Steady state of the KPZ equation on an interval and Liouville quantum mechanics

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G. Barraquand, PLD

Steady state of the KPZ equation on an interval
and Liouville quantum mechanics, arXiv2105.15178, EPL 2022

Stationary measures of the KPZ equation on an interval from
Enaud-Derrida's matrix product ansatz representation, arXiv2209.03131

+ more .. Liouville CFT !! (gourmet dish)

distribution of exponential functional of Brownian motion

integrals of geometric Brownian motion $Z_L^w = \int_0^L dx\, e^{B(x)-wx}$

$U(x) \equiv B(x) - wx$

$$\mathbb{E}[e^{-pZ_L^w}] =$$

$$\int_{-\infty}^{+\infty} dU_L \int_{U(0)=0}^{U(L)=U_L} \mathcal{D}U(x) \;\; e^{-\int_0^L dx \frac{1}{2}(\frac{dU}{dx}+w)^2 + pe^{U(x)}}$$

=> Liouville quantum mechanics

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=> Liouville quantum mechanics

$$= e^{-\frac{w^2 L}{2}} \int_{-\infty}^{+\infty} dU_L e^{-wU_L} \langle U_L | e^{-LH_p} | U_0 = 0 \rangle$$

$$H_p = -\frac{1}{2}\frac{d^2}{dU^2} + pe^U$$

$$= \int_0^{+\infty} dk \int_{-\infty}^{+\infty} dU_L \psi_k(U_L)\psi_k^*(0) e^{-wU_L - \frac{L}{8}(k^2+4w^2)}$$

$$H_p \psi_k(U) = \frac{k^2}{8}\psi_k(U)$$

$$\psi_k(U) = \frac{1}{\pi}\sqrt{k\sinh(\pi k)} K_{ik}(2\sqrt{2p}e^{U/2})$$

Comtet, Texier (1998) Matsumoto-Yor (2005)

Comtet, Monthus, Yor (1998)

Comtet, Monthus (1994)

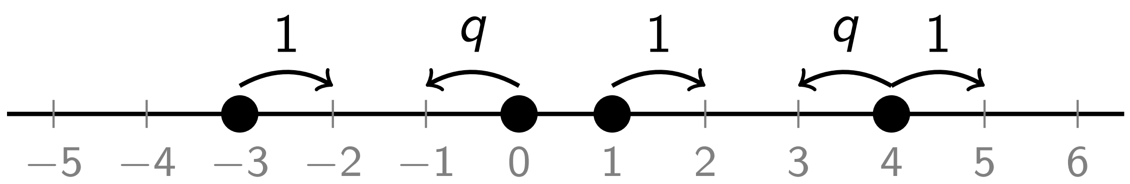
Tsvelik, Majumdar,..

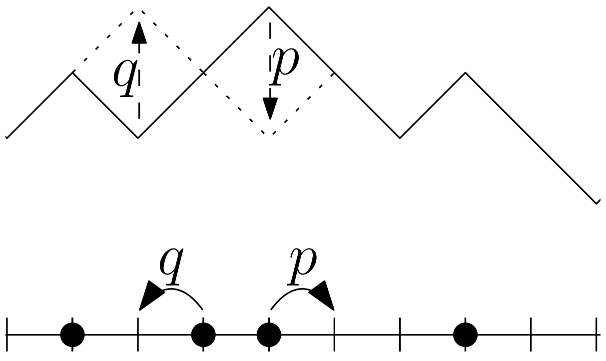
weak universality

weak asymmetry limit

convergence to the KPZ equation

asymmetric exclusion process on $\mathbb{Z}$





$p = 1$

height at time τ

$$\mathsf{h}(i,\tau) - \mathsf{h}(i-1,\tau) = \begin{cases} 1 & \text{site } i \text{ occupied} \\ -1 & \text{site } i \text{ empty} \end{cases}$$

$q = e^{-\epsilon}$

$\epsilon \to 0$

ASEP height function converges to a solution of the KPZ equation

$$\exp\left(\frac{\epsilon}{2}\mathsf{h}(4\epsilon^{-2}x, 16\epsilon^{-4}t) + c_\epsilon t\right) \underset{\epsilon \to 0}{\longrightarrow} Z(x,t)$$ solution of the SHE

Bertini–Giacomin 1997

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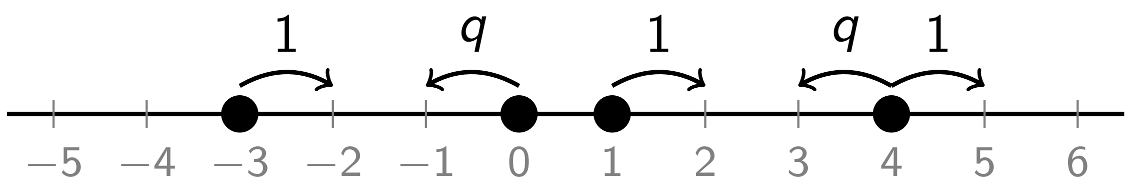

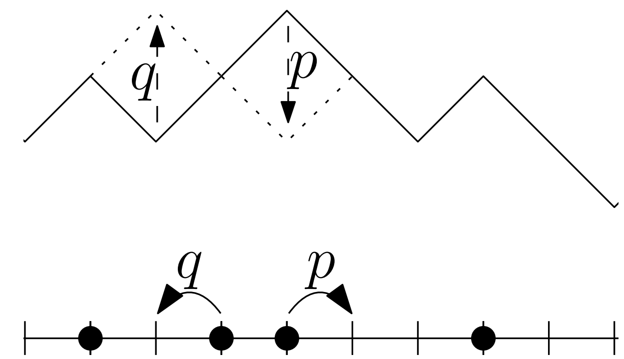

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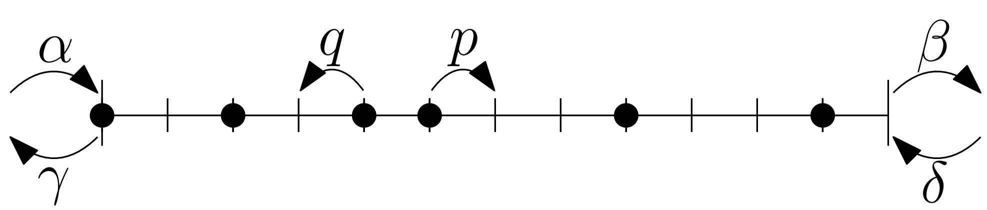
open ASEP



to KPZ equation on an interval

Corwin-Shen 2016

stationary measure for the KPZ equation

invariant measure

denote $H(x)$ stationary height field

if at t=t0 $h(x,t_0) - h(0,t_0) \equiv H(x)$

=> it remains true for all t>t0

since h(x,t) grows

$$h(0,t) \simeq v_\infty t + \chi\, t^{1/3}$$

only height differences

can be stationary

expect that h(x,t)-h(y,t)

becomes stationary $|x-y| \ll t^{2/3}$

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- on the full line $x \in \mathbb{R}$

$$H(x) = B(x) + ax$$

Funaki-Quastel 2014

Bertini-Giacomin 1997 ←

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full line ASEP stationary measures are i.i.d Bernoulli

sites occupied independently

w probability $0 < \rho < 1$

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- on the circle $x \in [0,1]$

$H(x)$ is Brownian bridge is unique invariant measure

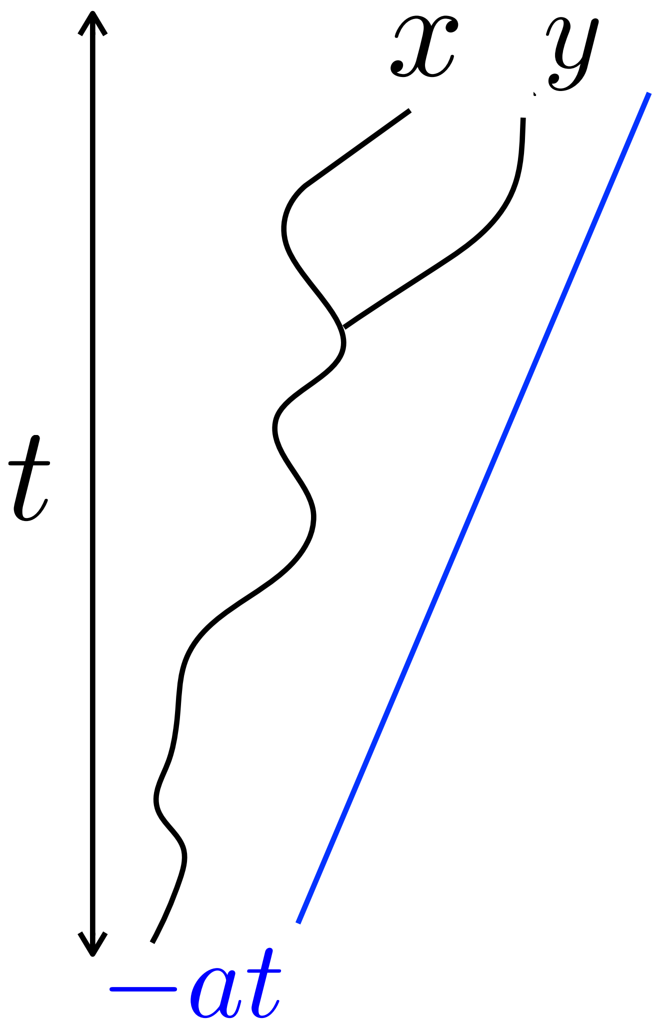
$$H(0) = H(1) = 0$$

Hairer-Mattingly 2016

$$\frac{Z(x,t)}{Z(y,t)} = e^{h(x,t)-h(y,t)} \underset{t\to+\infty}{\rightarrow} e^{B(x)-B(y)-a(x-y)}$$

$$|x-y| = O(1)$$

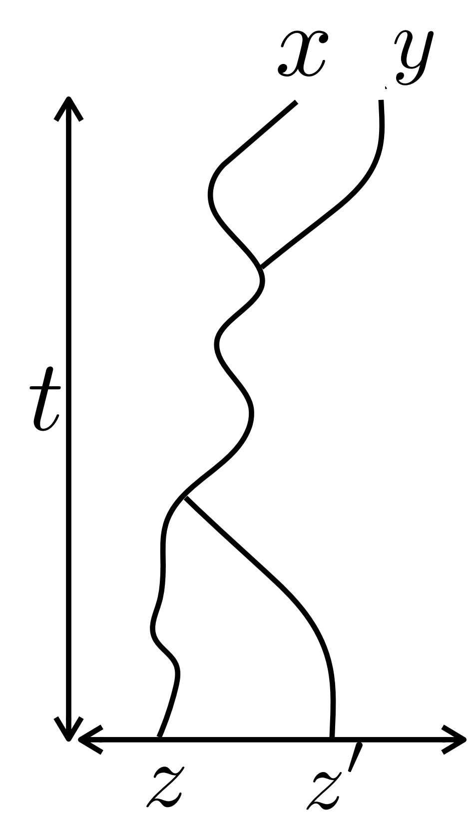
$$|x-y| \ll t^{2/3} \qquad x/t \to 0$$



very long polymer

with a slope

initial condition unimportant



KPZ equation on the half-line = directed polymer in half-space

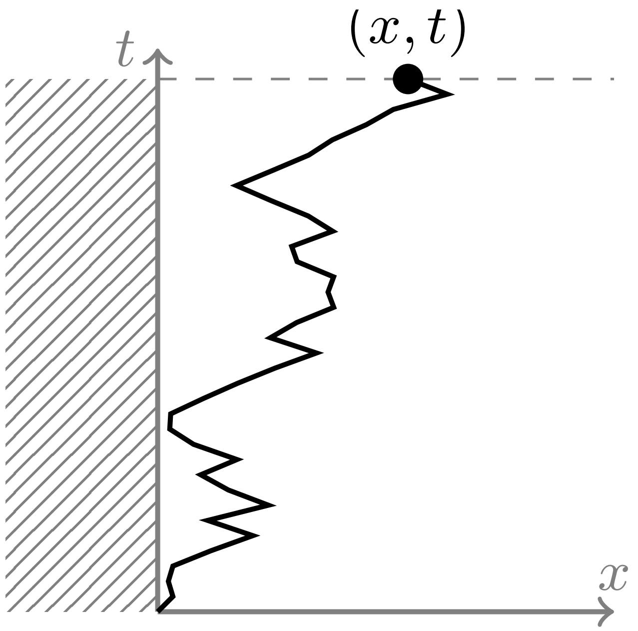
$$h(x,t) = \log Z(x,t) \qquad x > 0$$

$$\partial_t Z(x,t) = \partial_x^2 Z(x,t) + \sqrt{2}\,\eta(x,t) Z(x,t)$$

$$\partial_x Z(x,t)|_{x=0} = A\, Z(0,t)$$

$A > 0$ repulsive wall

$A < 0$ attractive wall



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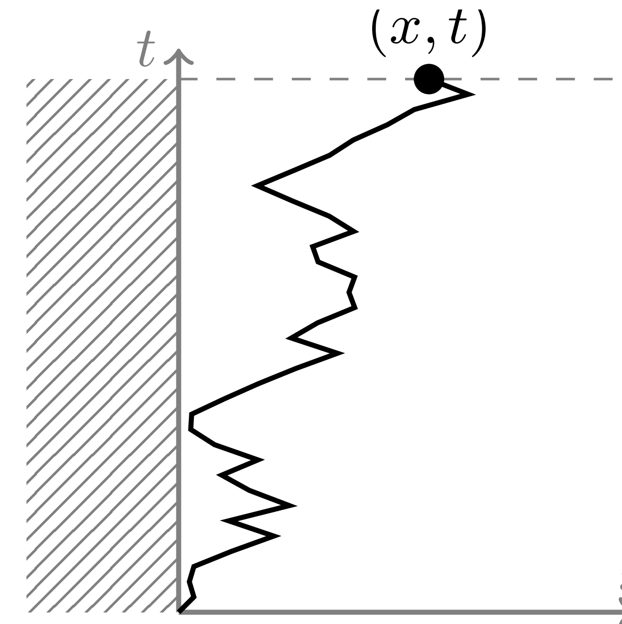
replica Bethe Ansatz

$A < 0$ attractive wall

ground state $\longrightarrow$ binding transition

$t \to +\infty$



$A < -1/2$ polymer bound to wall

$A > -1/2$ polymer unbound

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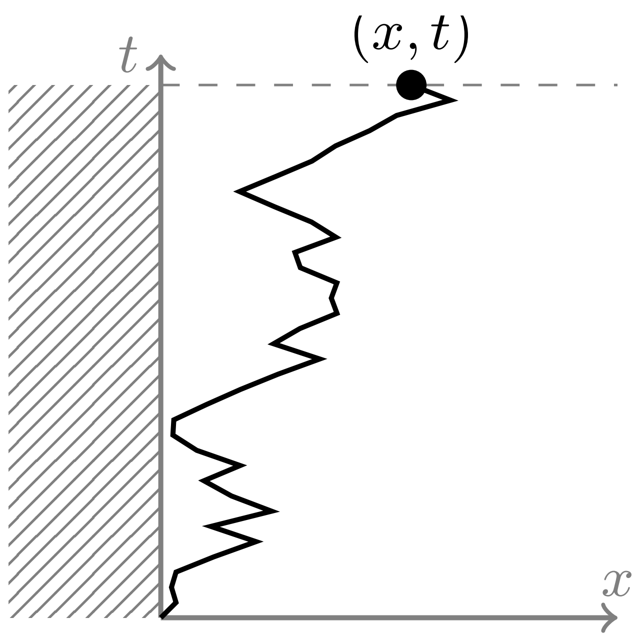

$A > 0$ repulsive wall — replica Bethe Ansatz

$A < 0$ attractive wall — ground state ⟶ binding transition $t \to +\infty$

$A < -1/2$ polymer bound to wall

$A > -1/2$ polymer unbound

Barraquand, Krajenbrink, PLD 2020

$$\frac{Z(x,t)}{Z(0,t)} = e^{B(x) + (A+\frac{1}{2})x}$$

is stationary in bound phase

⟶ stationary endpoint distribution $p(x) = \dfrac{e^{B(x)+(A+\frac{1}{2})x}}{\int_0^{+\infty} dy\, e^{B(y)+(A+\frac{1}{2})y}}$

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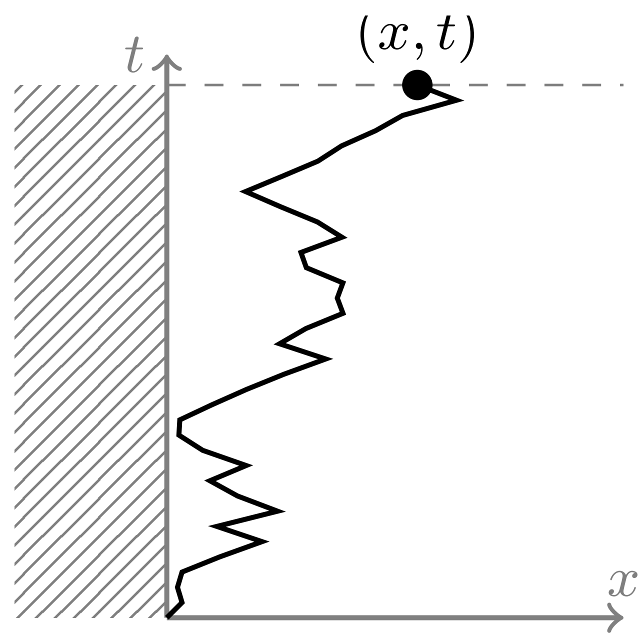
ground state → binding transition $t \to +\infty$



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Barraquand, Krajenbrink, PLD 2020

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Barraquand, PLD 2020

formula for $\overline{p(x_1)\dots p(x_k)}$ using Liouville QM

$\overline{\langle x^k\rangle^c} = -(-2)^k \psi^{(k)}(-2\epsilon)$ $\quad \overline{\langle x\rangle} \simeq \dfrac{1}{2\epsilon^2}$ $\quad \overline{\langle x^2\rangle^c} \simeq \dfrac{1}{\epsilon^3}$ $\quad \overline{\langle x\rangle^k} \simeq c_k \epsilon^{-2k}$ $\quad \epsilon = A + \dfrac{1}{2}$

KPZ stationary height profile on the interval

for any t0

$$h(x,t_0) - h(0,t_0) \equiv H(x)$$

$$\{H(x)\}_{x\in[0,L]}$$

$$H(0) = 0$$

$$\partial_x h|_{x=0} = u = A + \frac{1}{2}$$

$$\partial_x h|_{x=L} = -v$$

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Main result $$H(x) = \frac{1}{\sqrt{2}} W(x) + X(x)$$

$W(x)$ → one-sided standard Brownian motion

$X(x)$ → independent of W

$W(0)=0$
$W(L)$ free

measure of X(x) given by path integral

first form

any (u,v)

$X(0)=0$

$$\frac{\mathcal{D}X}{\mathcal{Z}_{u,v}} e^{-\int_0^L dx (\frac{dX(x)}{dx})^2} e^{-2vX(L)} \left(\int_0^L \mathrm{d}x\, e^{-2X(x)} \right)^{-(u+v)}$$

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second form

$$X(x) = U(x) - U(0)$$

$u + v > 0$

$$\frac{\mathcal{D}U}{\widetilde{\mathcal{Z}}_{u,v}} \exp\left(-2uU(0) - 2vU(L) - \int_0^L dx \left[\left(\frac{dU(x)}{dx} \right)^2 + e^{-2U(x)} \right] \right)$$

first obtained from second by integration over zero mode U(0)

how was that result obtained ?

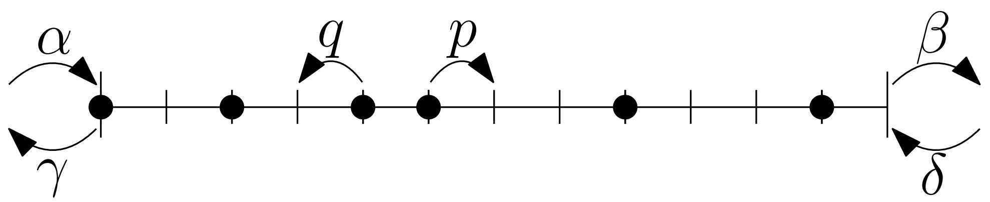
stationary measure

from matrix product ansatz (MPA)

Derrida, Evans, Hakim, Pasquier, 1993

$$P(\tau) = \frac{1}{Z_\ell(q)} \langle W| \prod_{i=1}^{\ell} (D\tau_i + E(1-\tau_i)) |V\rangle$$

from open ASEP



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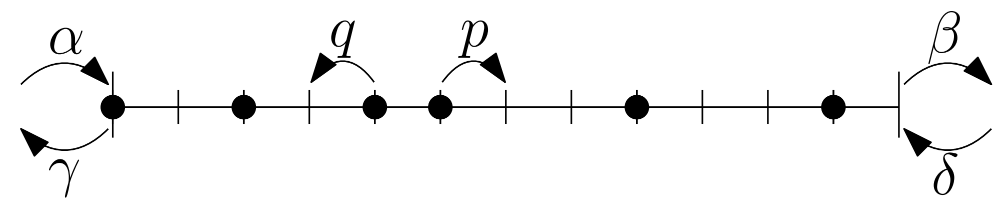

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→ representations of E,D,V,W

using Askey-Wilson orthogonal polynomials

Uchiyama,Sasamoto,Wadati 2003

↓

average of observables in open ASEP

from a Askey-Wilson (AW) process

Bryc,Wesolowski2018

↓

Corwin-Knizel 2021 KPZ limit of ASEP BW formula

$$\mathbb{E}\Big(\prod_{i=1}^{k} e^{-s_i(H(x_i)-H(x_{i-1}))}\Big) \qquad u+v>0$$

in terms of explicit transition proba

of a limit AW process

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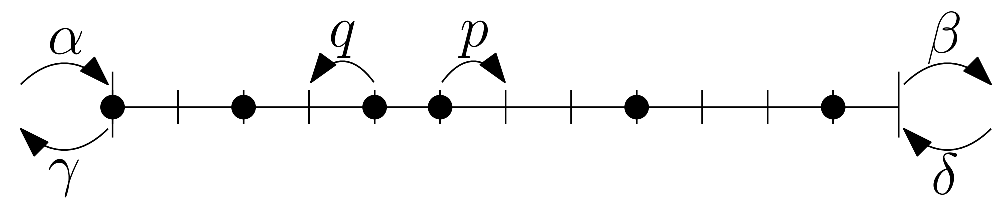

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Barraquand, PLD 2021

recognized formula

from Liouville QM

allows to perform

inverse Laplace transform ⟵

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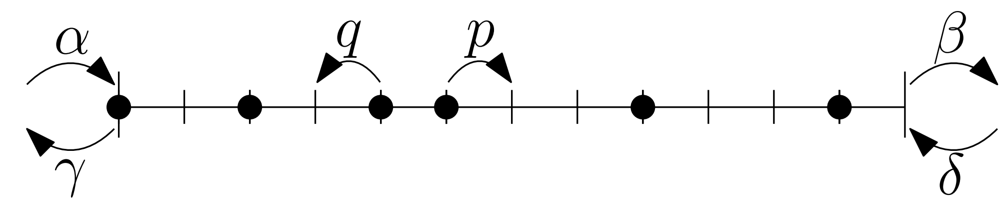

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in terms of explicit transition proba

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↓

Inverse Laplace from stochastic analysis

Markov process with transition proba

Bryc,Kuznetsov,Wang,Wesolowski 2021

↓

Bryc,Kuznetsov, 2021

proved quivalence of the two $u+v>0$

←

Barraquand, PLD 2021

recognized formula

from Liouville QM

allows to perform

inverse Laplace transform

↓

Barraquand, PLD 2022

more direct derivation using

Enaud-Derrida representation of the MPA

in terms of random walks

arXiv:2209.03131

some formula..

$$\mathbb{E}[e^{-\sum_{j=1}^{m} s_j(H(x_j)-H(x_{j-1}))}] = e^{\frac{1}{4}\sum_{j=1}^{m+1} s_j^2(x_j-x_{j-1})}\frac{J(\vec{s})}{J(0)}$$

$$\tilde{J}(\vec{s}) =$$

$$\frac{1}{2}\prod_{j=1}^{m+1}\int_0^{+\infty}\frac{dk_j}{4\pi|\Gamma(ik_j)|^2}\prod_{j=1}^{m}\frac{\Gamma_4\left(\frac{s_j-s_{j+1}}{2}\pm\frac{ik_j}{2}\pm\frac{ik_{j+1}}{2}\right)}{\Gamma(s_j-s_{j+1})}$$

$$\times\left|\Gamma\left(u-\frac{s_1}{2}+\frac{ik_1}{2}\right)\Gamma\left(v+\frac{ik_{m+1}}{2}\right)\right|^2 e^{\sum_{j=1}^{m+1}\frac{-k_j^2}{4}(x_j-x_{j-1})}$$

$$\Gamma_4(\alpha\pm x\pm y) := \prod_{\sigma,\tau=\pm 1}\Gamma(a+\sigma x+\tau y)$$

$$\langle k|e^{-2\alpha\hat{U}}|k'\rangle = \frac{N_k N_{k'}}{8\Gamma(2\alpha)}\Gamma_4\left(\alpha\pm\frac{ik}{2}\pm\frac{ik'}{2}\right)$$

The scaled process

$$\tilde{x} = x/L \in [0,1]$$

$$X(x) = \sqrt{L}\widetilde{X}(\tilde{x})$$

$$W(x) = \sqrt{L}\widetilde{W}(\tilde{x})$$

$$H(x) = \sqrt{L}\widetilde{H}(\tilde{x})$$

$\widetilde{W}(\tilde{x})$ also a one-sided standard Brownian motion

$$u = \tilde{u}/\sqrt{L}$$

$$v = \tilde{v}/\sqrt{L}$$

$$S[\widetilde{X}] = \int_0^1 d\tilde{x} \left(\frac{d\widetilde{X}(\tilde{x})}{d\tilde{x}} \right)^2 + \frac{\tilde{u}}{\sqrt{L}} \log \left(\int_0^1 \mathrm{d}\tilde{x}\; e^{-2\sqrt{L}\widetilde{X}(\tilde{x})} \right) + \frac{\tilde{v}}{\sqrt{L}} \log \left(\int_0^1 \mathrm{d}\tilde{x}\; e^{2\sqrt{L}(\widetilde{X}(1)-\widetilde{X}(\tilde{x}))} \right)$$

$\tilde{X}(0) = 0$ $\quad$ $\tilde{X}(1)$ free

$L \gg 1$ KPZ fixed point limit

$$\longrightarrow \quad \mathcal{D}\widetilde{X} e^{-\int_0^1 d\tilde{x} (\frac{d\tilde{X}(\tilde{x})}{d\tilde{x}})^2} e^{2\tilde{u}\min_{\tilde{x}}\{\widetilde{X}(\tilde{x})\} + 2\tilde{v}\min_{\tilde{x}}\{\widetilde{X}(\tilde{x}) - \widetilde{X}(1)\}}$$

$\tilde{u}, \tilde{v} \to +\infty \;\; (u, v > 0)$

$\min_{\tilde{x}} \widetilde{X}(\tilde{x}) = 0 \longrightarrow \tilde{X}(x) \geq 0$

$\min_{\tilde{x}} (\widetilde{X}(\tilde{x}) - \widetilde{X}(1)) = 0 \longrightarrow \tilde{X}(0) - \tilde{X}(1) \geq 0$

$\longrightarrow \tilde{X}(1) \leq 0$

$\longrightarrow \tilde{X}(1) = 0$

$\longrightarrow \quad \widetilde{X}(\tilde{x}) \Rightarrow \frac{1}{\sqrt{2}} E(\tilde{x})$

standard Brownian excursion

$\longrightarrow \quad \widetilde{H}(\tilde{x}) \Longrightarrow \frac{1}{\sqrt{2}} \widetilde{W}(\tilde{x}) + \frac{1}{\sqrt{2}} E(\tilde{x})$

as for TASEP (universality) Derrida,Enaud,Lebowitz2004

Stationary measures for half-line KPZ equation

Take the

$L \to \infty$ limit

with x=O(1) fixed

of

$$H(x) - H(0) = \frac{1}{\sqrt{2}} W(x) + X(x) \qquad Z_L[X] = \int_0^L dx e^{-2X(x)}$$

$$\frac{\mathcal{D}X}{\mathcal{Z}_{u,v}} e^{-\int_0^L dx (\frac{dX(x)}{dx})^2} e^{-2vX(L)} \, Z_L[X]^{-(u+v)}$$

the hard work is already done !

Y. Hariya, M. Yor (2004)

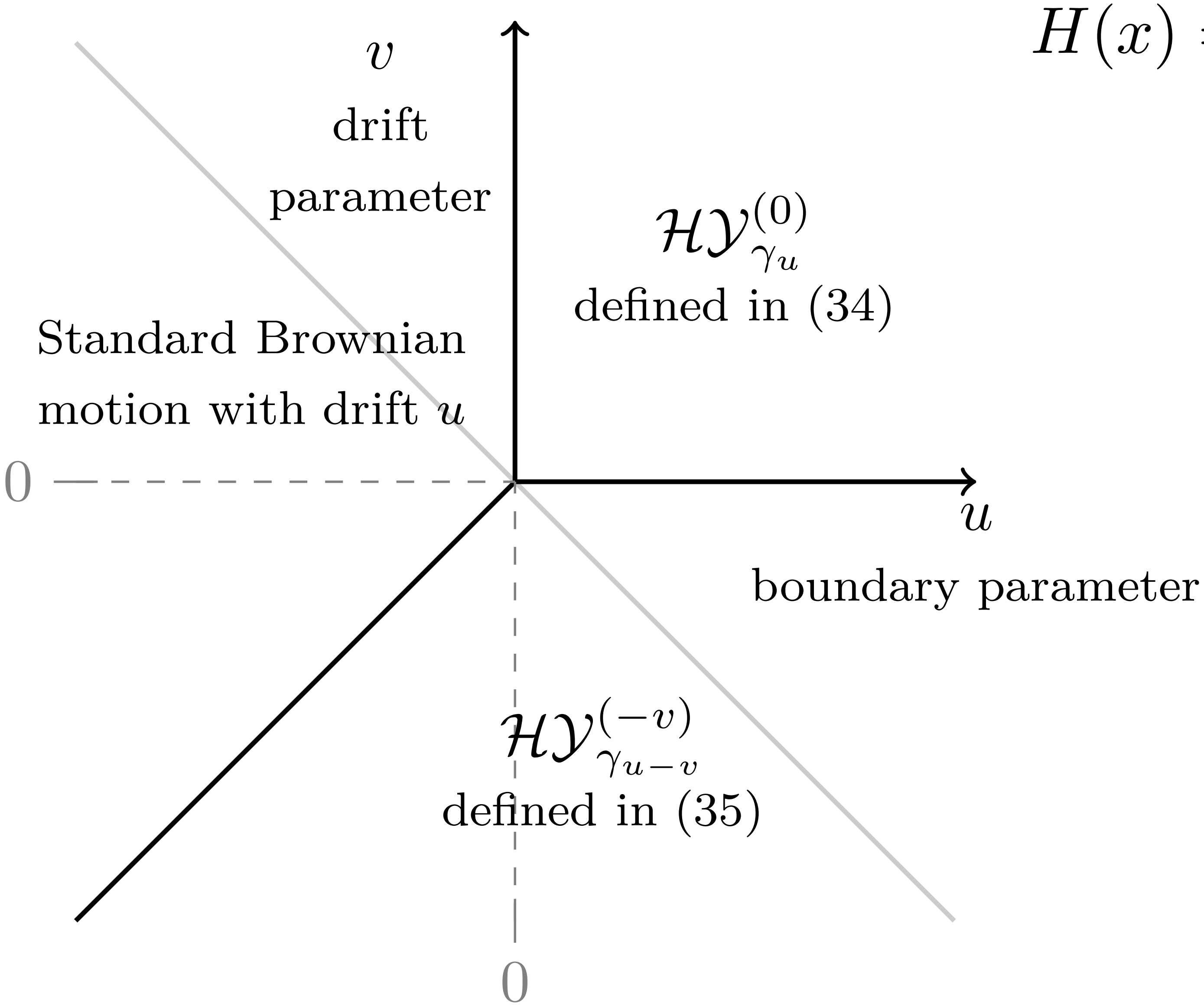
Limiting distributions associated with moments of exponential Brownian functionals

Stud. Sci. Math. Hung. 41 193 (2004)

Stationary measures for half-line KPZ equation

u= boundary parameter

v= drift parameter such that h(x,0) has drift - v at infinity



$\mathcal{HY}^{(-v)}_{\gamma_{u-v}}$ denotes

$$H(x) = B^{(1)}(x) + B^{(2)}(x) + vx$$
$$+ \log\left(1 + \gamma_{u-v}\int_0^x e^{-2B^{(2)}(z)-2vz}dz\right)$$

$$H(x) - H(0) \simeq -vx$$
$$x \to +\infty$$

$B^{(1)}(x), B^{(2)}(x)$ 2 independent Brownians with diffusion coeff = 1/2

γ_{u-v} independent gamma RV param. u-v

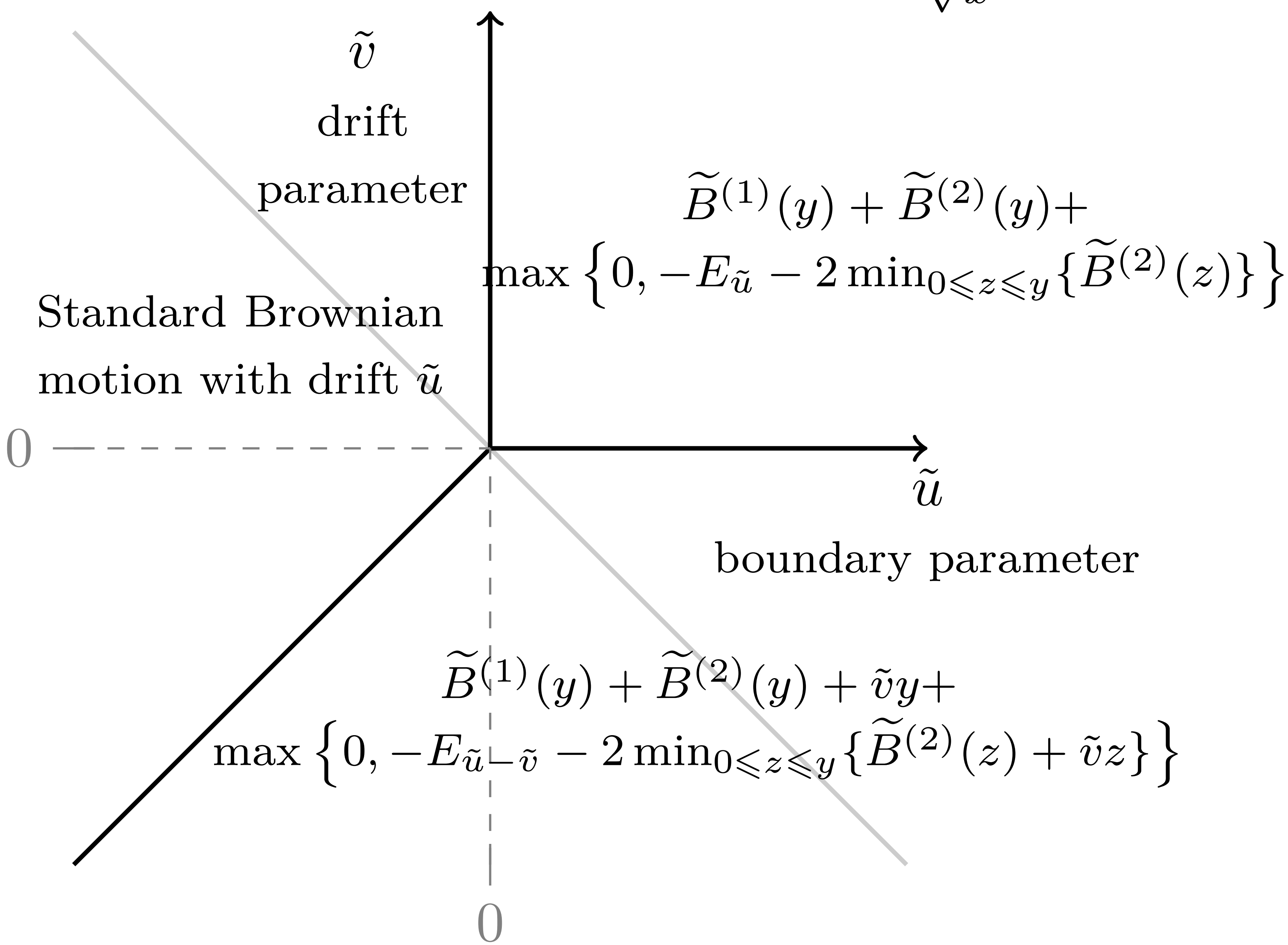
$$p(\gamma) \propto \gamma^{u-v-1}e^{-\gamma}$$

Stationary measures for half-line KPZ fixed point

look at large scale $x \to +\infty$

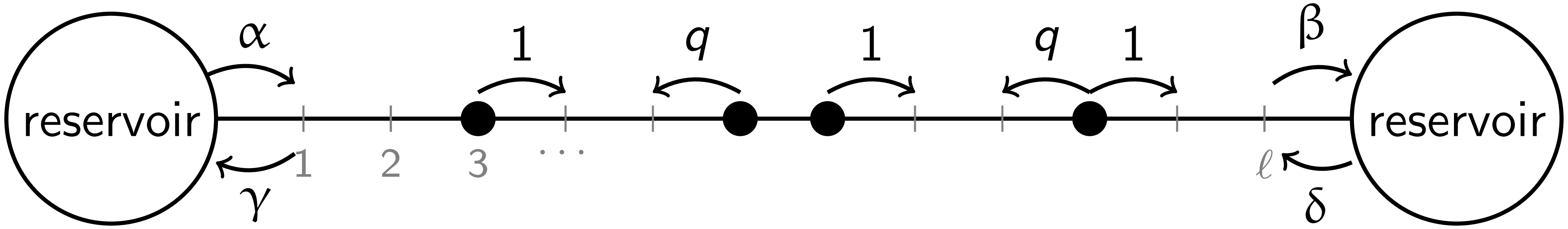
define scaled field/parameters $u = \frac{\tilde{u}}{\sqrt{x}}$ $v = \frac{\tilde{v}}{\sqrt{x}}$

$$\widetilde{H}(y) = \frac{1}{\sqrt{x}} H(xy)$$



Matrix product ansatz

Consider ASEP on $\{0,1\}^\ell$ with boundary parameters $\alpha, \beta, \gamma, \delta$.



We describe the state of the system by $\eta \in \{0,1\}^\ell$. The stationary measure $\mathbb{P}$ can be written as [Derrida-Evans-Hakim-Pasquier 1993]

$$\mathbb{P}(\eta) = \frac{1}{Z_\ell} \langle w| \prod_{i=1}^{\ell} (\eta_i D + (1-\eta_i)E) \, |v\rangle$$

where

$$Z_\ell = \langle w| (E+D)^\ell \, |v\rangle$$

and E, D are infinite matrices, and $\langle w|, |v\rangle$ are row/column vectors such that

$$\begin{aligned} DE - qED &= D + E \\ \langle w| (\alpha E - \gamma D) &= \langle w| \\ (\beta D - \delta E) |v\rangle &= |v\rangle \end{aligned}$$

Enaud-Derrida's representation

Enaud-Derrida found a very simple representation for any parameters $q, \alpha, \beta, \gamma, \delta$. Under Liggett's condition, it becomes :

$$D = \begin{pmatrix} [1]_q & [1]_q & 0 & 0 & 0 & \cdots \\ 0 & [2]_q & [2]_q & 0 & 0 & \cdots \\ 0 & 0 & [3]_q & [3]_q & 0 & \cdots \\ \vdots & \vdots & 0 & \ddots & \ddots & \ddots \end{pmatrix}, \quad E = \begin{pmatrix} [1]_q & 0 & 0 & 0 \\ [2]_q & [2]_q & 0 & 0 \\ 0 & [3]_q & [3]_q & 0 \\ 0 & 0 & \ddots & \ddots \end{pmatrix}$$

where $[n]_q = \frac{1-q^n}{1-q}$.

Denoting by $\{|n\rangle\}_{n\geqslant 1}$ the vectors of the associated basis, let

$$\langle w| = \sum_{n\geqslant 1} \left(\frac{1-\varrho_0}{\varrho_0}\right)^n \langle n|, \qquad |v\rangle = \sum_{n\geqslant 1} \left(\frac{\varrho_\ell}{1-\varrho_\ell}\right)^n [n]_q \, |n\rangle .$$

Then, $E, D, \langle w|, |v\rangle$ satisfy

$$DE - qED = D + E$$
$$\langle w| (\alpha E - \gamma D) = \langle w|$$
$$(\beta D - \delta E) |v\rangle = |v\rangle$$

Sum over paths

Due to the bidiagonal structure, the normalization constant $Z_\ell = \langle w| (D+E)^\ell |v\rangle$ can be written as a sum over lattice paths $\vec{n} = (n_0, n_1, \ldots, n_\ell) \in \mathbb{N}^\ell$ of the form

$$Z_\ell = \sum_{\vec{n}} \Omega(\vec{n})$$

where

$$\Omega(\vec{n}) = \left(\frac{1-\varrho_0}{\varrho_0}\right)^{n_0} \left(\frac{\varrho_\ell}{1-\varrho_\ell}\right)^{n_\ell} \prod_{i=1}^{\ell} v(n_{i-1}, n_i) \prod_{i=0}^{\ell} [n_i]_q,$$

with

$$v(n,n') = \begin{cases} 2 & \text{if } n = n', \\ 1 & \text{if } |n-n'| = 1 \\ 0 & \text{else.} \end{cases}$$

- This introduces a natural probability measure on random walk paths $\vec{n}$. The stationary measure $\mathbb{P}(\eta)$ can be recovered from this measure.

Open ASEP invariant measure

Following arguments similar as [Derrida-Enaud-Lebowitz 2004], one arrives at

Theorem ([B.-Le Doussal 2022])

Under the stationary measure $\mathbb{P}(\tau)$, ASEP height function $H(x) = \sum_{j=1}^{x}(2\eta_i - 1)$ is such that

$$(H(i))_{1\leqslant i\leqslant \ell} \overset{(d)}{=} (n_i - n_0 + m_i)_{1\leqslant i\leqslant \ell},$$

where $(n_i, m_i)_{0\leqslant i\leqslant \ell}$ is a two dimensional random walk on $\mathbb{Z}^2$, starting from $(n_0, 0)$, distributed as

$$P(\vec{n}, \vec{m}) = \frac{\mathbb{1}_{n_0>0}}{4^{-\ell} Z_\ell} \left(\frac{1-\varrho_0}{\varrho_0}\right)^{n_0} \left(\frac{\varrho_\ell}{1-\varrho_\ell}\right)^{n_\ell} \prod_{i=0}^{\ell}[n_i]_q \times P^{SSRW}_{n_0,0}(\vec{n}, \vec{m}),$$

where $P^{SSRW}_{n_0,0}$ denotes the probability measure of the symmetric simple random walk (SSRW) on $\mathbb{Z}^2$ starting from $(n_0, 0)$.

Scaling limit to the KPZ equation

Under the scalings such that ASEP's height function converges to KPZ, in particular

$$q = 1 - \varepsilon, \;\; \ell = \varepsilon^{-2}, \;\; \varrho_0 = \frac{1}{2}(1 + u\varepsilon), \;\; \varrho_\ell = \frac{1}{2}(1 - v\varepsilon)$$

we find, denoting by Y_x the rescaled version of the random walk n_i

$$\prod_{i=0}^{\ell} [n_i]_q \to e^{-\int_0^L e^{-2Y_s} ds}$$

$$\left(\frac{1 - \varrho_a}{\varrho_a}\right)^{n_0} \left(\frac{\varrho_b}{1 - \varrho_b}\right)^{n_\ell} \to e^{-2uY_0 - 2vY_L}$$

so that

$$(m_i, n_i) \Longrightarrow (W_x, Y_x)$$

where W_x is a Brownian motion and Y_x is absolutely continuous to the Brownian measure with Radon Nikodym derivative

$$\frac{1}{\mathcal{Z}_{u,v}} e^{-2uY_0 - 2vY_L} e^{-\int_0^L e^{-2Y_s} ds}.$$

Liouville field theory in dimension 1

Theorem

The KPZ equation on $[0, L]$ with boundary parameters u and v with $u + v > 0$ has a unique stationary measure

$$h^L_{u,v}(x) = W_x + Y_x - Y_0,$$

where

- *W is a Brownian motion,*
- *Y is independent from W, and its law is absolutely continuous w.r.t. to that of a Brownian motion with free starting point. The Radon-Nikodym derivative is*

$$\frac{1}{\mathcal{Z}_{u,v}} \exp\left(-2uY_0 - 2vY_L - \int_0^L e^{-2Y_s} ds\right)$$

It was originally proved by [Bryc-Kuznetsov-Wang-Wesołowski 2021], [B.-Le Doussal 2021] using results from [Corwin-Knizel 2021]. Uniqueness was later proved by [Knizel-Matetski 2022].

stationary measure for 2 non-crossing polymers ?

on real line

stationary measure for 2 non-crossing polymers

in a random potential (on the line)

G. Barraquand,PLD, arXiv:2205.08023

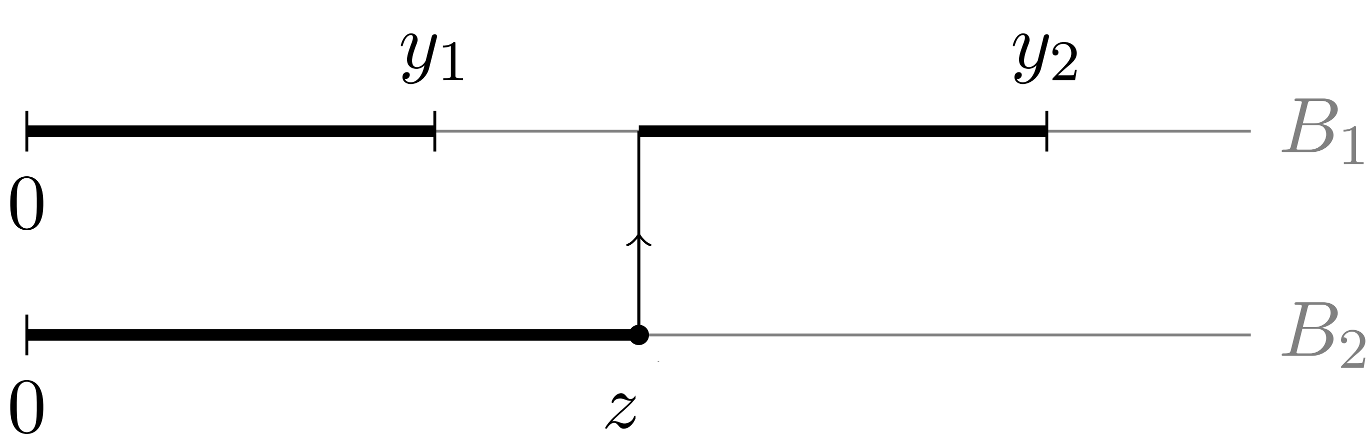
1 polymer $\quad Z_1^{\text{stat}}(y) = e^{B(y)}$

$B(0) = 0 \quad dB(y)^2 = dy$

2 NC polymers

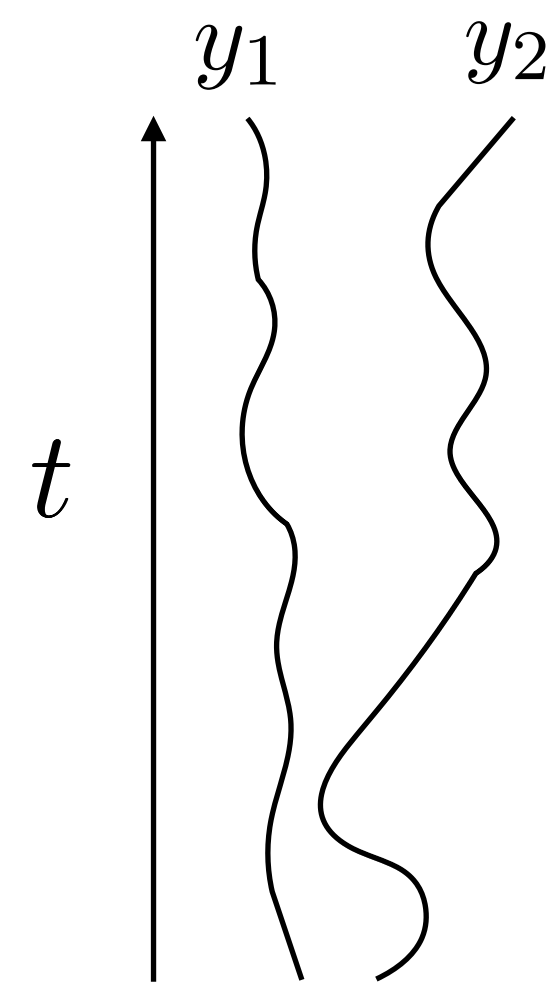
$$Z_2^{\text{stat}}(y_1, y_2) = e^{B_1(y_1)+B_1(y_2)} \int_{y_1}^{y_2} dz e^{-B_1(z)+B_2(z)}$$

$y_1 < y_2$



equal to partition sum of

2 NC semi-discrete O'Connel-Yor polymers



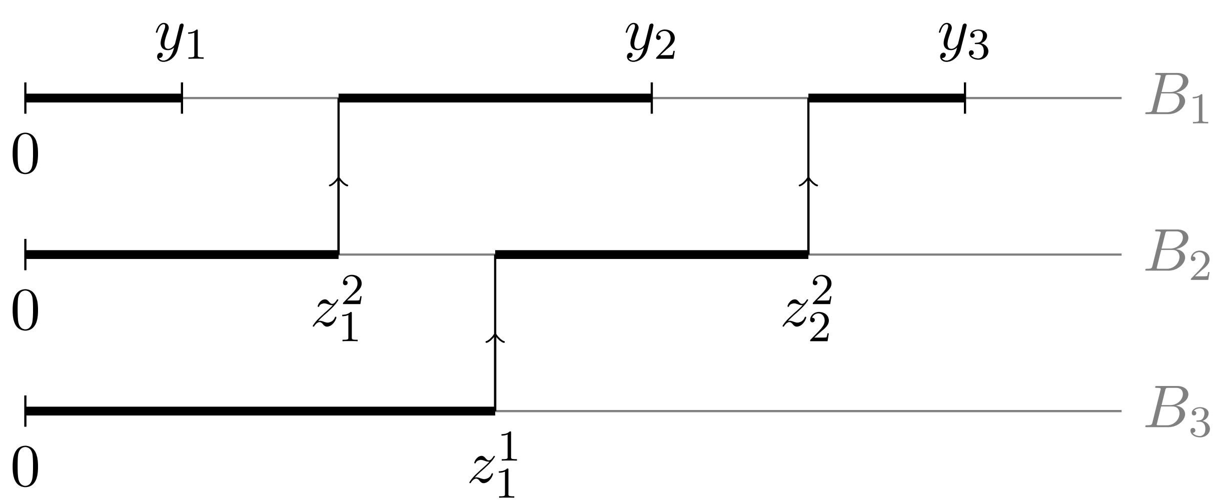
$t \to +\infty$

$$\frac{Z_\ell(\vec{x}; -t|\vec{y}; 0)}{Z_\ell(\vec{x}; -t|\vec{z}; 0)} \overset{(d)}{\simeq} \frac{Z_\ell^{\text{stat}}(\vec{y})}{Z_\ell^{\text{stat}}(\vec{z})}$$

stationary measure for ℓ non-crossing polymers

3 NC polymers

$$Z_3^{\text{stat}}(y_1,y_2,y_3) = e^{B_1(y_1)+B_1(y_2)+B_1(y_3)}$$
$$\times \int_{y_1<z_1^2<y_2<z_2^2<y_3} dz_1^2 dz_2^2 e^{B_2(z_1^2)-B_1(z_1^2)+B_2(z_2^2)-B_1(z_2^2)} \int_{z_1^2<z_1^1<z_2^2} dz_1^1 e^{B_3(z_1^1)-B_2(z_1^1)}$$



ℓ NC polymers

$$Z_\ell^{\text{stat}}(\vec{y}) = \int_{GT(\vec{y})} \prod_{k=1}^{\ell}\prod_{i=1}^{k} e^{B_{\ell-k+1}(z_i^k)-B_{\ell-k+1}(z_{i-1}^{k-1})} \prod_{k=1}^{\ell-1}\prod_{i=1}^{k} \mathrm{d}z_i^k$$

$$GT(\vec{y}) = \{(z_i^k)_{1\leqslant i\leqslant k\leqslant \ell} : z_i^{k+1} \leqslant z_i^k \leqslant z_{i+1}^{k+1} \text{ for } 1\leqslant i\leqslant k\leqslant \ell-1, \text{ and } z_i^\ell = y_i \text{ for } 1\leqslant i\leqslant \ell\}$$

Gelfand-Tsetlin pattern

interlaced set of $\ell(\ell-1)/2$ auxiliary variables

Liouville field theory and log-correlated Random Energy Models

X. Cao, A. Rosso, R. Santachiara, P. Le Doussal arXiv:1611.02193

An exact mapping is established between the $c \geq 25$ Liouville field theory (LFT) and the Gibbs measure statistics of a thermal particle in a 2D Gaussian Free Field plus a logarithmic confining potential. The probability distribution of the position of the minimum of the energy landscape is obtained exactly by combining the conformal bootstrap and one-step replica symmetry breaking methods. Operator product expansions in LFT allow to unveil novel universal behaviours of the log-correlated Random Energy class. High precision numerical tests are given.

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Log-Random Energy Model(REM)

normalized Gibbs measure of a particle
in log-correlated field

$$p_\beta(z) \stackrel{\text{def}}{=} \frac{1}{Z} e^{-\beta(\phi(z)+U(z))}, \; z \in \mathbb{C}$$

$$Z \stackrel{\text{def}}{=} \int_{\mathbb{C}} e^{-\beta(\phi(z)+U(z))} \mathrm{d}^2 z$$

$$\overline{\phi(z)\phi(w)} = 4\ln(R/|z-w|)$$

$$\overline{\phi(z)^2} = 4\ln(R/\epsilon) \qquad \epsilon \to 0, R \to \infty$$

$$U(z) \stackrel{\text{def}}{=} 4a_1 \ln|z| + 4a_2 \ln|z-1|\,, a_1, a_2 > 0$$

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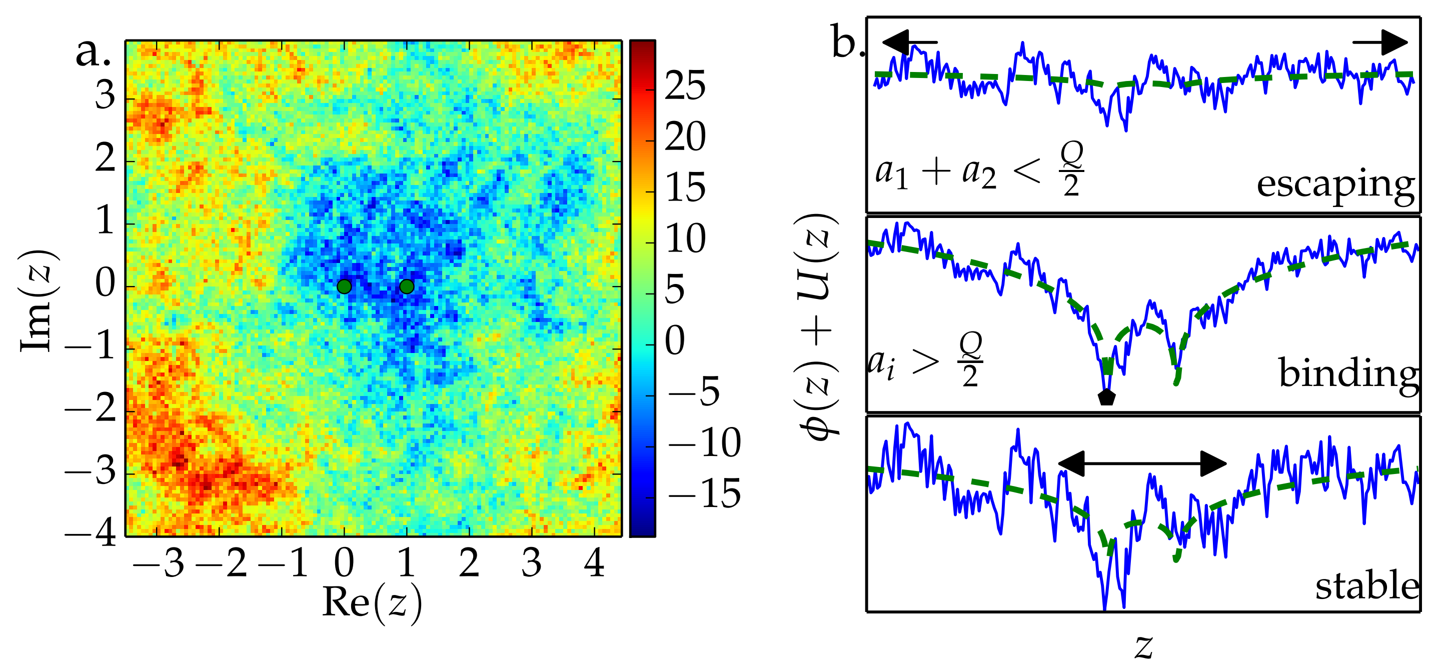

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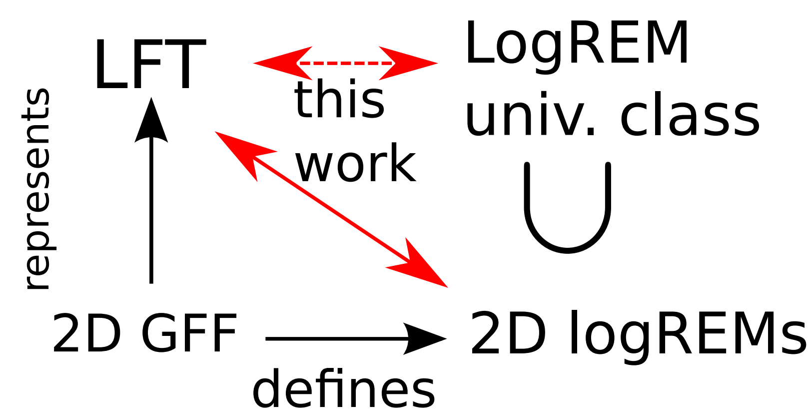

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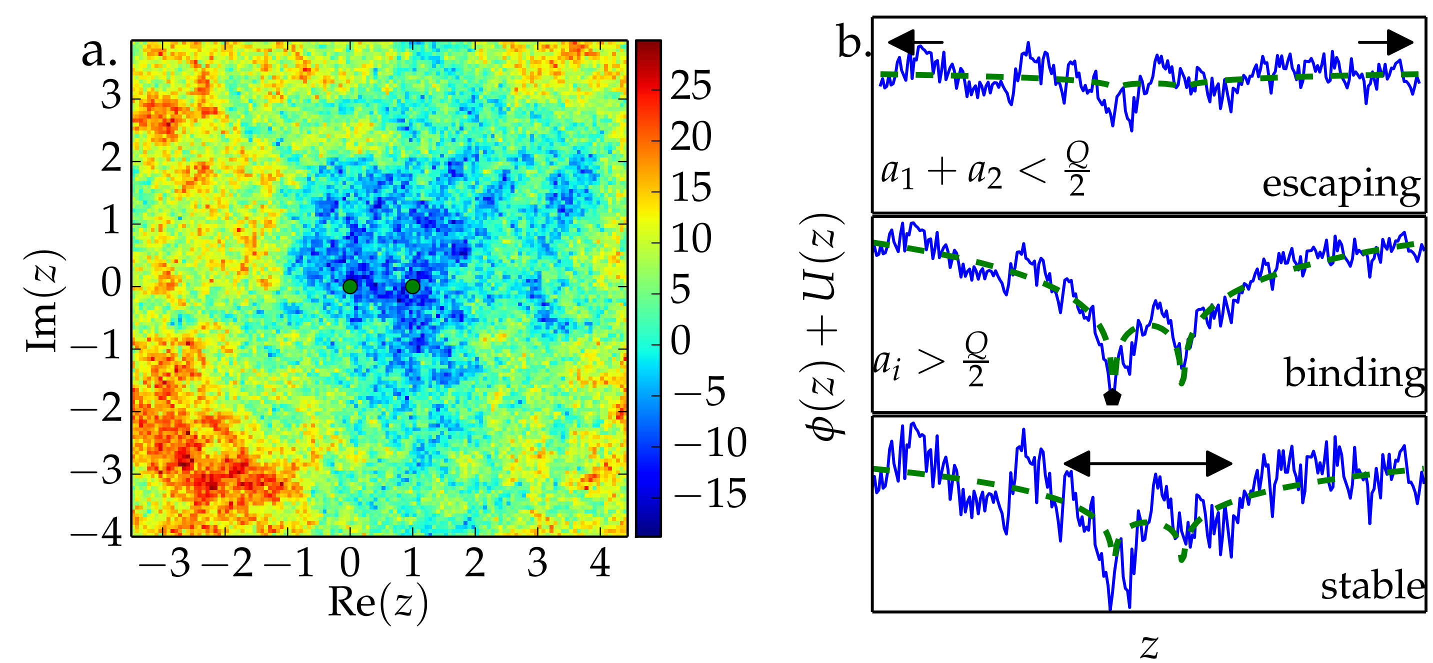

A. M. Polyakov, Physics Letters B **103**, 207 (1981)

Liouville CFT

$$\mathcal{S}_b = \int_\Sigma \left[\frac{1}{16\pi} (\nabla\varphi)^2 - \frac{1}{8\pi} Q\hat{R}\varphi + \mu e^{-b\varphi} \right] \mathrm{d}A$$

$\Sigma = \mathbb{C} \cup \{\infty\}$

$\hat{R}(z) = 8\pi\delta^2(z-\infty), \mathrm{d}A = \mathrm{d}^2 z.$

$Q = b + b^{-1}$ $\qquad c = 1 + 6Q^2$

$\mathcal{V}_a(w) \rightsquigarrow e^{-a\varphi(w)}$ $\qquad \Delta_a = a(Q - a)$

rigorous probabilistic construction of LCFT path integral

David, Kupiainen, Rhodes, Vargas, arXiv:1410.7318

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invariant under duality $b \to 1/b$

freezing duality conjecture

Fyodorov, Le Doussal, Rosso 2009

=>

$$\overline{p_{\beta>1}} = \overline{p_1} \textit{ freezes}$$

predicts PDF of position of the minimum of

$$\phi(z) + U(z)$$

Left = Test of $\overline{p_\beta(z)} \overset{\beta<1}{\propto} \langle \mathcal{V}_{a_1}(0)\mathcal{V}_{a_2}(1)\mathcal{V}_b(z)\mathcal{V}_{a_3}(\infty)\rangle_b$

Right = Test of PDF of position of the minimum of $\phi(z) + U(z)$

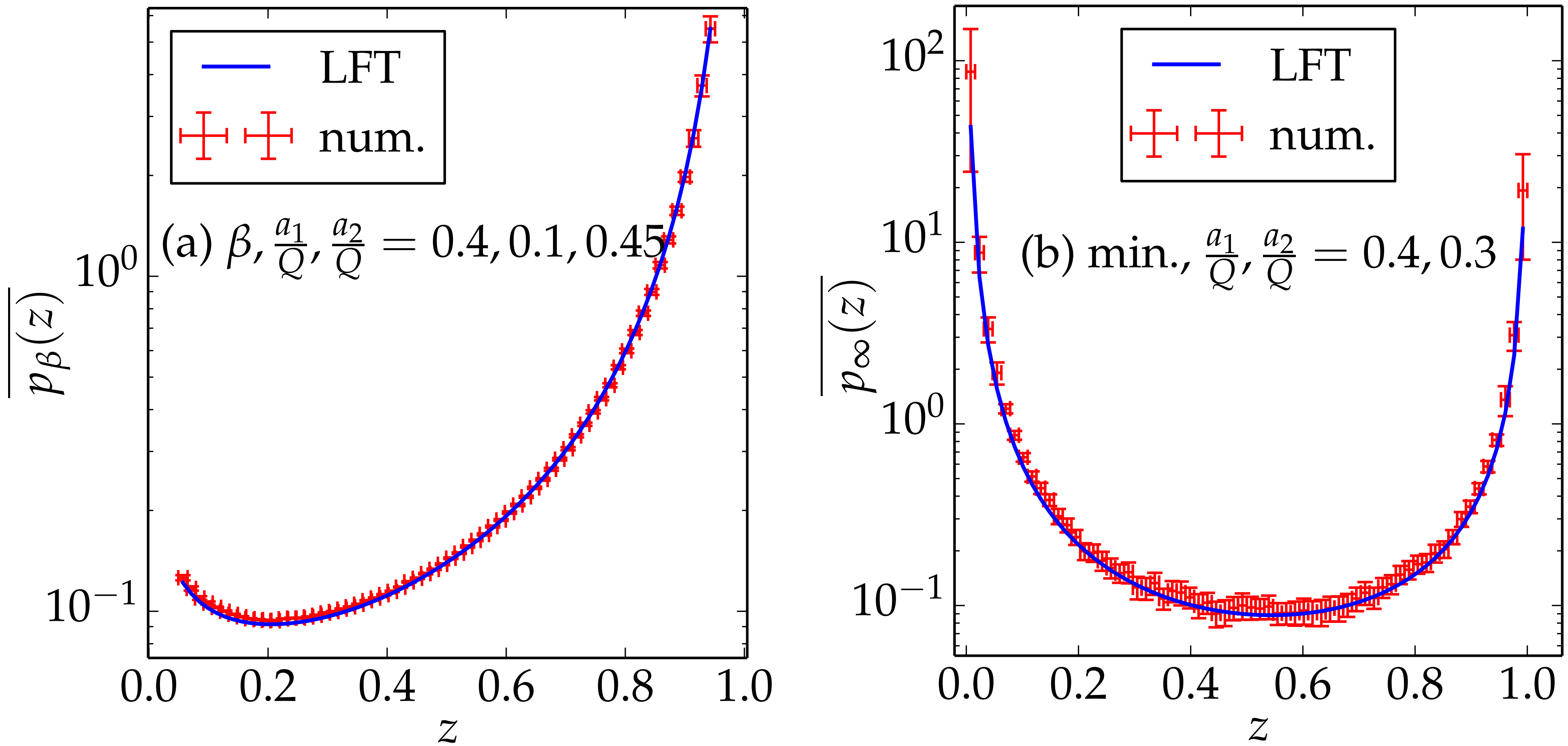
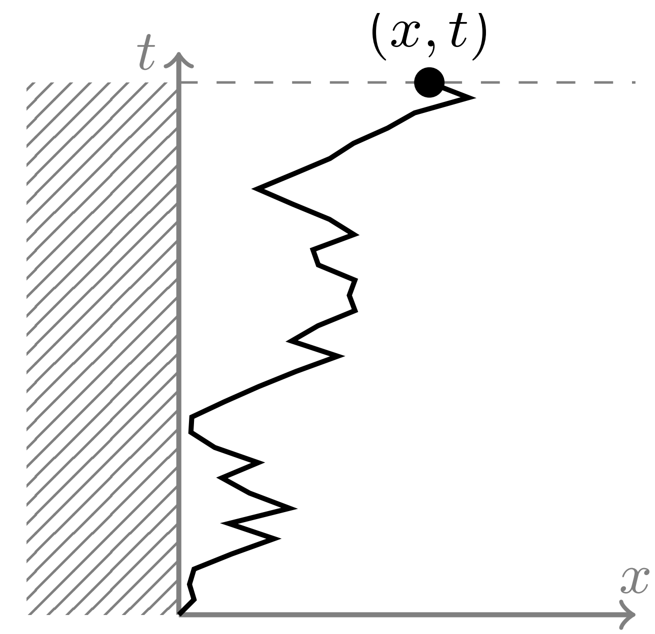


Figure 3. (Color online) Test of (9) on the segment $z \in [0, 1]$. (a) High-T regime ($\beta = .4$). (b) Minimum position distribution versus LFT with $b = 1$. Numerical parameters: $L = 2^{12}, \epsilon = 2^{-9}$, 5×10^6 independent samples.

KPZ equation on the half-line = directed polymer in half-space



$$h(x,t) = \log Z(x,t) \qquad x > 0$$

$$\partial_t Z(x,t) = \partial_x^2 Z(x,t) + \sqrt{2}\,\eta(x,t) Z(x,t)$$

$$\partial_x Z(x,t)|_{x=0} = A\, Z(0,t)$$

$A > 0$ repulsive wall

$A < 0$ attractive wall

Kardar 1985

replica Bethe Ansatz

ground state $A < \frac{n-1}{2}$

$t \to +\infty$ binding transition

$$\Psi_0(\vec{x}) = c_{n,A}\, e^{\sum_{j=1}^n (A-j+1)x_j} \qquad 0 \leq x_1 \leq \dots \leq x_n$$

$$x_i \to x_i + y$$

$$e^{-\frac{n}{2}(n-(1+2A))y}$$

$A < -1/2$ polymer bound to wall

$A > -1/2$ polymer unbound

$t \to +\infty$

ground state dominance

Barraquand, Krajenbrink, PLD 2020

$$\overline{Z(x_1,t)\dots Z(x_n,t)} = \overline{Z(0,t)^n}\; e^{\sum_{j=1}^n (A-j+1)x_j}$$

$$\frac{Z(x,t)}{Z(0,t)} = e^{B(x)+(A+\frac{1}{2})x}$$

is stationary in bound phase →

$$\overline{\frac{Z(x_1,t)}{Z(0,t)} \dots \frac{Z(x_n,t)}{Z(0,t)}} = \overline{e^{\sum_j B(x_j)+(A+\frac{1}{2})x_j}}$$

$$= e^{\sum_{j=1}^n (A+n-j+1)x_j}$$

stationary endpoint distribution

$$p(x) = \frac{e^{B(x)+(A+\frac{1}{2})x}}{\int_0^{+\infty} dy\, e^{B(y)+(A+\frac{1}{2})y}}$$

Barraquand, PLD 2020

$$\overline{\langle x^k \rangle^c} = -(-2)^k \psi^{(k)}(-2\epsilon) \qquad \epsilon = A + \frac{1}{2}$$

$$\overline{\langle x \rangle} \simeq \frac{1}{2\epsilon^2} \qquad \overline{\langle x^2 \rangle^c} \simeq \frac{1}{\epsilon^3} \qquad \overline{\langle x \rangle^k} \simeq c_k \epsilon^{-2k}$$

formula for $\overline{p(x_1)\dots p(x_k)}$ using Liouville QM

back to 2 non-crossing polymers and Dyson BM

$y_1 < y_2$

$$Z_2^{\text{stat}}(y_1, y_2) = e^{B_1(y_1)+B_1(y_2)} \int_{y_1}^{y_2} dz e^{-B_1(z)+B_2(z)}$$

short scale behavior

$y_j \approx y$

$$Z_\ell^{\text{stat}}(\vec{y}) \simeq \prod_{i=1}^{\ell} \frac{e^{B_i(y)}}{(i-1)!} \Delta(\vec{y}) \qquad \Delta(\vec{y}) := \prod_{i<j}(y_j - y_i)$$

large scale behavior

$y_i - y_j \gg 1$

$$\lim_{x\to\infty} \frac{1}{\sqrt{x}} \log Z_2^{\text{stat}}(xy_1, xy_2) \overset{(d)}{=} B_1(y_1) + B_2(y_1) + \Lambda_1(y_2 - y_1)$$

$\Lambda_1(y)$ is independent of $B_{1,2}(y_1)$

largest eigenvalue of the

GUE(2) Dyson Brownian motion

if we condition the first polymer
to end up in atypical position with slope b

=> the two endpoints are "bound"

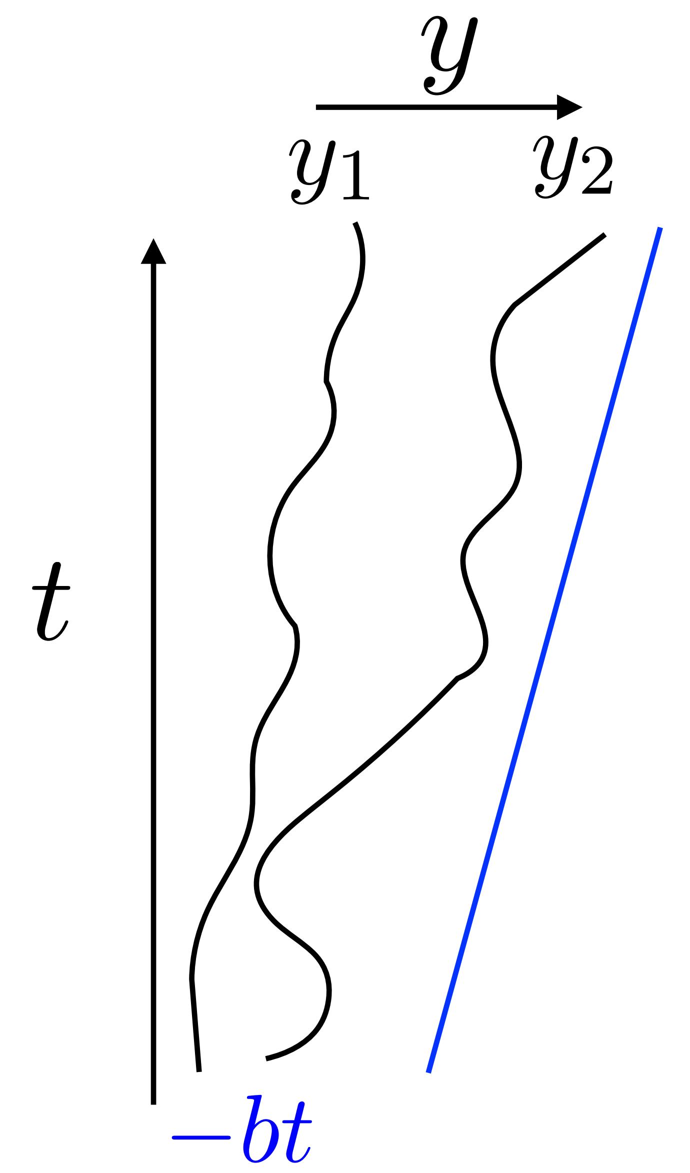
$$P(y) = \frac{Z(y)}{\int_0^{+\infty} dy Z(y)} \qquad y = y_2 - y_1$$

$$Z(y) := Z_\ell^{\text{stat}}(0, y; -b, -b)$$

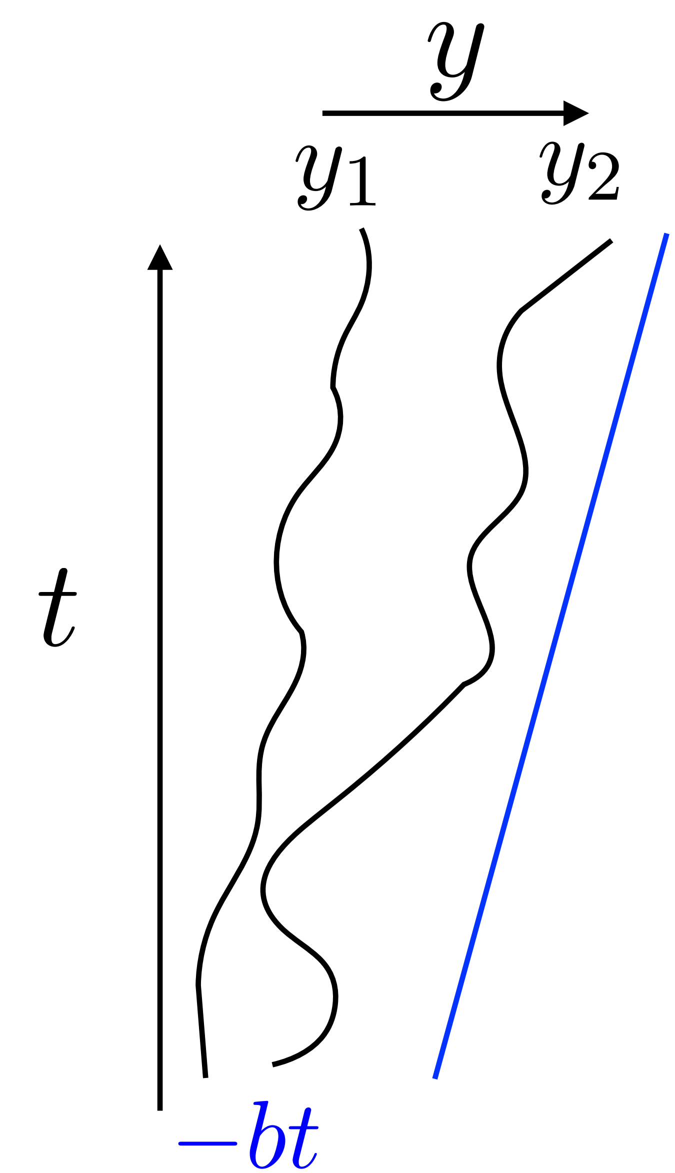
$b \ll 1$ small slope => large scale $y = \tilde{y}/b^2$

$$\tilde{y} = \text{argmax}_{z \in \mathbb{R}_+} (\Lambda_1(z) - z)$$

GUE(2) DBM



if we condition the first polymer
to end up in atypical position with slope b



=> the two endpoints are "bound"

$$P(y) = \frac{Z(y)}{\int_0^{+\infty} dy Z(y)} \qquad y = y_2 - y_1$$

$$Z(y) := Z_\ell^{\mathrm{stat}}(0, y; -b, -b)$$

$b \ll 1$ small slope => large scale $y = \tilde{y}/b^2$

$$\tilde{y} = \mathrm{argmax}_{z \in \mathbb{R}_+} (\Lambda_1(z) - z)$$

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$$\longrightarrow \quad \mathbb{E}\left[\langle y \rangle\right] \simeq \frac{5}{4b^2} \qquad \mathbb{E}\left[\langle y^2 \rangle\right] \simeq \frac{29}{8b^4} \qquad \mathbb{E}\left[\langle y^2 \rangle - \langle y \rangle^2\right] \simeq \frac{5}{2b^3}$$

any slope $b = O(1)$

$$\mathbb{E}\left[\langle y^p \rangle^c\right] = (-2)^p (2^p \psi_p(4b) - 3\psi_p(2b))$$