Quantum Length of SLE

KPZ meets KPZ, Fields Institute, Toronto
4th March 2024

Ellen Powell, *Durham University*.
Based on joint work with Avelio Sepúlveda

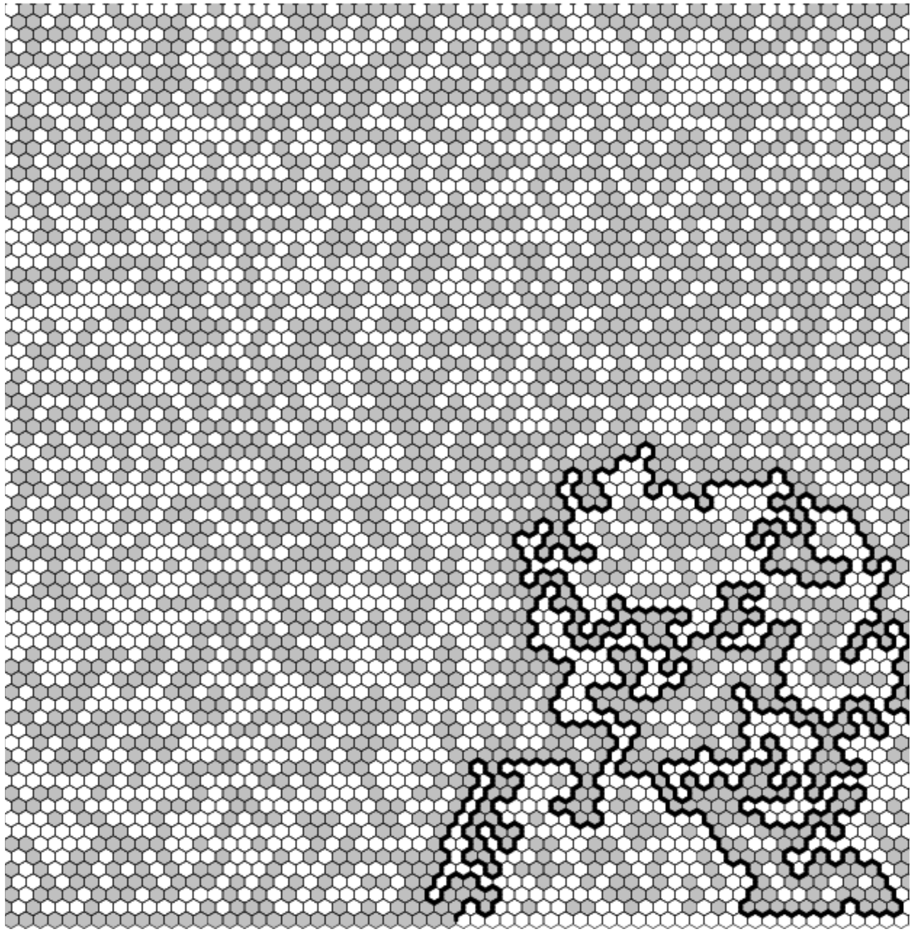
Aim of the talk

- What are **Schramm-Loewner Evolutions (SLE)**?
- What's the **natural parametrisation**?
- Why are we interested in **SLE on Liouville Quantum Gravity (LQG)?**
- **Quantum length of SLE:** how to construct it and we care

Schramm-Loewner Evolution (SLE)

Motivation

- Statistical mechanics models at **critical point** of continuous phase transition should satisfy **conformal invariance** in the **scaling limit**
- Can study **correlations,** or other **macroscopic observables** such as **interfaces**
- Scaling limits of interfaces should be **conformally invariant** and satisfy a **domain Markov property**



Critical Percolation Interface (Schramm)

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Critical Ising Interface (Kennedy)

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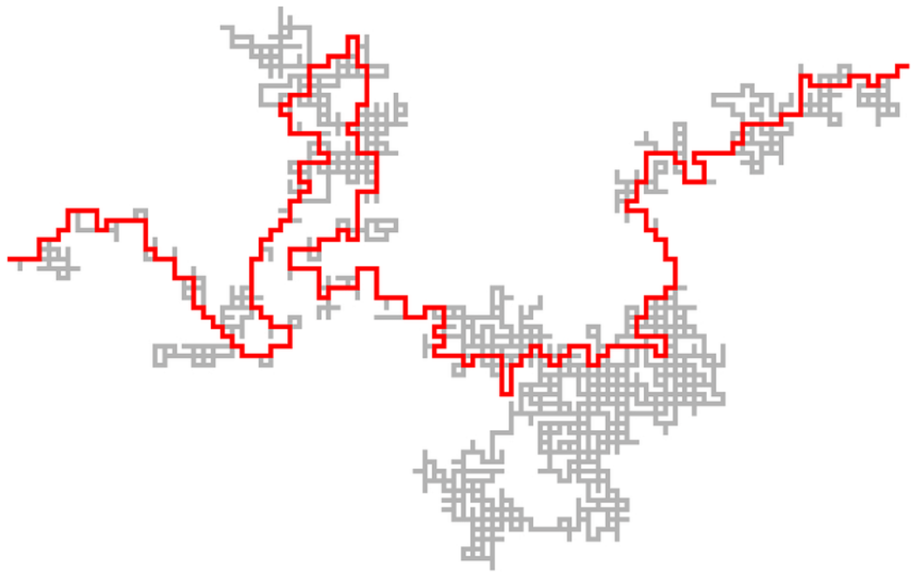

Loop-Erased Random Walk (Karrila-Kytola-Peltola)

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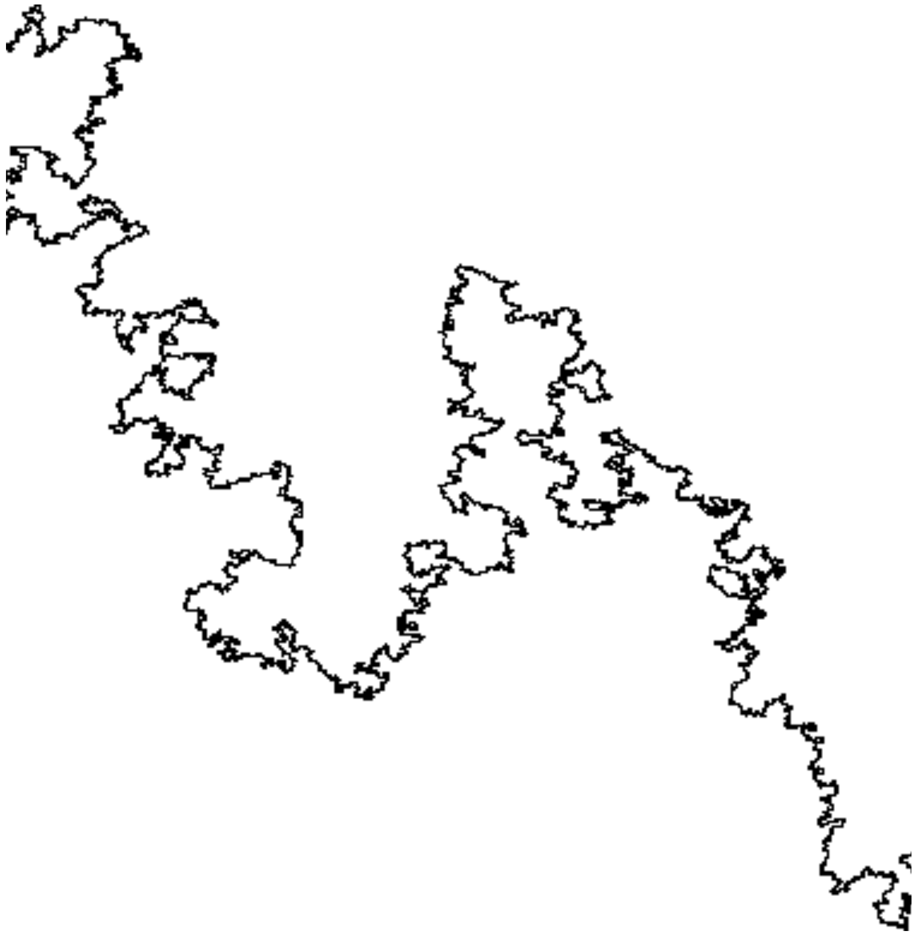
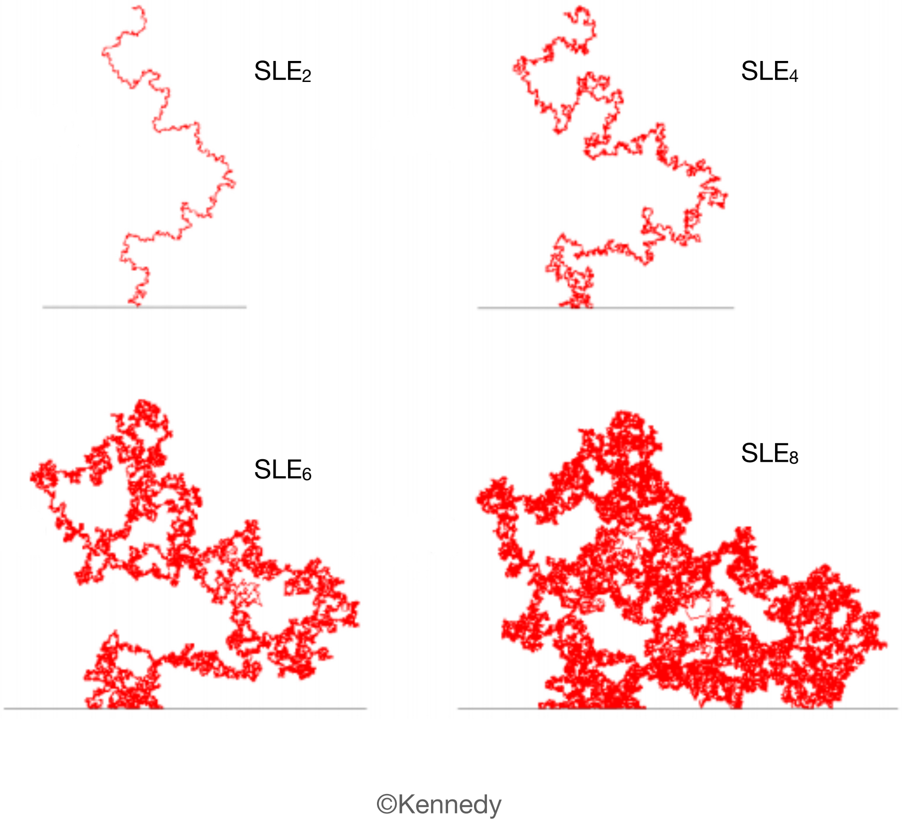


Self-Avoiding Random Walk (Kennedy)



Definition: SLE

- Schramm defined (chordal) SLE_κ for $\kappa \geq 0$ as a family of laws
$$\mu_\kappa^{D,a,b}$$
on curves in simply connected domains D from $a \in \partial D$ to $b \in \partial D$

- These laws satisfy **conformal invariance (CI)** and **domain Markov property (DMP)**

- When $(D, a, b) = (\mathbb{H}, 0, \infty)$, the curve is described via the **Loewner equation** with driving function
$$(\sqrt{\kappa} B_t)_{t \geq 0}$$

Scaling limits

- Schramm proved that any collection of laws on curves satisfying **CI and DMP** must be an SLE

- $\Rightarrow$ SLE are **only candidates for scaling limits of critical discrete interfaces**

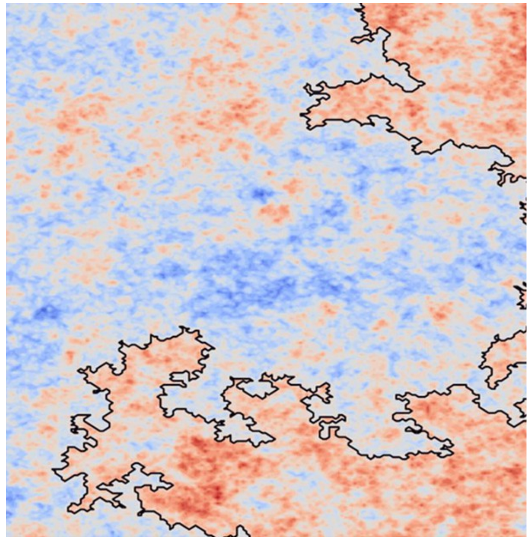
- Proven for

  $(\kappa = 2{,}8)$ **Loop-Erased Random Walk & Uniform Spanning Tree** (Lawler, Sheffield & Werner)

  $(\kappa = 6)$ **Percolation Interface** (Smirnov)

  $(\kappa = 3{,}16/3)$ **Ising Interface** (Chelkak, Hongler, Duminil-Copin, Kempainnen & Smirnov) and **FK Ising** (Garban & Wu)

  $(\kappa = 4)$ **Level line of discrete GFF** (Schramm & Sheffield)



GFF level line ©Aru

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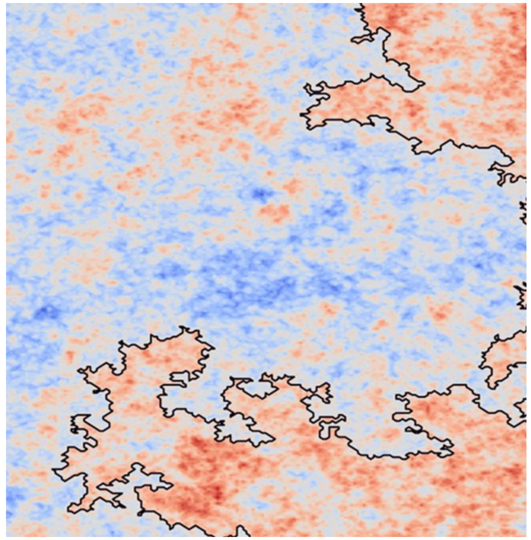
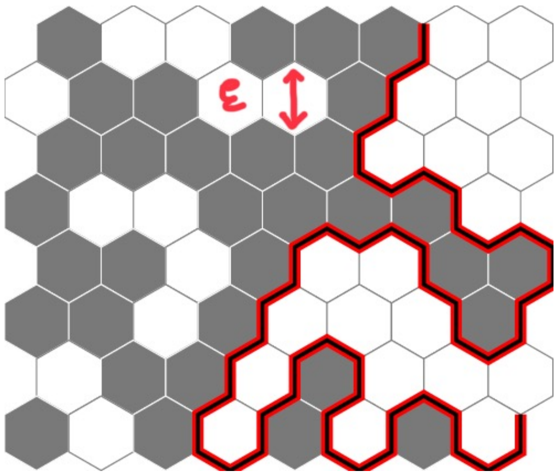


GFF level line ©Aru

All other κ: open!



Percolation Interface ©Schramm-Steif

Natural Length: I

- SLE_κ a.s. has Hausdorff dim

$$d = d_\kappa^{\text{SLE}} = \min(2, 1 + \kappa/8)$$

- On ε-diameter grid, expect discrete interfaces to have $\text{O}(\varepsilon^{-d})$ "steps".

- **Reparametrise time** so ε^{-d} steps made in time interval length one

$\rightarrow$ **parametrised** limit curve?

- This should be **SLE in the "natural parametrisation"**

Natural length: II

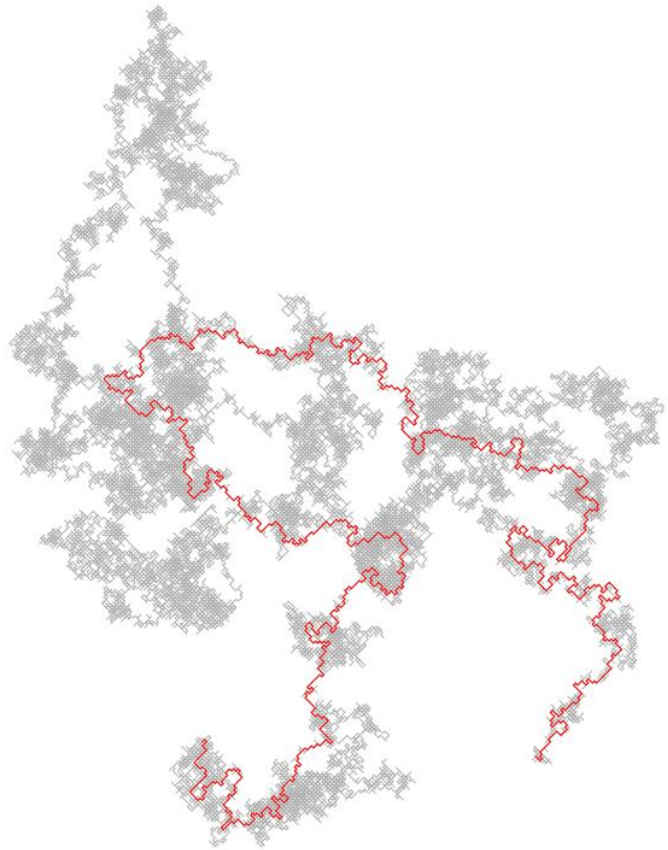
- The **natural parametrisation** of SLE (if it exists) can be defined by an **axiomatic characterisation** (Lawler & Sheffield)

- **Existence** was first shown by Lawler & Sheffield for $\kappa \leq \kappa_0 \approx 5$

- Then extended to all $\kappa < 8$ by Lawler & Zhou

- Shown by Lawler & Rezaei to coincide with d_κ^{SLE}-**Minkowski content**

- Convergence of **LERW in natural parametrisation** shown by Lawler & Viklund



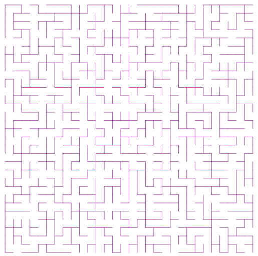
Loop Erased Random Walk (Red) ©Viklund

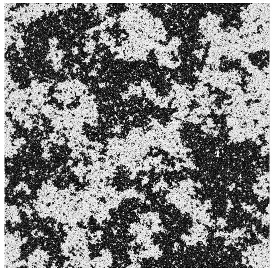
SLE on Liouville Quantum Gravity (LQG)

Motivation

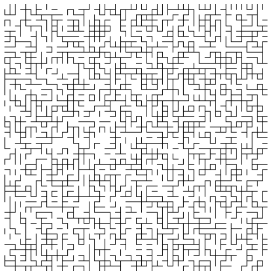
Critical statistical physics models… … on random graphs “maps”

Uniform Spanning Tree © Kassel

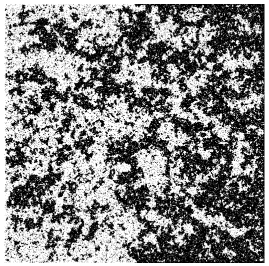




Ising model ©Cerf

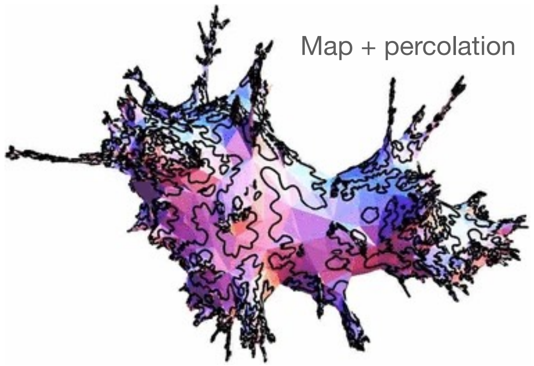


Bond percolation ©Wiki



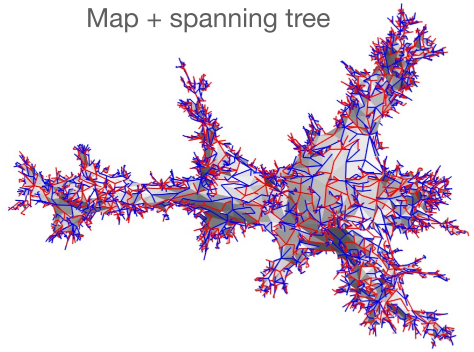
Random cluster model ©Pete

Map + percolation



©Curien

Map + spanning tree



©Budd

“Random planar map” has law weighted by the partition function of the model

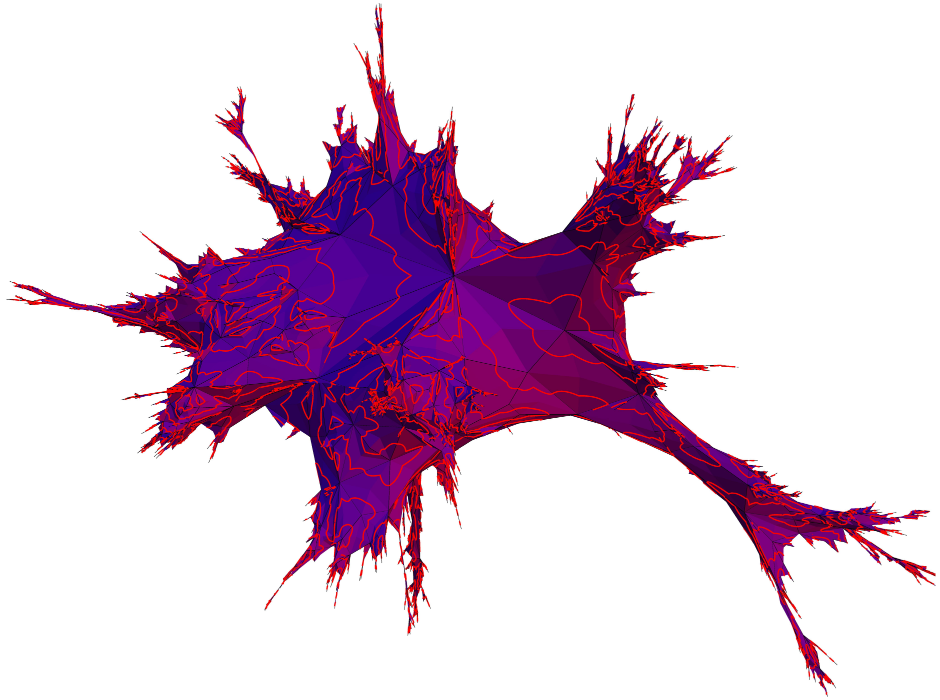
Conjecture

In an appropriate scaling limit…

- Random map → **Liouville quantum gravity surface (LQG)**

- Loops/interfaces → **SLE or conformal loop ensemble (CLE)**

- **Independent** of each other!



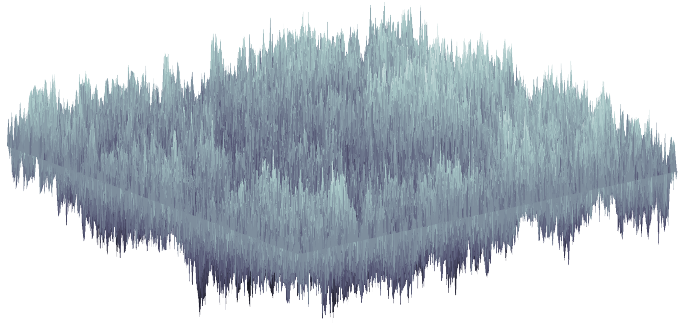
FK-cluster model weighted map ©Bettinelli-Laslier

LQG

- Random metric on $D = \mathbb{C}, \mathbb{D}, \mathbb{H}, \mathbb{S}^2 \ldots$

$$D_h = \exp(\gamma h(z))\, dz^2; \quad \gamma \in (0,2]$$

for h a **Gaussian free field** (GFF) on D



Approximation of a Gaussian free field on a square

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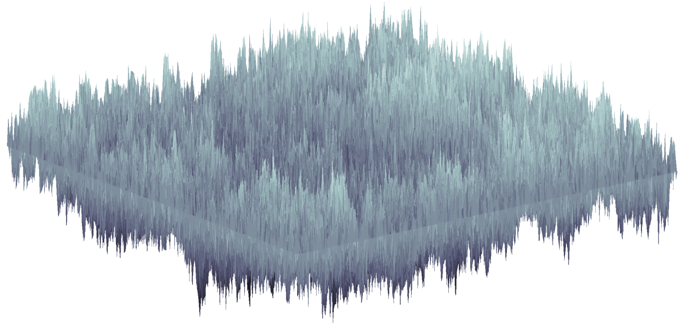

Approximation of a Gaussian free field on a square

- Associated (LQG) **volume measure** μ_h defined first using **Gaussian multiplicative chaos** regularisation (Kahane, Robert & Vargas, Duplantier & Sheffield, Berestycki …)

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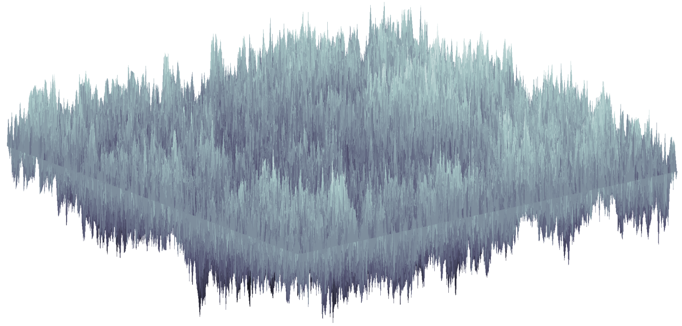

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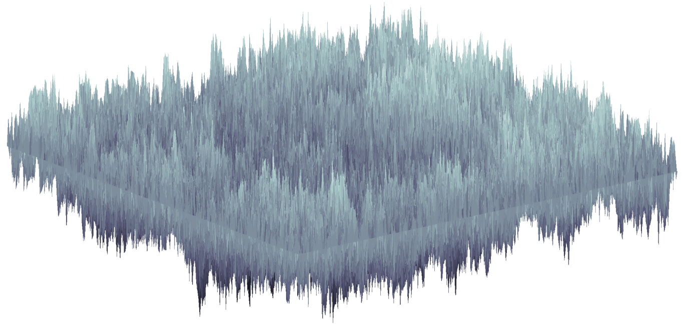
LQG



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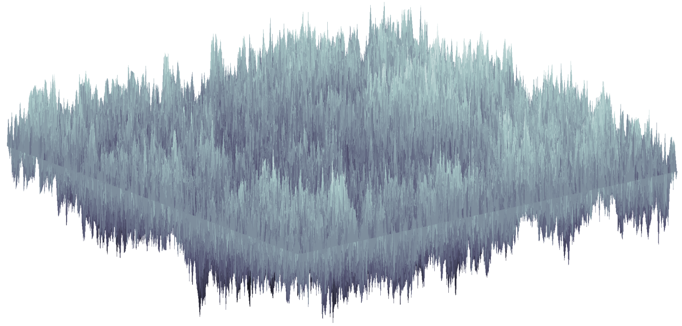

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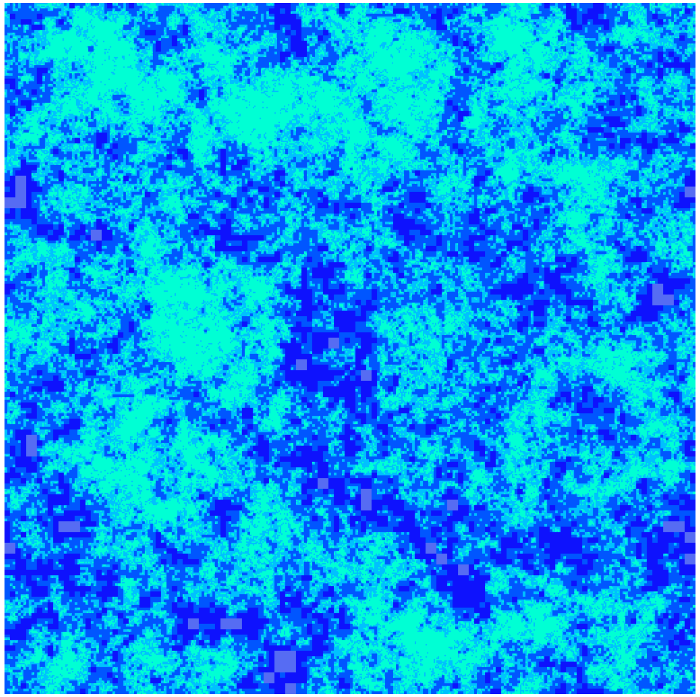
- Now, **metric** D_h also defined (Ding, Dunlap, Dubédat & Falconet, Gwynne & Miller)
- D_h has dimension $d_\gamma^{\mathrm{LQG}} > 2$, and μ_h is a d_γ^{LQG}-diml Minkowski content measure for D_h (Ang, Falconet & Sun, Gwynne & Sung)

Back to conjecture

- **Conformally embed** the random map plus interfaces in $\mathbb{C}, \mathbb{D}, \mathbb{H}, \mathbb{S}^2 \dots$
- Consider images of interfaces plus
  - Rescaled **counting measure** on faces/vertices
  - Rescaled **graph distance** between points
- Should be γ-LQG metric-measure space built from a GFF-type field *h*, plus **independent** SLE_κ
- γ depends on discrete model and $\kappa = \gamma^2$ or $16/\gamma^2$



LQG density ©Miller

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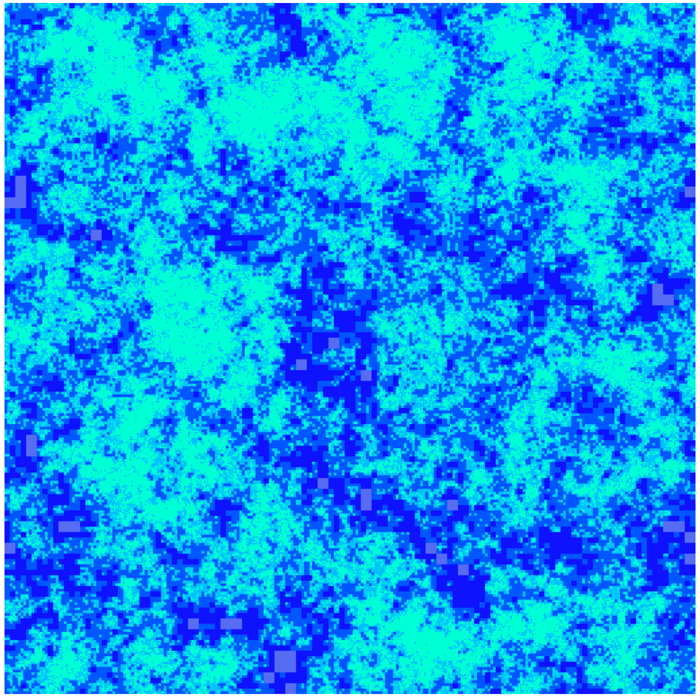
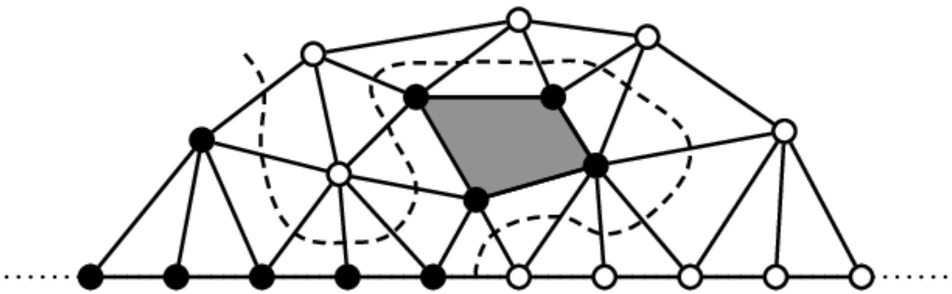


LQG density ©Miller

Now proven in some special cases: Holden & Sun, Gwynne & Miller, Gwynne & Miller & Sheffield

Quantum Length of SLE

Quantum Length



Percolation on random planar map ©Angel

- If discrete interface is naturally parameterised by # "steps" in the graph…
- Limit after rescaling → SLE in "**quantum length parametrisation"**
  - **d-diml content measure** with respect to the LQG metric D_h?

    (**KPZ**: implies $d = d_\gamma^{\mathrm{LQG}}/2$!)
  - **"**$e^{\alpha h(z)} m(dz)$**"** for m natural length measure on the SLE and h the LQG field?

Can any of this be made sense of directly in the continuum?

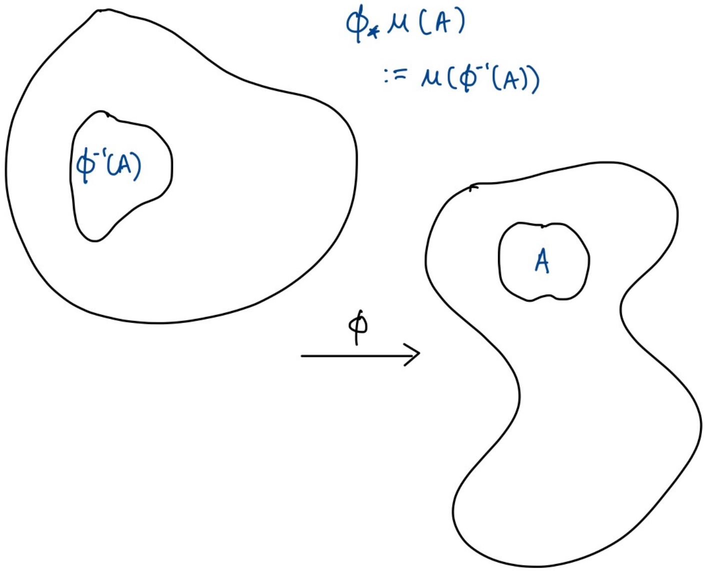
Sheffield's Approach

Conformal Covariance

Suppose h is a free boundary GFF-type field on $D \subset \mathbb{C}$ with (D_h, μ_h) its LQG metric-measure space for some γ, and $\phi : D \to D'$ conformal.

Then $\phi_* \mu_h = \mu_{\phi(h)}$ with

$$\phi(h) = h \circ \phi^{-1} + (\frac{2}{\gamma} + \frac{\gamma}{2}) \log |(\phi^{-1})'|$$



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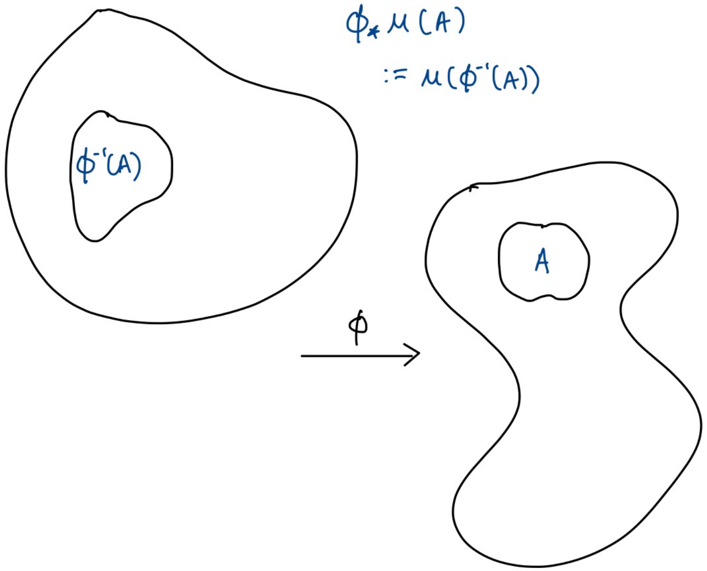

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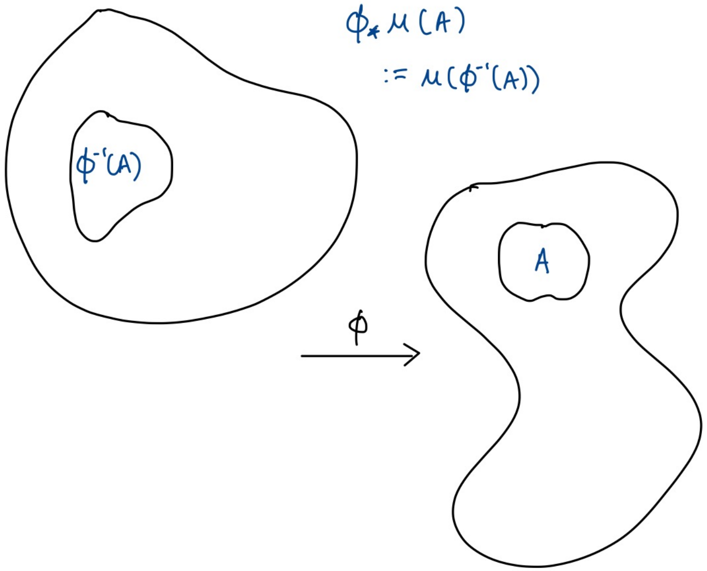

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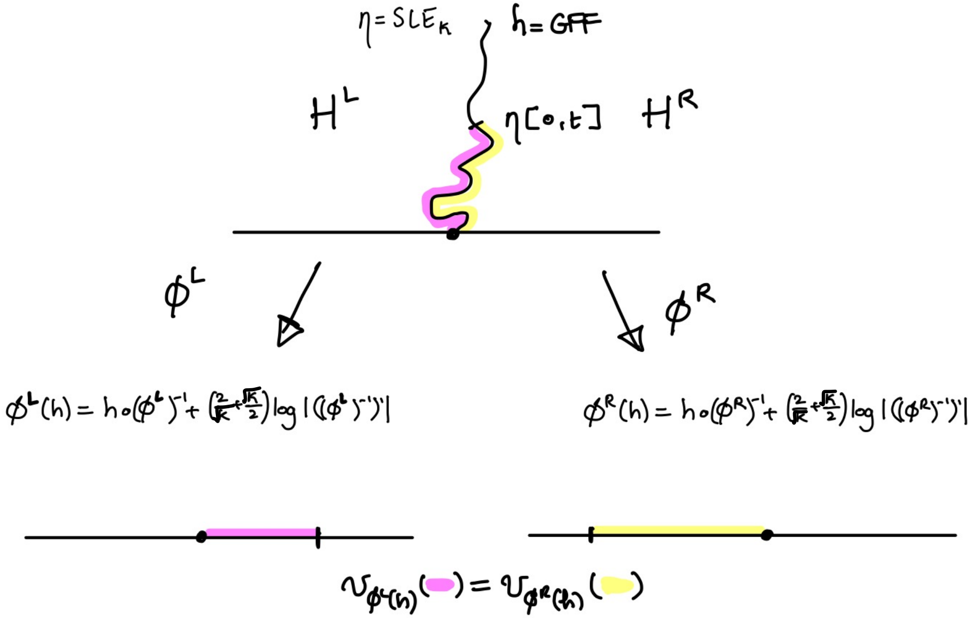
* Holds for D_h & $D_{\phi(h)}$ if $D = \mathbb{C}$ and ϕ is translation/scaling. **Expected to hold in general.**



Sheffield's Approach

Construction for $\kappa \in (0,4)$

- Define $\phi^L(h)$, $\phi^R(h)$ as previously
- Define the **quantum length** on the **left** and the **right** sides of the SLE via $\nu_{\phi^L(h)}$ and $\nu_{\phi^R(h)}$
- **Question:** Do the left/right measures agree??

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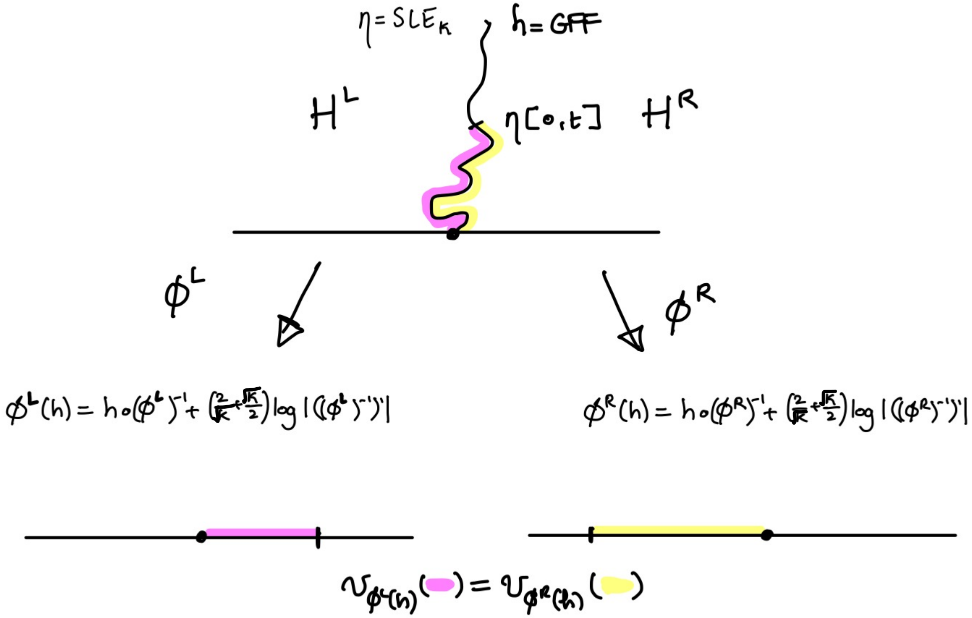

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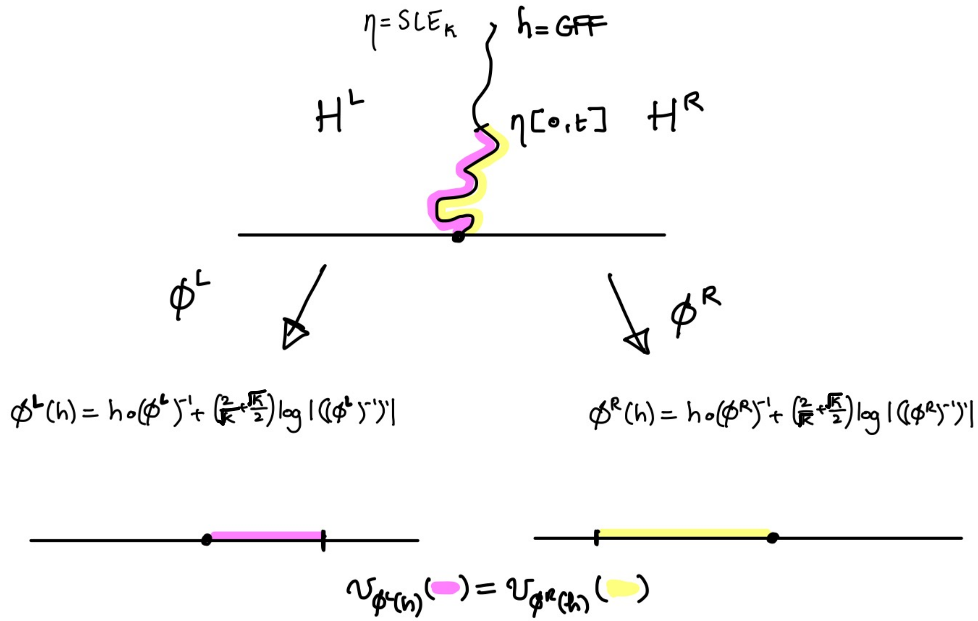

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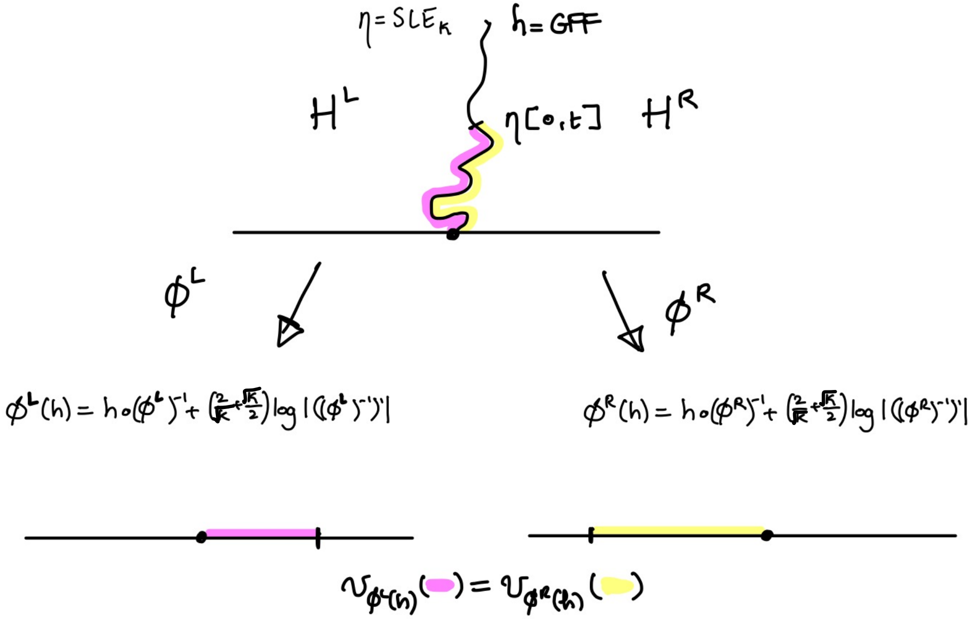

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… but proof not so easy, uses the **quantum gravity zipper**

Applications

- **Conformal welding:** Sheffield also showed that an LQG surface with an independent SLE can be viewed as the conformal welding **according to quantum boundary length** of two independent LQG surfaces

- Generalisations developed in **mating-of-trees** theory of Duplantier & Miller & Sheffields makes scaling limit intuition that **“independent SLE/CLE decomposes LQG into small independent pieces”** rigorous

- **Powerful tool** for studying LQG surfaces and for studying SLE!

SLE exploration of LQG ©Miller

Direct approaches: for $\kappa \leq 4$

Questions

* Can we construct quantum length of an SLE with respect to an independent GFF h as a $(d_\gamma^{\mathrm{LQG}}/2)$-**dimensional Minkowski content measure** wrt metric D_h?

* Or as "$e^{\alpha h(z)} m(dz)$" (defined via regularisation) where m is the natural length of the SLE?

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Answers

- Not known ??

- Yes! First shown by Benoist with $\alpha = \gamma/2$ (same as the boundary length measure for D_h) and $\kappa \in (0,4)$.

[**Drawback:** Benoist's proof of (2) relies on the delicate **stationary quantum zipper** construction of Sheffield, plus an ergodicity argument. Equivalence with Sheffield's quantum length up to an **unknown multiplicative constant.**]

(Even more) direct approach for $\kappa \leq 4$

With Avelio Sepúlveda

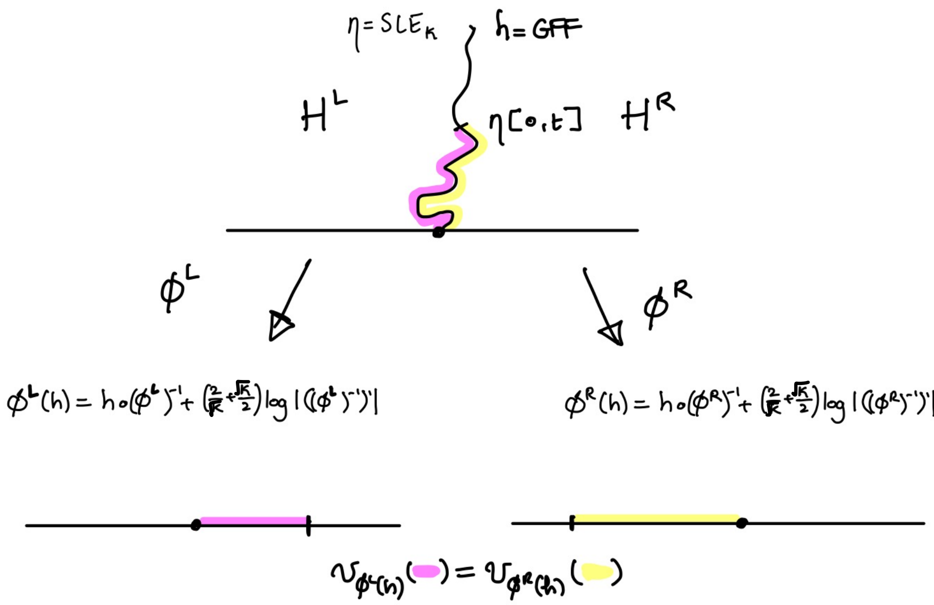
- **Elementary construction** of the quantum length for SLE_κ ($\kappa < 4$) with respect to an independent GFF h

$$\nu_h^{(\hat{m})} = \text{“}e^{(\gamma/2)h(x)}\hat{m}(dx)\text{”}; \qquad \gamma = \sqrt{\kappa}$$

via standard Gaussian multiplicative chaos theory. $\hat{m} = cm$ **conformal Minkowski content**

- **Direct proof** that after conformal mapping with ϕ^L, ϕ^R, for any subset A of the SLE

$$\nu_{\phi^L(h)}(\phi^L(A)) = \nu_{\phi^R(h)}(\phi^R(A)) = \frac{4-\kappa}{4}\nu_h^{(\hat{m})}(A)$$



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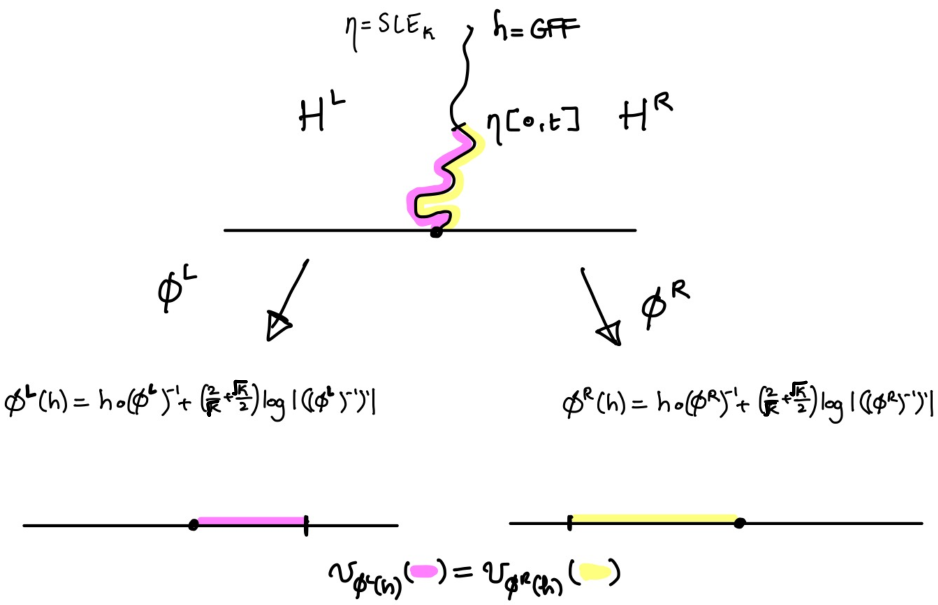


- That is, **equivalence with Sheffield's length** up to a **known multiplicative constant**

(Even more) direct approach for $\kappa \leq 4$

With Avelio Sepúlveda

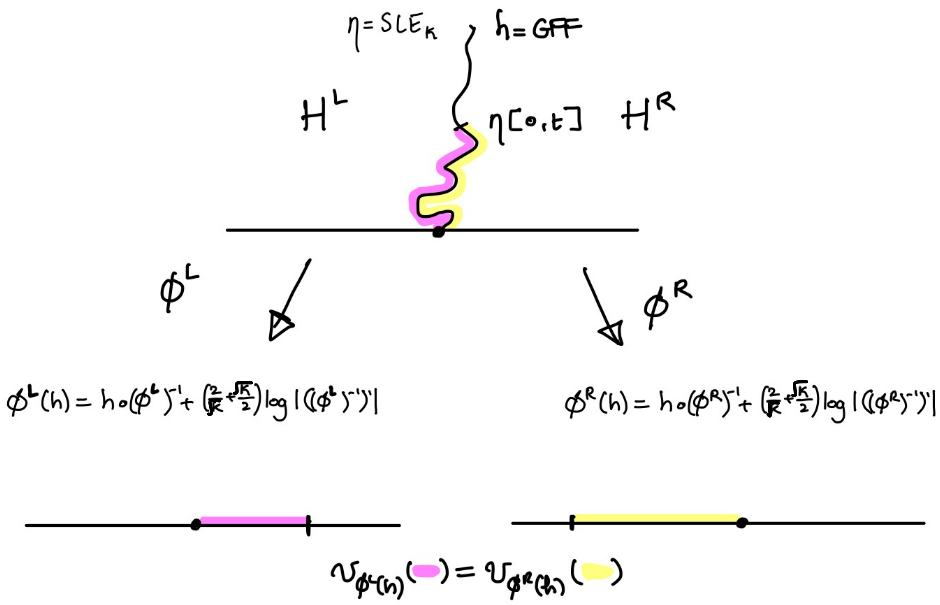
- **Ingredient:** new approximation

$$\hat{m}(dx) = 2 \lim_{\delta \to 0} \delta \mathrm{CR}(z, D \backslash \eta)^{(\kappa/8)-1+\delta} 1_{z \in H^i}$$

  to $\hat{m}$ for both $i = L, R$

- This approximation **transforms very well** under applying ϕ^L or ϕ^R …

- … it becomes an approximation of **Lebesgue measure on the boundary**



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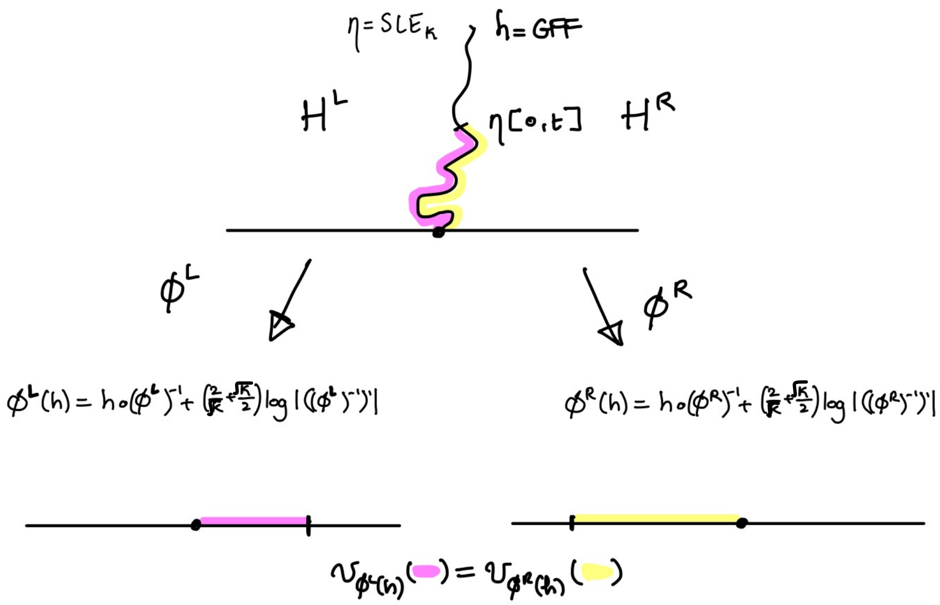

- Identifying the ratio with Sheffield's quantum length explicitly allows us to take $\gamma \uparrow 2$, $\kappa \uparrow 4$ and get same result for $\gamma = 2$, $\kappa = 4$

Thanks!