fact a countable intersection of projective sets.

Descriptive complexity of 2nd-order theories

Lemma

Let S be a countable signature, and let T be a second-order S-theory. Then $\mathrm{Mod}_S(T)$ is σ-projective; it is in fact a countable intersection of projective sets.

Proposition 1

Let S be a countable signature, and let T be an S-theory of some logic. The complexity of the equivalence relation $\cong_T$ (on $\mathcal{P}(\mathbb{R})$) is the minimum complexity that includes both $\mathbf{\Sigma}^1_1$ and the complexity of the complement of $\mathrm{Mod}_S(T)$.

Descriptive complexity of 2nd-order theories

Lemma

Let S be a countable signature, and let T be a second-order S-theory. Then $\mathrm{Mod}_S(T)$ is σ-projective; it is in fact a countable intersection of projective sets.

Proposition 1

Let S be a countable signature, and let T be an S-theory of some logic. The complexity of the equivalence relation $\cong_T$ (on $\mathcal{P}(\mathbb{R})$) is the minimum complexity that includes both $\mathbf{\Sigma}^1_1$ and the complexity of the complement of $\mathrm{Mod}_S(T)$.

Proposition 2

If T is a second-order theory of bounded quantifier complexity, then $\cong_T$ is a projective equivalence relation. Omitting "bounded", then $\cong_T$ is σ-projective.

σ-projective sets

Definition

The collection of σ-projective sets is the smallest σ-algebra containing the open subsets (of $\mathbb{R}$) and closed under projections (continuous real-valued images).

σ-projective sets

Definition

The collection of σ-projective sets is the smallest σ-algebra containing the open subsets (of $\mathbb{R}$) and closed under projections (continuous real-valued images).

Lemma

Let S be a countable signature, and let T be a second-order S-theory. Then $\mathrm{Mod}_S(T)$ is σ-projective; it is in fact a countable intersection of projective sets.

Second-order Morley

Morley's Theorem fails consistently for second-order logic

Idea of proof.

Adjoin $\aleph_2$ Cohen reals to L. Then add $\aleph_3$ random reals. In the resulting model $2^{\aleph_0} = \aleph_3$, but there are still only $\aleph_2$ reals Cohen over L.
We exhibit a second-order theory T whose countable models are all isomorphic to some $\langle \omega, <, x \rangle$, where x is Cohen over L. □

Second-order Morley

Morley's Theorem fails consistently for second-order logic

Idea of proof.

Adjoin $\aleph_2$ Cohen reals to L. Then add $\aleph_3$ random reals. In the resulting model $2^{\aleph_0} = \aleph_3$, but there are still only $\aleph_2$ reals Cohen over L.
We exhibit a second-order theory T whose countable models are all isomorphic to some $\langle \omega, <, x \rangle$, where x is Cohen over L. □

Definition

A second-order formula is $\forall_n$ if it is equivalent to a prenex formula that begins with a second-order universal quantifier and has a total of n blocks of quantifiers. Similarly for $\exists_n$.

Second-order Morley

Morley's Theorem fails consistently for second-order logic

Definition

A second-order formula is $\forall_n$ if it is equivalent to a prenex formula that begins with a second-order universal quantifier and has a total of n blocks of quantifiers. Similarly for $\exists_n$.

Lemma

Let $S = \{+, \cdot, <, 0, 1, R\}$, where R is a unary relation symbol. Suppose that $A \subseteq 2^\omega$ and A is Σ^1_n (respectively, Π^1_n) for some $n \geq 2$. Then there is a $\exists_n$ (respectively, $\forall_n$) S-sentence such that every model of σ is isomorphic to $(\omega, +, \cdot, <, 0, 1, R)$ for some $R \in A$, and moreover $(\omega, +, \cdot, <, 0, 1, R) \cong (\omega, +, \cdot, <, 0, 1, R')$ if and only if $R = R'$. In particular, the number of isomorphism classes of models of σ is $|A|$.

Second-order Morley

Morley's Theorem fails consistently for second-order logic

Lemma

Let $S = \{+, \cdot, <, 0, 1, R\}$, where R is a unary relation symbol. Suppose that $A \subseteq 2^\omega$ and A is Σ^1_n (respectively, Π^1_n) for some $n \geq 2$. Then there is a $\exists_n$ (respectively, $\forall_n$) S-sentence such that every model of σ is isomorphic to $(\omega, +, \cdot, <, 0, 1, R)$ for some $R \in A$, and moreover $(\omega, +, \cdot, <, 0, 1, R) \cong (\omega, +, \cdot, <, 0, 1, R')$ if and only if $R = R'$. In particular, the number of isomorphism classes of models of σ is $|A|$.

Proof. We give the proof for Σ^1_n; the proof for Π^1_n is similar.

Proof. We give the proof for Σ^1_n; the proof for Π^1_n is similar.

Let $\mathrm{PA}_{\mathrm{II}}$ be second-order Peano arithmetic, which can be expressed as a $\forall_1$ sentence $S \setminus \{R\}$. It is well-known that $\mathrm{PA}_{\mathrm{II}}$ is categorical, and (up to isomorphism) the only $S \setminus \{R\}$ structure satisfying $\mathrm{PA}_{\mathrm{II}}$ is $(\omega, +, \cdot, <, 0, 1)$.

Proof. We give the proof for Σ^1_n; the proof for Π^1_n is similar.

Let $\mathrm{PA_{II}}$ be second-order Peano arithmetic, which can be expressed as a $\forall_1$ sentence $S \setminus \{R\}$. It is well-known that $\mathrm{PA_{II}}$ is categorical, and (up to isomorphism) the only $S \setminus \{R\}$ structure satisfying $\mathrm{PA_{II}}$ is $(\omega, +, \cdot, <, 0, 1)$.

Since A is Σ^1_n, there is an $\exists_n$ formula $\phi(X)$, in one second-order variable X, such that for every $a \in 2^\omega$, $a \in A$ if and only if $(\omega, +, \cdot, <, 0, 1) \models \phi(a)$ (refer to Moschovakis's *Descriptive Set Theory* for the details).

Proof. We give the proof for Σ^1_n; the proof for Π^1_n is similar.

Let $\mathrm{PA}_{\mathrm{II}}$ be second-order Peano arithmetic, which can be expressed as a $\forall_1$ sentence $S \setminus \{R\}$. It is well-known that $\mathrm{PA}_{\mathrm{II}}$ is categorical, and (up to isomorphism) the only $S \setminus \{R\}$ structure satisfying $\mathrm{PA}_{\mathrm{II}}$ is $(\omega, +, \cdot, <, 0, 1)$.

Since A is Σ^1_n, there is an $\exists_n$ formula $\phi(X)$, in one second-order variable X, such that for every $a \in 2^\omega$, $a \in A$ if and only if $(\omega, +, \cdot, <, 0, 1) \models \phi(a)$ (refer to Moschovakis's *Descriptive Set Theory* for the details).

Let σ be $\mathrm{PA}_{\mathrm{II}} \wedge \phi(R)$; σ is $\exists_n$ because $\mathrm{PA}_{\mathrm{II}}$ is $\forall_1$ and ϕ is $\exists_n$ with $n \geq 2$. As noted above, every model of σ is isomorphic to one of the form $(\omega, +, \cdot, <, 0, 1, R)$ for some $R \in A$.

Proof. We give the proof for Σ^1_n; the proof for Π^1_n is similar.

Let $\mathrm{PA}_{\mathrm{II}}$ be second-order Peano arithmetic, which can be expressed as a $\forall_1$ sentence $S \setminus \{R\}$. It is well-known that $\mathrm{PA}_{\mathrm{II}}$ is categorical, and (up to isomorphism) the only $S \setminus \{R\}$ structure satisfying $\mathrm{PA}_{\mathrm{II}}$ is $(\omega, +, \cdot, <, 0, 1)$.

Since A is Σ^1_n, there is an $\exists_n$ formula $\phi(X)$, in one second-order variable X, such that for every $a \in 2^\omega$, $a \in A$ if and only if $(\omega, +, \cdot, <, 0, 1) \models \phi(a)$ (refer to Moschovakis's *Descriptive Set Theory* for the details).

Let σ be $\mathrm{PA}_{\mathrm{II}} \wedge \phi(R)$; σ is $\exists_n$ because $\mathrm{PA}_{\mathrm{II}}$ is $\forall_1$ and ϕ is $\exists_n$ with $n \geq 2$. As noted above, every model of σ is isomorphic to one of the form $(\omega, +, \cdot, <, 0, 1, R)$ for some $R \in A$.

Finally, $(\omega, <)$ has no non-trivial automorphisms, so the only possible isomorphism from $(\omega, +, \cdot, <, 0, 1, R)$ to $(\omega, +, \cdot, <, 0, 1, R')$ is the identity map. □

Theorem

It is consistent with ZFC *that there exists a second-order sentence with exactly* $\aleph_2$ *non-isomorphic countable models while the continuum is* $\aleph_3$*.*

Theorem

It is consistent with ZFC that there exists a second-order sentence with exactly $\aleph_2$ non-isomorphic countable models while the continuum is $\aleph_3$.

Proof.

Force over L to add $\aleph_2$ Cohen reals, and then force over the resulting model to add $\aleph_3$ random reals. Let C be the set of reals in this model that are Cohen over L. Then $|C| = \aleph_2$, while $2^{\aleph_0} = \aleph_3$. It is a folklore result that C is Π^1_2. Thus, the Lemma provides a $\forall_2$ sentence with exactly $\aleph_2$ models (all of which are countable). □

Second-order Morley

Morley's Theorem is consistently true (modulo a large cardinal) for second-order logic

Second-order Morley

Morley's Theorem is consistently true (modulo a large cardinal) for second-order logic

Theorem

If it is consistent that there is a supercompact cardinal, then it is consistent that $\neg$ CH *and every equivalence relation on the power set of* $\mathbb{R}$ *that is in* $L(\mathbb{R})$ *has* $\leq \aleph_1$ *or a perfect set of inequivalent elements.*

Second-order Morley

Morley's Theorem is consistently true (modulo a large cardinal) for second-order logic

Theorem

If it is consistent that there is a supercompact cardinal, then it is consistent that $\neg$ CH *and every equivalence relation on the power set of* $\mathbb{R}$ *that is in* $L(\mathbb{R})$ *has* $\leq \aleph_1$ *or a perfect set of inequivalent elements.*

Proof. This is implicit in [FM 95] who use the usual model for PFA .

Second-order Morley

Theorem

If it is consistent that there is a supercompact cardinal, then it is consistent that $\neg$ CH *and every equivalence relation on the power set of* $\mathbb{R}$ *that is in* $L(\mathbb{R})$ *has* $\leq \aleph_1$ *or a perfect set of inequivalent elements.*

Proof. This is implicit in [FM 95] who use the usual model for PFA .

[FM95] M. Foreman and M. Magidor. Large cardinals and definable counterexamples to the continuum hypothesis. *Annals of Pure and Applied Logic*, 76(1):47–97, 1995.

Second-order Morley

Theorem

If it is consistent that there is a supercompact cardinal, then it is consistent that $\neg$ CH *and every equivalence relation on the power set of* $\mathbb{R}$ *that is in* $L(\mathbb{R})$ *has* $\leq \aleph_1$ *or a perfect set of inequivalent elements.*

Proof. This is implicit in [FM 95] who use the usual model for PFA .

Corollary

Second-order Morley holds in this model since σ*-projective sets are in* $L(\mathbb{R})$*.*

Idea of [FM95] proof in the σ-projective case

Definition (Thin)

An equivalence relation E on a Polish space X is *thin* if there is no perfect set of pairwise E-inequivalent elements of X.

Idea of [FM95] proof in the σ-projective case

Definition (Thin)

An equivalence relation E on a Polish space X is *thin* if there is no perfect set of pairwise E-inequivalent elements of X.

Consider the usual model for PFA. The code for a σ-projective set of reals is a real and appears at an initial stage of the iteration.

Idea of [FM95] proof in the σ-projective case

Definition (Thin)

An equivalence relation E on a Polish space X is *thin* if there is no perfect set of pairwise E-inequivalent elements of X.

Consider the usual model for PFA. The code for a σ-projective set of reals is a real and appears at an initial stage of the iteration.

If an equivalence relation in $L(\mathbb{R})$ is thin, then FM show (via supercompactness) that the interpretation of that code when it appears is also thin ("downwards generic absoluteness").

Idea of [FM95] proof in the σ-projective case

Definition (Thin)

An equivalence relation E on a Polish space X is *thin* if there is no perfect set of pairwise E-inequivalent elements of X.

Consider the usual model for PFA. The code for a σ-projective set of reals is a real and appears at an initial stage of the iteration.

If an equivalence relation in $L(\mathbb{R})$ is thin, then FM show (via supercompactness) that the interpretation of that code when it appears is also thin ("downwards generic absoluteness"). Without loss of generality, CH holds at that stage, since it holds cofinally often.

Idea of [FM95] proof in the σ-projective case

Definition (Thin)

An equivalence relation E on a Polish space X is *thin* if there is no perfect set of pairwise E-inequivalent elements of X.

Consider the usual model for PFA. The code for a σ-projective set of reals is a real and appears at an initial stage of the iteration.

If an equivalence relation in $L(\mathbb{R})$ is thin, then FM show (via supercompactness) that the interpretation of that code when it appears is also thin ("downwards generic absoluteness"). Without loss of generality, CH holds at that stage, since it holds cofinally often.

They then prove via supercompactness that, because the interpretation of the code at that stage is thin, and because the equivalence relation is in $L(\mathbb{R})$, the rest of the forcing adds no new equivalence classes ("upwards generic absoluteness").

Idea of [FM95] proof in the σ-projective case

Consider the usual model for PFA. The code for a σ-projective set of reals is a real and appears at an initial stage of the iteration.

If an equivalence relation in $L(\mathbb{R})$ is thin, then FM show (via supercompactness) that the interpretation of that code when it appears is also thin ("downwards generic absoluteness"). Without loss of generality, CH holds at that stage, since it holds cofinally often. Thus, at that stage, the interpretation of the code for the equivalence relation has at most $\aleph_1$ equivalence classes.

They then prove via supercompactness that, because the interpretation of the code at that stage is thin, and because the equivalence relation is in $L(\mathbb{R})$, the rest of the forcing adds no new equivalence classes ("upwards generic absoluteness"). But the interpretation of the code at the end of the forcing is just the equivalence relation we started with, so indeed, it has no more than $\aleph_1$ equivalence classes. □

Strengthening FM

With more modern methods, we can greatly reduce the strength of the large cardinal hypothesis from [FM95], at least for σ-projective equivalence relations and hence for second-order Morley.

Strengthening FM

Theorem (C)

Suppose there are infinitely many Woodin cardinals. Then there is a model of $\neg\,\mathrm{CH}$ *in which every* σ*-projective equivalence relation on the power set of* $\mathbb{R}$ *has* $\leq \aleph_1$ *or a perfect set of inequivalent elements.*

Strengthening FM

Theorem (C)

Suppose there are infinitely many Woodin cardinals. Then there is a model of $\neg$ CH in which every σ-projective equivalence relation on the power set of $\mathbb{R}$ has $\leq \aleph_1$ or a perfect set of inequivalent elements.

Definition (Woodin)

(i) Let $\kappa < \delta$ be ordinals and $A \subseteq V_\delta$. Then κ is called *A-reflecting in δ* if and only if for all $\eta < \delta$ there is an elementary embedding $i : V \to M$ with critical point κ such that $i(\kappa) > \eta$ and $i(A) \cap V_\eta = A \cap V_\eta$.

Strengthening FM

Theorem (C)

Suppose there are infinitely many Woodin cardinals. Then there is a model of $\neg$ CH *in which every* σ*-projective equivalence relation on the power set of* $\mathbb{R}$ *has* $\leq \aleph_1$ *or a perfect set of inequivalent elements.*

Definition (Woodin)

(i) Let $\kappa < \delta$ be ordinals and $A \subseteq V_\delta$. Then κ is called *A*-reflecting in δ if and only if for all $\eta < \delta$ there is an elementary embedding $i : V \to M$ with critical point κ such that $i(\kappa) > \eta$ and $i(A) \cap V_\eta = A \cap V_\eta$.

(ii) A cardinal δ is a Woodin cardinal if and only if for all $A \subseteq \delta$ there is some $\kappa < \delta$ that is A-reflecting in δ.

Strengthening FM

Theorem (C)

Suppose there are infinitely many Woodin cardinals. Then there is a model of $\neg$CH *in which every* σ*-projective equivalence relation on the power set of* $\mathbb{R}$ *has* $\leq \aleph_1$ *or a perfect set of inequivalent elements.*

Definition (Woodin)

(i) Let $\kappa < \delta$ be ordinals and $A \subseteq V_\delta$. Then κ is called *A-reflecting in* δ if and only if for all $\eta < \delta$ there is an elementary embedding $i : V \to M$ with critical point κ such that $i(\kappa) > \eta$ and $i(A) \cap V_\eta = A \cap V_\eta$.

(ii) A cardinal δ is a Woodin cardinal if and only if for all $A \subseteq \delta$ there is some $\kappa < \delta$ that is A-reflecting in δ.

In terms of consistency strength, measurable < Woodin < a measurable above infinitely many Woodins < supercompact.

By work of Martin, Steele, Woodin, many applications of supercompactness can be replaced by the assumption of (a measurable above) infinitely many Woodins.

By work of Martin, Steele, Woodin, many applications of supercompactness can be replaced by the assumption of (a measurable above) infinitely many Woodins.

The proof of the Theorem is technical but similar to the [**FM**] supercompact proof sketched earlier.

By work of Martin, Steele, Woodin, many applications of supercompactness can be replaced by the assumption of (a measurable above) infinitely many Woodins.

The proof of the Theorem is technical but similar to the [**FM**] supercompact proof sketched earlier.

One simplification is that we don't need PFA ; instead we just iteratively blow up the continuum and collapse it down to $\aleph_1$, entailing that in the final model $2^{\aleph_0} = \aleph_2$.

By work of Martin, Steele, Woodin, many applications of supercompactness can be replaced by the assumption of (a measurable above) infinitely many Woodins.

The proof of the Theorem is technical but similar to the [**FM**] supercompact proof sketched earlier.

One simplification is that we don't need PFA ; instead we just iteratively blow up the continuum and collapse it down to $\aleph_1$, entailing that in the final model $2^{\aleph_0} = \aleph_2$.

Surprising us, Jing Zhang [**TZ**, *Arch. Math Logic*, just appeared] later proved:

Theorem (D)

Adjoin at least $\aleph_2$ Cohen reals to a model of CH *. Then Absolute Morley holds.*

As in the proofs of Theorems B and C, generic absoluteness is key. It turns out that because Cohen real forcing is so simple and homogeneous and because adding one Cohen real by forcing adds perfectly many Cohen reals, we don't need the large cardinal. This latter observation substitutes for upwards generic absoluteness.

As in the proofs of Theorems B and C, generic absoluteness is key. It turns out that because Cohen real forcing is so simple and homogeneous and because adding one Cohen real by forcing adds perfectly many Cohen reals, we don't need the large cardinal. This latter observation substitutes for upwards generic absoluteness.

If we add $\aleph_2$-many Cohen reals, the code for a σ-projective set appeared at a stage where CH holds.

As in the proofs of Theorems B and C, generic absoluteness is key. It turns out that because Cohen real forcing is so simple and homogeneous and because adding one Cohen real by forcing adds perfectly many Cohen reals, we don't need the large cardinal. This latter observation substitutes for upwards generic absoluteness.

If we add $\aleph_2$-many Cohen reals, the code for a σ-projective set appeared at a stage where CH holds.

In the case where we add more than $\aleph_2$ Cohen reals, we need to apply an automorphism argument to get that without loss of generality we may assume that the real that codes the σ-projective set appears in the first ω_1 stages.

As in the proofs of Theorems B and C, generic absoluteness is key. It turns out that because Cohen real forcing is so simple and homogeneous and because adding one Cohen real by forcing adds perfectly many Cohen reals, we don't need the large cardinal. This latter observation substitutes for upwards generic absoluteness.

If we add $\aleph_2$-many Cohen reals, the code for a σ-projective set appeared at a stage where CH holds.

In the case where we add more than $\aleph_2$ Cohen reals, we need to apply an automorphism argument to get that without loss of generality we may assume that the real that codes the σ-projective set appears in the first ω_1 stages.

The required downward generic absoluteness is proved by induction on the complexity of the σ-projective formulas that define our equivalence relations. □

Proof of Theorem C

Proof of Theorem C. Let M be a model of ZFC with a sequence of Woodin cardinals $(\delta_i : i < \omega)$ and let κ be the least inaccessible cardinal in M. In particular, $\kappa < \delta_0$.

Proof of Theorem C

Proof of Theorem C. Let M be a model of ZFC with a sequence of Woodin cardinals $(\delta_i : i < \omega)$ and let κ be the least inaccessible cardinal in M. In particular, $\kappa < \delta_0$. We consider a generic extension $M[G]$ of M via the following forcing:

Proof of Theorem C

Proof of Theorem C. Let M be a model of ZFC with a sequence of Woodin cardinals $(\delta_i : i < \omega)$ and let κ be the least inaccessible cardinal in M. In particular, $\kappa < \delta_0$. We consider a generic extension $M[G]$ of M via the following forcing:

Consider the countable support iteration $\mathbb{P}_\kappa$ of length κ of the partial orders $\{\dot{\mathbb{Q}}_\alpha : \alpha < \kappa\}$.

Proof of Theorem C

Proof of Theorem C. Let M be a model of ZFC with a sequence of Woodin cardinals $(\delta_i : i < \omega)$ and let κ be the least inaccessible cardinal in M. In particular, $\kappa < \delta_0$. We consider a generic extension $M[G]$ of M via the following forcing:

Consider the countable support iteration $\mathbb{P}_\kappa$ of length κ of the partial orders $\{\dot{\mathbb{Q}}_\alpha : \alpha < \kappa\}$.

At an even stage $\alpha < \kappa$ that is not a limit stage of cofinality ω, let $\dot{\mathbb{Q}}_\alpha$ be the usual countably closed collapse of the continuum (of the current stage of the iteration) to ω_1.

Proof of Theorem C

Proof of Theorem C. Let M be a model of ZFC with a sequence of Woodin cardinals $(\delta_i : i < \omega)$ and let κ be the least inaccessible cardinal in M. In particular, $\kappa < \delta_0$. We consider a generic extension $M[G]$ of M via the following forcing:

Consider the countable support iteration $\mathbb{P}_\kappa$ of length κ of the partial orders $\{\dot{\mathbb{Q}}_\alpha : \alpha < \kappa\}$.

At an even stage $\alpha < \kappa$ that is not a limit stage of cofinality ω, let $\dot{\mathbb{Q}}_\alpha$ be the usual countably closed collapse of the continuum (of the current stage of the iteration) to ω_1. At limit stages $\alpha < \kappa$ of cofinality ω, let $\dot{\mathbb{Q}}_\alpha$ be the trivial forcing.

Proof of Theorem C

Proof of Theorem C. Let M be a model of ZFC with a sequence of Woodin cardinals $(\delta_i : i < \omega)$ and let κ be the least inaccessible cardinal in M. In particular, $\kappa < \delta_0$. We consider a generic extension $M[G]$ of M via the following forcing:

Consider the countable support iteration $\mathbb{P}_\kappa$ of length κ of the partial orders $\{\dot{\mathbb{Q}}_\alpha : \alpha < \kappa\}$.

At an even stage $\alpha < \kappa$ that is not a limit stage of cofinality ω, let $\dot{\mathbb{Q}}_\alpha$ be the usual countably closed collapse of the continuum (of the current stage of the iteration) to ω_1. At limit stages $\alpha < \kappa$ of cofinality ω, let $\dot{\mathbb{Q}}_\alpha$ be the trivial forcing. At an odd stage $\beta < \kappa$, let $\dot{\mathbb{Q}}_\beta$ add β Cohen reals.

Proof of Theorem C

Proof of Theorem C. Let M be a model of ZFC with a sequence of Woodin cardinals $(\delta_i : i < \omega)$ and let κ be the least inaccessible cardinal in M. In particular, $\kappa < \delta_0$. We consider a generic extension $M[G]$ of M via the following forcing:

Consider the countable support iteration $\mathbb{P}_\kappa$ of length κ of the partial orders $\{\dot{\mathbb{Q}}_\alpha : \alpha < \kappa\}$.

At an even stage $\alpha < \kappa$ that is not a limit stage of cofinality ω, let $\dot{\mathbb{Q}}_\alpha$ be the usual countably closed collapse of the continuum (of the current stage of the iteration) to ω_1. At limit stages $\alpha < \kappa$ of cofinality ω, let $\dot{\mathbb{Q}}_\alpha$ be the trivial forcing. At an odd stage $\beta < \kappa$, let $\dot{\mathbb{Q}}_\beta$ add β Cohen reals.

Note that each individual forcing $\dot{\mathbb{Q}}_\alpha$ and hence the whole iteration $\mathbb{P}_\kappa$ is proper and in particular preserves $\aleph_1$.

Proof of Theorem C

Consider the countable support iteration $\mathbb{P}_\kappa$ of length κ of the partial orders $\{\dot{\mathbb{Q}}_\alpha : \alpha < \kappa\}$.

At an even stage $\alpha < \kappa$ that is not a limit stage of cofinality ω, let $\dot{\mathbb{Q}}_\alpha$ be the usual countably closed collapse of the continuum (of the current stage of the iteration) to ω_1. At limit stages $\alpha < \kappa$ of cofinality ω, let $\dot{\mathbb{Q}}_\alpha$ be the trivial forcing. At an odd stage $\beta < \kappa$, let $\dot{\mathbb{Q}}_\beta$ add β Cohen reals.

Note that each individual forcing $\dot{\mathbb{Q}}_\alpha$ and hence the whole iteration $\mathbb{P}_\kappa$ is proper and in particular preserves $\aleph_1$.

The final model $M[G]$ satisfies $2^{\aleph_0} = \aleph_2$ but CH holds at cofinally many initial segments $M[G\restriction_\alpha]$ of the iteration. (Here $G\restriction_\alpha$ denotes the canonical restriction of the generic G to the initial segment $\mathbb{P}_\alpha$ of the iteration.)

Proof of Theorem C

Consider the countable support iteration $\mathbb{P}_\kappa$ of length κ of the partial orders $\{\dot{\mathbb{Q}}_\alpha : \alpha < \kappa\}$.

At an even stage $\alpha < \kappa$ that is not a limit stage of cofinality ω, let $\dot{\mathbb{Q}}_\alpha$ be the usual countably closed collapse of the continuum (of the current stage of the iteration) to ω_1. At limit stages $\alpha < \kappa$ of cofinality ω, let $\dot{\mathbb{Q}}_\alpha$ be the trivial forcing. At an odd stage $\beta < \kappa$, let $\dot{\mathbb{Q}}_\beta$ add β Cohen reals.

Note that each individual forcing $\dot{\mathbb{Q}}_\alpha$ and hence the whole iteration $\mathbb{P}_\kappa$ is proper and in particular preserves $\aleph_1$.

The final model $M[G]$ satisfies $2^{\aleph_0} = \aleph_2$ but CH holds at cofinally many initial segments $M[G{\restriction}_\alpha]$ of the iteration. (Here $G{\restriction}_\alpha$ denotes the canonical restriction of the generic G to the initial segment $\mathbb{P}_\alpha$ of the iteration.)

We claim that in $M[G]$ every σ-projective equivalence relation on the power set of $\mathbb{R}$ has $\leq \aleph_1$ or a perfect set of inequivalent elements.

Proof components

The proof that the above model works uses ideas similar to those of [FM95]. The key definitions and fact are:

Proof components

Definition (Universally Baire [FMW92])

Let A be a set of reals. A is *λ-universally Baire* if for any topological space X with a regular open basis of cardinality $\leq \lambda$, and any continuous $f : X \to \mathbb{R}$, the preimage $f^{-1}(A)$ has the property of Baire.
A is universally Baire if it is λ-universally Baire for every infinite λ.

Proof components

Definition (Universally Baire [FMW92])

Let A be a set of reals. A is *λ-universally Baire* if for any topological space X with a regular open basis of cardinality $\leq \lambda$, and any continuous $f : X \to \mathbb{R}$, the preimage $f^{-1}(A)$ has the property of Baire.
A is universally Baire if it is λ-universally Baire for every infinite λ.

Lemma (Woodin)

Let M be a model of ZFC *with a sequence of Woodin cardinals $(\delta_i : i < \omega)$. Then all σ-projective sets in M are $< \delta_0$-universally Baire.*

Proof components

Definition (Universally Baire [FMW92])

Let A be a set of reals. A is *λ-universally Baire* if for any topological space X with a regular open basis of cardinality $\leq \lambda$, and any continuous $f : X \to \mathbb{R}$, the preimage $f^{-1}(A)$ has the property of Baire.
A is universally Baire if it is λ-universally Baire for every infinite λ.

Lemma (Woodin)

Let M be a model of ZFC *with a sequence of Woodin cardinals $(\delta_i : i < \omega)$. Then all σ-projective sets in M are $< \delta_0$-universally Baire.*

[FMW92] Q. Feng, M. Magidor, and H. Woodin. Universally Baire Sets of Reals.
In H. Judah, W. Just, and H. Woodin, editors, *Set Theory of the Continuum*, pages 203–242, New York, NY, 1992. Springer US

Proof components

Definition (σ-projective)

A formula $\varphi(v)$ in the language $L_{\omega_1,\omega}$ is called *σ-projective* if and only if it is Σ_α for some $\alpha < \omega_1$.

Proof components

Definition (σ-projective)

A formula $\varphi(v)$ in the language $L_{\omega_1,\omega}$ is called *σ-projective* if and only if it is Σ_α for some $\alpha < \omega_1$.

Note that a set of reals A is σ-projective iff there is a Σ_α-formula φ for some $\alpha < \omega_1$ and a parameter $z \in {}^\omega\omega$ such that A is $\Sigma_\alpha(z)$ definable in second-order arithmetic in z

Proof components

Definition (σ-projective)

A formula $\varphi(v)$ in the language $L_{\omega_1,\omega}$ is called *σ-projective* if and only if it is Σ_α for some $\alpha < \omega_1$.

Note that a set of reals A is σ-projective iff there is a Σ_α-formula φ for some $\alpha < \omega_1$ and a parameter $z \in {}^\omega\omega$ such that A is $\Sigma_\alpha(z)$ definable in second-order arithmetic in z, i.e.,

$$A = \{x \in {}^\omega\omega : \mathcal{A}^2(z) \models \varphi(z)\},$$

where $\mathcal{A}^2(z)$ is the two-sorted structure

$$(\omega, {}^\omega\omega, ap, +, \times, \exp, <, 0, 1, z).$$

Here $ap : {}^\omega\omega \to \omega$ denotes the binary operation of *application*, i.e., $ap(x, n) = x(n)$.

Proof components

Definition (σ-projective)

A formula $\varphi(v)$ in the language $L_{\omega_1,\omega}$ is called *σ-projective* if and only if it is Σ_α for some $\alpha < \omega_1$.

Lemma

Let M be a model of ZFC with a sequence of Woodin cardinals $(\delta_i : i < \omega)$. Let $\mathbb{P}$ be a partial order in M with $|\mathbb{P}| < \delta_0$, and let G be $\mathbb{P}$-generic over M.
Then for every σ-projective formula $\varphi(v)$ and every $x \in ({}^\omega\omega)^M$,

$$(\mathcal{A}^2(x))^M \models \varphi(x) \text{ if and only if } (\mathcal{A}^2(x))^{M[G]} \models \varphi(x).$$

Proof components

Lemma

Let M be a model of ZFC with a sequence of Woodin cardinals $(\delta_i : i < \omega)$. Let $\mathbb{P}$ be a partial order in M with $|\mathbb{P}| < \delta_0$, and let G be $\mathbb{P}$-generic over M.
Then for every σ-projective formula $\varphi(v)$ and every $x \in ({}^{\omega}\omega)^M$,

$$(\mathcal{A}^2(x))^M \models \varphi(x) \text{ if and only if } (\mathcal{A}^2(x))^{M[G]} \models \varphi(x).$$

We now have everything we need in order to carry out the FM proof for σ-projective sets with much weaker large cardinal hypotheses.

Proof overview

We now proceed as in FM, arguing that if the (σ-projective) set of isomorphism classes of countable models for second-order theory is thin in the final model, then its code appears in an intermediate model in which there is an infinite sequence of Woodins. In that model, that code codes a σ-projective set, which, by the Woodins is universally Baire.

Proof overview

We now proceed as in FM, arguing that if the (σ-projective) set of isomorphism classes of countable models for second-order theory is thin in the final model, then its code appears in an intermediate model in which there is an infinite sequence of Woodins. In that model, that code codes a σ-projective set, which, by the Woodins is universally Baire.

That model can be assumed to be a model of CH, so there are only $\aleph_1$ isomorphism classes there.

Proof overview

We now proceed as in FM, arguing that if the (σ-projective) set of isomorphism classes of countable models for second-order theory is thin in the final model, then its code appears in an intermediate model in which there is an infinite sequence of Woodins. In that model, that code codes a σ-projective set, which, by the Woodins is universally Baire.

That model can be assumed to be a model of CH, so there are only $\aleph_1$ isomorphism classes there.

The universal Baireness assures that no equivalence class added after that is equivalent to those in the thin set, so the thin set doesn't grow in size as its code becomes the code in the final model for our original thin set. This last argument about universal Baireness is carried out in [FM95]. □

Definition (Absolutely complementing trees)

Let (S, T) be trees on $\omega \times \kappa$ for some ordinal κ, and let η be an ordinal. We say (S, T) is η-absolutely complementing if and only if $p[S] = {}^{\omega}\omega \setminus p[T]$ in every $\mathrm{Col}(\omega, \eta)$-generic extension of V.

Definition (Absolutely complementing trees)

Let (S, T) be trees on $\omega \times \kappa$ for some ordinal κ, and let η be an ordinal. We say (S, T) is η-absolutely complementing if and only if $p[S] = {}^{\omega}\omega \setminus p[T]$ in every $\mathrm{Col}(\omega, \eta)$-generic extension of V.

We don't have time to define $p[T]$ here. The basic idea is that it's a projection of T into ${}^{\omega}\omega$. The trees S, T act as codes to provide absoluteness for complicated sets of reals. For details, see [FM95].

Definition (Absolutely complementing trees)

Let (S, T) be trees on $\omega \times \kappa$ for some ordinal κ, and let η be an ordinal. We say (S, T) is η-absolutely complementing if and only if $p[S] = {}^{\omega}\omega \setminus p[T]$ in every $\mathrm{Col}(\omega, \eta)$-generic extension of V.

We don't have time to define $p[T]$ here. The basic idea is that it's a projection of T into ${}^{\omega}\omega$. The trees S, T act as codes to provide absoluteness for complicated sets of reals. For details, see [FM95].

[FM95] M. Foreman and M. Magidor. Large cardinals and definable counterexamples to the continuum hypothesis.
Annals of Pure and Applied Logic, 76(1):47–97, 1995.
doi:https://doi.org/10.1016/0168-0072(94)00031-W

Definition (Absolutely complementing trees)

Let (S, T) be trees on $\omega \times \kappa$ for some ordinal κ, and let η be an ordinal. We say (S, T) is η-absolutely complementing if and only if $p[S] = {}^{\omega}\omega \setminus p[T]$ in every $\mathrm{Col}(\omega, \eta)$-generic extension of V.

Definition (Universally Baire, FMW92)

Let A be a set of reals. We say that A is *$< \eta$-universally Baire* ($< \eta$-uB) if for every ordinal $\nu < \eta$, there are ν-absolutely complementing trees (S, T) with $p[S] = A$.
We say A is *universally Baire* (uB) if it is $< \eta$-universally Baire for every ordinal η.

Definition (Universally Baire, FMW92)

Let A be a set of reals. We say that A is *$< \eta$-universally Baire* ($< \eta$-uB) if for every ordinal $\nu < \eta$, there are ν-absolutely complementing trees (S, T) with $p[S] = A$.
We say A is *universally Baire* (uB) if it is $< \eta$-universally Baire for every ordinal η.

Proposition 3 ([FMW92])

The two definitions of $< \eta$-universally Baire are equivalent.

Definition (Universally Baire, FMW92)

Let A be a set of reals. We say that A is *$<\eta$-universally Baire* ($<\eta$-uB) if for every ordinal $\nu < \eta$, there are ν-absolutely complementing trees (S, T) with $p[S] = A$.
We say A is *universally Baire* (uB) if it is $<\eta$-universally Baire for every ordinal η.

Lemma

Let M be a model of ZFC *with a sequence of Woodin cardinals $(\delta_i : i < \omega)$. Then all σ-projective sets in M are $<\delta_0$-universally-Baire.*

Thin Equivalence Relations

Can one prove in ZFC restricted versions of Absolute Morley?

Thin Equivalence Relations

Can one prove in ZFC restricted versions of Absolute Morley?

Yes for Σ^1_1 thin equivalence relations and hence for universal second order sentences. But

Thin Equivalence Relations

Can one prove in ZFC restricted versions of Absolute Morley?

Yes for Σ^1_1 thin equivalence relations and hence for universal second order sentences. But

Theorem

In ZFC there is no upper bound below $2^{\aleph_0}$ for the number of equivalence classes of thin Σ^1_2 equivalence relations.

Thin Equivalence Relations

Can one prove in ZFC restricted versions of Absolute Morley?

Yes for Σ^1_1 thin equivalence relations and hence for universal second order sentences. But

Theorem

In ZFC there is no upper bound below $2^{\aleph_0}$ for the number of equivalence classes of thin Σ^1_2 equivalence relations.

Can one prove upper bounds from large cardinals rather than from generic extensions involving large cardinals?

Thin Equivalence Relations

Can one prove in ZFC restricted versions of Absolute Morley?

Yes for Σ^1_1 thin equivalence relations and hence for universal second order sentences. But

Theorem

In ZFC there is no upper bound below $2^{\aleph_0}$ for the number of equivalence classes of thin Σ^1_2 equivalence relations.

Can one prove upper bounds from large cardinals rather than from generic extensions involving large cardinals?

Theorem

Assume the Axiom of Projective Determinacy (which follows from the existence of a supercompact cardinal). Then thin Σ^1_2 equivalence relations have at most $\aleph_1$ equivalence classes.

Thin Equivalence Relations

Can one prove in ZFC restricted versions of Absolute Morley?

Yes for Σ^1_1 thin equivalence relations and hence for universal second order sentences. But

Theorem

In ZFC there is no upper bound below $2^{\aleph_0}$ for the number of equivalence classes of thin Σ^1_2 equivalence relations.

Can one prove upper bounds from large cardinals rather than from generic extensions involving large cardinals?

Theorem

Assume the Axiom of Projective Determinacy (which follows from the existence of a supercompact cardinal). Then thin Σ^1_2 equivalence relations have at most $\aleph_1$ equivalence classes.

Other logics, e.g. with game quantifiers.