# Introduction to Liouville quantum gravity

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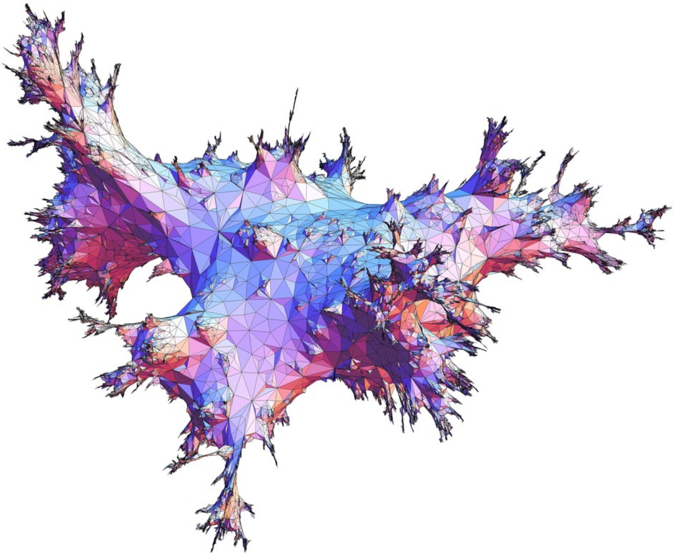
KPZ meets KPZ workshop, Fields Institute

# Introduction to Liouville quantum gravity (LQG)

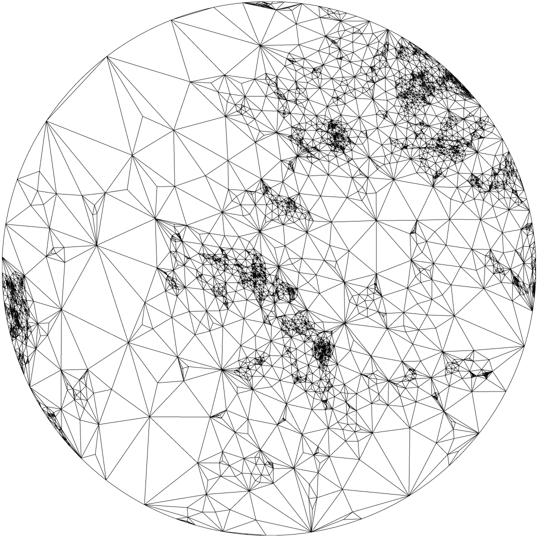
1. Motivation: Random planar maps
2. Construction and properties of LQG
3. Relationships between LQG and other topics

# Random planar map simulation



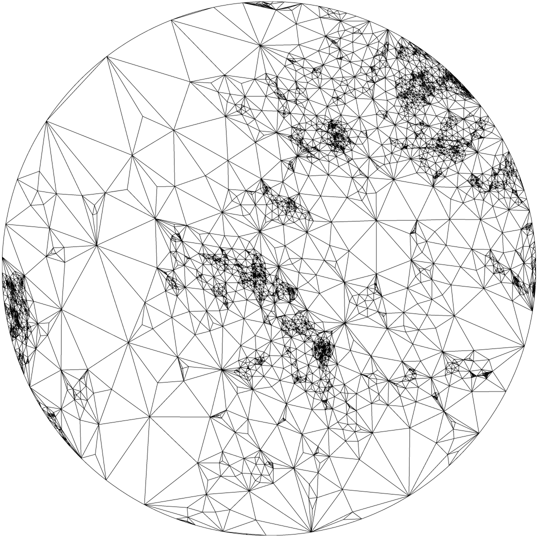
Random planar map, spring embedding in $\mathbb{R}^3$ (Thomas Budzinski).

# Random planar map simulation



Random planar map, harmonic embedding in $\mathbb{D}$ (Jason Miller).

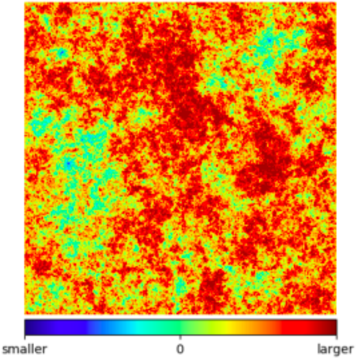
# Random planar map simulation



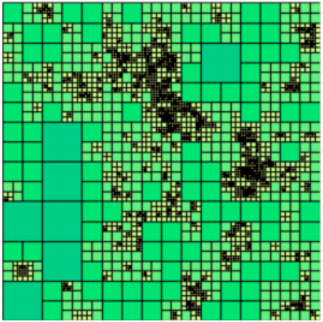
Random planar map, harmonic embedding in $\mathbb{D}$ (Jason Miller).
Scaling limit of this random geometry is **Liouville quantum gravity**.

# Liouville quantum gravity (LQG)

LQG parameter $\gamma \in (0, 2]$.




Random generalized function
"$h : \mathbb{C} \to \mathbb{R}$"



Random geometry
"$e^{\gamma h}(dx^2 + dy^2)$"

Simulation of discretization of LQG (Minjae Park).

LQG is a random continuum geometry endowed with:
**metric**, **measure**, **conformal structure**.

# Examples and simulations

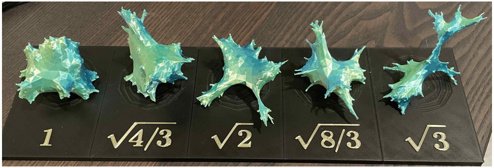
$\gamma = \sqrt{2}$: Tree-decorated random planar map.

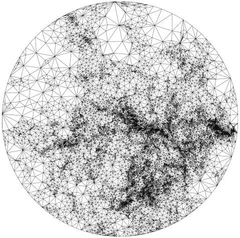
$\gamma = \sqrt{8/3}$: Uniform random planar map.

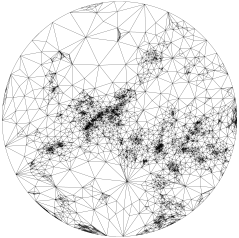
$\gamma = \sqrt{3}$: Ising-decorated random planar map.

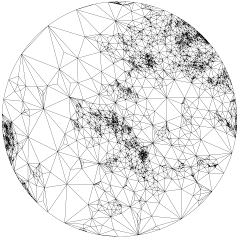
$\gamma = 2$: GFF-decorated random planar map

"Larger $\gamma \implies$ rougher surface".







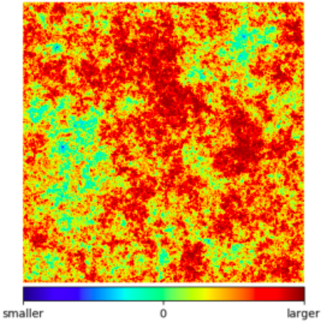
# Gaussian free field

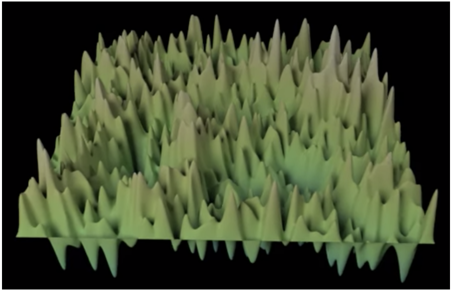
Centered Gaussian field $h$ on the plane with covariance

$$\mathbb{E}[h(z)h(w)] = -\log|z-w| + O(1) \quad \text{as } z \to w.$$

$h$ is a **distribution** or **generalized function**:

- $h(z)$ is not well-defined;
- $\int_{\mathbb{C}} h(z)\rho(dz)$ is defined when $\rho$ is sufficiently regular, e.g., $\rho(dz) = f(z)dz$ for smooth compactly supported $f$.



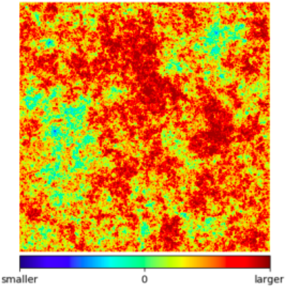
(Simulations by Minjae Park, Henry Jackson.)

# Liouville quantum gravity area measure

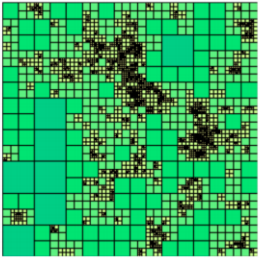
Let $h$ be a Gaussian free field.
Let $h_\varepsilon(z)$ be the average of $h$ on the radius-$\varepsilon$ circle around $z$.
Let $\gamma \in (0,2)$. The $\gamma$-**LQG area measure** is

$$A_h^\gamma(dz) := \lim_{\varepsilon \to 0} \varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(z)}\, dz.$$

This is an example of **Gaussian multiplicative chaos**.
Kahane '85, Robert-Vargas '08, Duplantier-Sheffield '08




Gaussian free field $h$
Images by Minjae Park.



Discretization of $A_h$,
squares have comparable $A_h$-mass.

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When $\gamma = 2$, have log-correction
[Duplantier-Sheffield-Rhodes-Vargas '14]:

$$A_h^{\gamma=2}(dz) := \lim_{\varepsilon \to 0} \sqrt{\log(1/\varepsilon)}\varepsilon^2 e^{2h_\varepsilon(z)}\, dz.$$

When $\gamma > 2$, the LQG area measure does not exist: $A_h^\gamma(dz) \equiv 0$.

# Properties of the LQG area measure

## Weyl scaling

If $f$ is a continuous function (not necessarily indep. of $h$), then

$$A^\gamma_{h+f}(dz) = e^{\gamma f(z)} A^\gamma_h(dz).$$

## Moments

$\mathbb{E}[A^\gamma_h([0,1]^2)^p] < \infty$ if and only if $p < \frac{4}{\gamma^2}$.

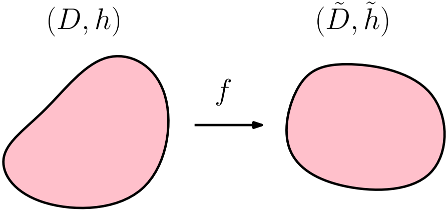
We say $z$ is an $\alpha$-thick point of $h$ if $\lim_{\varepsilon \to 0} \frac{\log h_\varepsilon(z)}{\log(1/\varepsilon)} = \alpha$, i.e., $h$ blows up like $-\alpha \log|\cdot - z|$ near $z$.

## Singularity of $A^\gamma_h$-typical points

The LQG measure $A^\gamma_h$ is supported on the set of $\gamma$-thick points.

# Properties of the LQG area measure: coordinate change

Let $f : D \to \tilde{D}$ be a conformal map. Let $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$.




$$h = \tilde{h} \circ f + (\tfrac{\gamma}{2} + \tfrac{2}{\gamma}) \log |f'|$$

In above setting, we have

$$f_* A_h^\gamma = A_{\tilde{h}}^\gamma.$$

We think of $(D, h)$ and $(\tilde{D}, \tilde{h})$ as describing the same LQG surface!
A **quantum surface** is an equivalence class of pairs $(D, h)$.

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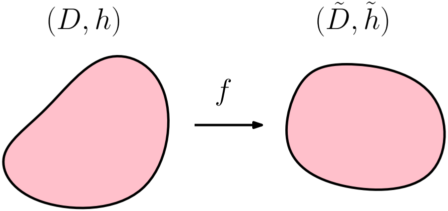


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A **quantum surface** is an equivalence class of pairs $(D, h)$.

Any reasonable notion of LQG metric $D_h^\gamma$ will also satisfy

$$f_* D_h^\gamma = D_{\tilde{h}}^\gamma.$$

# The Liouville quantum gravity metric

There is a constant $d_\gamma$ called the **fractal dimension of $\gamma$-LQG**.

Will discuss $d_\gamma$ in more detail later.

We want a metric $D_h^\gamma$ satisfying the following properties:

1. (Length space)
2. (Locality)
3. (Weyl scaling)
4. (Coordinate change)

# The Liouville quantum gravity metric

There is a constant $d_\gamma$ called the **fractal dimension of $\gamma$-LQG**. We want a metric $D_h^\gamma$ satisfying the following properties:

1. (Length space) For a path $P : [0,1] \to \mathbb{C}$, define

$$\mathrm{len}(P; D_h^\gamma) = \sup \sum_{i=1}^{n} D_h^\gamma(P(t_i), P(t_{i-1}))$$

   where the supremum is taken over all partitions $0 = t_0 < t_1 < \cdots < t_n = 1$. Then

$$D_h^\gamma = \inf_P \mathrm{len}(P; D_h^\gamma).$$

2. (Locality)
3. (Weyl scaling)
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# The Liouville quantum gravity metric

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$$D_h^\gamma = \inf_P \mathrm{len}(P; D_h^\gamma).$$
2. (Locality) For deterministic open $U$, define internal metric
$$D_h^\gamma(z, w; U) = \inf\{\mathrm{len}(P; D_h^\gamma) : P \text{ is a path from } z \text{ to } w \text{ in } U\}.$$
Then $D_h^\gamma(z, w; U)$ is a measurable function of $h|_U$.
3. (Weyl scaling)
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1. (Length space)
$$D_h^\gamma = \inf_P \operatorname{len}(P; D_h^\gamma).$$
2. (Locality) Internal metric $D_h^\gamma(z, w; U)$ is determined by $h|_U$.
3. (Weyl scaling) Let $\xi = \frac{\gamma}{d_\gamma}$. For continuous $f : \mathbb{C} \to \mathbb{R}$, define
$$(e^{\xi f} \cdot D_h^\gamma)(z, w) = \inf_{P: z \to w} \int_0^{\operatorname{len}(P; D_h^\gamma)} e^{\xi f(P(t))}\, dt$$
where the infimum is over paths parametrized by $D_h^\gamma$-length. Then a.s.
$$e^{\xi f} \cdot D_h^\gamma = D_{h+f}^\gamma.$$
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$$D_{h+f}^\gamma(z, w) = \inf_{P: z \to w} \int_0^{\operatorname{len}(P; D_h^\gamma)} e^{\xi f(P(t))} \, dt.$$
4. (Coordinate change for scaling and rotation) Let $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$.
Let $r > 0$ and $z \in \mathbb{C}$. Almost surely, for all $u, v \in \mathbb{C}$
$$D_h^\gamma(ru + z, rv + z) = D_{h(r\cdot+z)+Q\log r}(u, v).$$

# The Liouville quantum gravity metric

**Theorem** — Ding-Dubédat-Dunlap-Falconet '19

There exists an LQG metric.

Construction via $\varepsilon \to 0$ subsequential limit of
$\varepsilon$-**Liouville first passage percolation**,
the metric corresponding to $e^{\xi h_\varepsilon}(dx^2 + dy^2)$.

**Theorem** — Gwynne-Miller '19

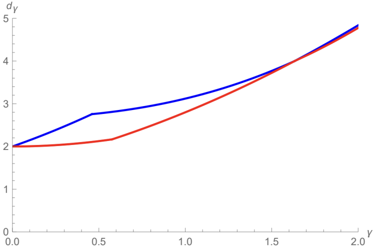
The LQG metric is unique up to multiplicative constant.

# Fractal dimension of the LQG metric

Denote the **fractal dimension** of $\gamma$-LQG by $d_\gamma$ [Ding-Gwynne '18].

Minkowski dimension of $\gamma$-LQG $= d_\gamma$ [A.-Falconet-Sun '20],
$d_\gamma$-Minkowski content of $D_h^\gamma$ is $A_h^\gamma$ [Gwynne-Sung '22].

Describes distances in random planar maps
[Gwynne-Holden-Sun '17, Ding-Gwynne '18].



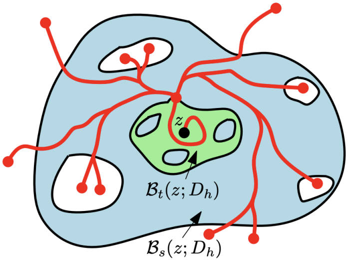

Upper and lower bounds for $d_\gamma$.
Only known values are $d_0 = 2$ and $d_{\sqrt{8/3}} = 4$.

**Open problem: Compute $d_\gamma$.**

# Geodesics of the LQG metric

**Confluence of geodesics** Gwynne-Miller '19

Fix $z \in \mathbb{C}$. Almost surely, for each radius $s > 0$ there exists a radius $t \in (0, s)$ such that any two $D_h^\gamma$-geodesics from $z$ to points outside the $\gamma$-LQG metric ball $B_s(z; D_h^\gamma)$ coincide on the time interval $[0, t]$.




Much is unknown about geodesics, e.g., Euclidean dimension.

# The Knizhik-Polyakov-Zamolodchikov relation

The Hausdorff dimension of a set $X$ with respect to metric $D$ is

$$\inf\{\sum_{j=1}^{\infty} r_j^{\Delta} : \text{exists covering of } X \text{ by } D\text{-metric balls with radii } r_j\}.$$

(e.g. Hausdorff dim of point, line, plane w.r.t. Euc. metric is $0, 1, 2$.)

Recall $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$.

**KPZ formula** — Gwynne-Pfeffer '19

Let $X$ be a random fractal set independent of $h$.
$\Delta_0 =$ Hausdorff dimension of $X$ w.r.t. Euclidean metric,
$\Delta_h =$ Hausdorff dimension of $X$ w.r.t. $D_h^\gamma$. Then a.s.,

$$\Delta_h = \frac{d_\gamma}{\gamma}(Q - \sqrt{Q^2 - 2\Delta_0}).$$

More elementary formulations using $A_h^\gamma$ rather than $D_h^\gamma$ (Duplantier-Sheffield '08, Rhodes-Vargas '08, Aru '15).

# The KPZ relation: some history

- Early evidence for "planar maps $\longleftrightarrow e^{\gamma h}(dx^2 + dy^2)$".

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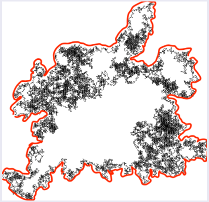
- Physics derivation of Mandelbrot's conjecture! [Duplantier '98]

**Mandelbrot's conjecture**

The dimension of the outer boundary of 2D Brownian motion is $4/3$.



- Compute dimension of Brownian motion outer boundary on $\gamma = \sqrt{8/3}$-LQG, using enumeration of planar maps.
- Compute dimension of Brownian motion outer boundary on plane, using KPZ relation.

# The KPZ relation: some history

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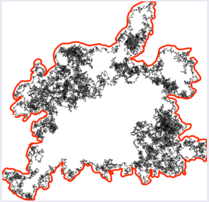

  - Compute dimension of Brownian motion outer boundary on $\gamma = \sqrt{8/3}$-LQG, using enumeration of planar maps.
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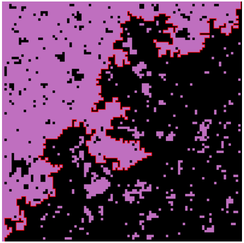
- Eventually Mandelbrot's conjecture was rigorously proved by Lawler-Schramm-Werner via SLE.

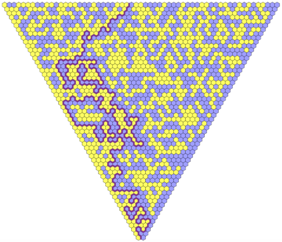
# Introduction to Liouville quantum gravity (LQG)

# Schramm-Loewner evolution

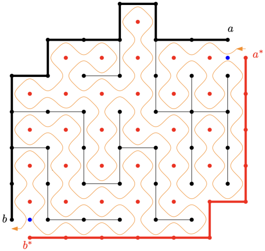
Cutting LQG by independent SLE gives independent surfaces.
See Powell's upcoming talk!



Critical Ising model

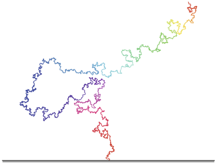


Critical percolation

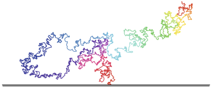



Uniform spanning tree

Images by Peltola, Duminil-Copin, Yong Han, Mingchang Liu, Hao Wu '20.



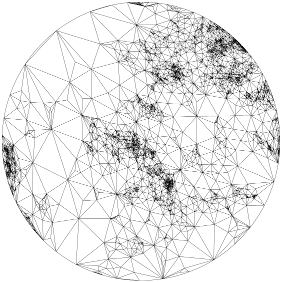
$\mathrm{SLE}_3$



$\mathrm{SLE}_6$

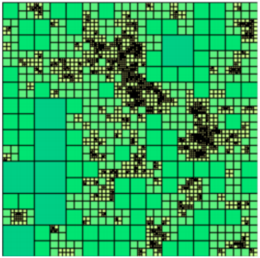
$\mathrm{SLE}_{32}$

# Random planar maps



***n*-face random planar map**

Image by Jason Miller



**Liouville quantum gravity**

Image by Minjae Park

Discrete area measure $A_n$, boundary measure $L_n$, metric $d_n$.
General conjecture: $(A_n, L_n, d_n) \to (A_h, L_h, d_h)$!

Important special cases:
Gwynne-Miller-Sheffield '17 (mated-CRT map, harmonic embedding),
Holden-Sun '19 (uniform RPM, Cardy embedding).
Bertacco-Gwynne-Sheffield '23 (mated-CRT map, Smith embedding).

**Open problem:**
**convergence for RPM weighted by statistical physics model.**

# Random planar maps: convergence of observables

Conformal loop ensemble (CLE) = random collection of SLE loops.

- Random planar maps with Fortuin-Kasteleyn random cluster
  $\longrightarrow$ LQG + independent CLE.

  Convergence of "boundary length process".

  Sheffield '11, Duplantier-Miller-Sheffield '14, Gwynne-Mao-Sun '15.

- Random planar maps with $O(n)$ loop model
  $\longrightarrow$ LQG + independent CLE.

  Convergence of loop lengths.

  Bertoin-Budd-Curien-Kortchemski '16,

  Chen-Curien-Maillard '17, Miller-Sheffield-Werner '20.

  Convergence of area.

  Aïdekon, Da Silva, Hu '24.

# Liouville conformal field theory

Lots of recent progress on Liouville conformal field theory.
Why should we care?

For probability theorists:
The laws of LQG areas and boundary lengths of natural random surfaces are encapsulated by correlation functions of Liouville CFT.
These correlation functions have explicit expressions.

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For mathematicians:
Making rigorous sense of quantum field theories (path integrals) is a major driving force in mathematics.

For physicists:
Derivations of Liouville CFT correlation functions were lacking rigor even by physicists standards.

# Liouville conformal field theory (*imprecise slide)

A **quantum field theory** is a collection of numbers called **correlation functions**, that arise as expectations of a random field.

A **conformal field theory** is a QFT with conformal symmetries.

Liouville conformal field theory was introduced by Polyakov '81 in the context of bosonic string theory.

One of the most fundamental 2D CFTs.

Mathematically constructed by

- David-Kupiainen-Rhodes-Vargas '14 (sphere)
- Huang-Rhodes-Vargas '15 (disk)
- Guillarmou-Rhodes-Vargas '16 (closed surfaces)
- Remy '17 (annulus)

# Liouville conformal field theory (*imprecise slide)

Very, very roughly speaking, the Liouville CFT correlation functions are something like

$$\langle\prod_{i=1}^{n} e^{\alpha_i h(z_i)}\rangle_\mu := \text{``}\mathbb{E}[\prod_{i=1}^{n} e^{\alpha_i h(z_i)} e^{-\mu A_h^\gamma(\mathbb{C})}]\text{''}$$

where $h$ is a Gaussian free field, $\alpha_i \in \mathbb{R}$ are weights, $z_i \in \mathbb{C}$ are points, and $\mu > 0$.

Describes law of LQG area via Laplace transform.

“Solving LCFT” $=$ computing all correlation functions.

# Correlation functions of Liouville CFT

"Solving LCFT" = computing all correlation functions.

**Three-point correlation function** Kupiainen-Rhodes-Vargas '17

$\langle \prod_{i=1}^{3} e^{\alpha_i h(z_i)} \rangle_\mu$ has the explicit formula

$$C_\gamma^{\mathrm{DOZZ}}(\alpha_1, \alpha_2, \alpha_3) \times \frac{\mu^{(2Q - \sum_i \alpha_i)/\gamma}}{2} \prod_{i=1}^{3} |z_i - z_{i+1}|^{-2(\Delta_i + \Delta_{i+1} - \Delta_{i+2})}$$

where $\Delta_i = \frac{\alpha_i}{2}(Q - \frac{\alpha_i}{2})$.

# Correlation functions of Liouville CFT

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where $\Delta_i = \frac{\alpha_i}{2}(Q - \frac{\alpha_i}{2})$.

Given three-point correlation function, can inductively solve for higher order correlation functions via **conformal bootstrap**.

Guillarmou-Kupiainen-Rhodes-Vargas '20

# DOZZ formula (jump-scare)

Write $\overline{\alpha} = \sum \alpha_i$.

$$C_\gamma^{\mathrm{DOZZ}}(\alpha_1,\alpha_2,\alpha_3) := \left(\pi \frac{\Gamma(\frac{\gamma^2}{4})}{\Gamma(1-\frac{\gamma^2}{4})}(\frac{\gamma}{2})^{2-\gamma^2/2}\right)^{\frac{2}{\gamma}(Q-\overline{\alpha})} \times \frac{\Upsilon'_{\frac{\gamma}{2}}(0)\Upsilon_{\frac{\gamma}{2}}(\alpha_1)\Upsilon_{\frac{\gamma}{2}}(\alpha_2)\Upsilon_{\frac{\gamma}{2}}(\alpha_3)}{\Upsilon_{\frac{\gamma}{2}}(\frac{\overline{\alpha}}{2}-Q)\Upsilon_{\frac{\gamma}{2}}(\frac{\overline{\alpha}}{2}-\alpha_1)\Upsilon_{\frac{\gamma}{2}}(\frac{\overline{\alpha}}{2}-\alpha_2)\Upsilon_{\frac{\gamma}{2}}(\frac{\overline{\alpha}}{2}-\alpha_3)},$$

$$\ln \Upsilon_{\frac{\gamma}{2}}(z) := \int_0^\infty \left((\frac{Q}{2}-z)^2 e^{-t} - \frac{(\sinh((\frac{Q}{2}-z)\frac{t}{2})^2)}{\sinh(\frac{t\gamma}{4})\sinh(\frac{t}{\gamma})}\right)\frac{dt}{t}.$$

# Introduction to Liouville quantum gravity (LQG)

# Application of LQG to critical percolation

Uses all discussed connections of LQG and other topics!

**Coupling of LQG and SLE/CLE**

- Miller-Sheffield-Werner '20
- A.-Holden-Sun '21
- A.-Holden-Sun-Yu '23

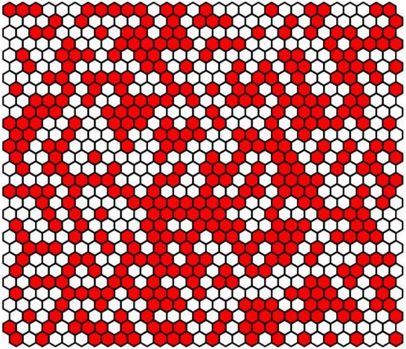
**Random planar maps + $O(n)$ loop model $\longrightarrow$ LQG + CLE**

- Bertoin-Budd-Curien-Kortchemski '16
- Chen-Curien-Maillard '17

**Liouville conformal field theory solvability**

- Kupiainen-Rhodes-Vargas '17

# Application of LQG to critical percolation



Critical site percolation on triangular lattice with mesh size $\delta$.

$P_n^\delta(z_1, \ldots, z_n)$ = probability $n$ points lie in the same cluster.

Conformally invariant scaling limit [Camia '23]:

$$P_n(z_1, \ldots, z_n) := \lim_{\delta \to 0} \pi_1(\delta)^{-n} P_n^\delta(z_1, \ldots, z_n).$$

# Application of LQG to critical percolation

$$P_n(z_1,\ldots,z_n) := \lim_{\delta\to 0} \pi_1(\delta)^{-n} P_n^\delta(z_1,\ldots,z_n).$$

Three-point connectivity constant

$$\frac{P_3(z_1,z_2,z_3)}{\sqrt{P_2(z_1,z_2)P_2(z_2,z_3)P_2(z_3,z_1)}} \overset{?}{=}$$

$$\sqrt{2}\pi^{-\frac{1}{8}}\left(\frac{2}{3}\right)^{-\frac{1}{2}}\left(\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}\right)^{\frac{3}{8}}\left(\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\right)^{\frac{3}{2}}\frac{1}{C^{\mathrm{DOZZ}}_{\sqrt{\frac{8}{3}}}\left(\frac{7\sqrt{6}}{12},\frac{7\sqrt{6}}{12},\frac{7\sqrt{6}}{12}\right)} \approx 1.022,$$

$C_\gamma^{\mathrm{DOZZ}}$ = DOZZ formula from $\gamma$-Liouville conformal field theory.

**Theorem (Delfino-Viti conjecture)** A.-Cai-Sun-Wu '24+

Above conjecture for the three-point connectivity constant holds.

# References


*Gaussian free field and Liouville quantum gravity*
Berëstycki-Powell '24+ book in preparation.
https://sites.google.com/view/ellenpowell/research

*Introduction to the Liouville quantum gravity metric*
Ding-Dubédat-Gwynne '23.

*Mating of trees for random planar maps and Liouville quantum gravity: a survey*
Gwynne-Holden-Sun '19.

# Liouville CFT: the Liouville field on $\mathbb{D}$

Motivation: probabilistic interpretation of Polyakov's path integral formulation of LCFT [David-Kupiainen-Rhodes-Vargas '14].

Liouville field depends implicitly on $\gamma \in (0,2)$. Let $Q := \frac{\gamma}{2} + \frac{2}{\gamma}$.
Let $P_{\mathbb{D}}$ be the law of the GFF on $\mathbb{D}$ with average zero on $\partial\mathbb{D}$.

**Liouville field on $\mathbb{D}$** — Huang-Rhodes-Vargas '15

Sample $(h, \mathbf{c})$ from $P_{\mathbb{D}} \times [e^{-Qc}dc]$ on $H^{-1}(\mathbb{D}) \times \mathbb{R}$.
The Liouville field is $\phi = h + \mathbf{c}$.
Let $\mathrm{LF}_{\mathbb{D}}$ be the measure on $H^{-1}(\mathbb{D})$ describing the law of $\phi$.

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Liouville field with insertion: $\mathrm{LF}_{\mathbb{D}}^{(\alpha,0)} := e^{\alpha\phi(0)} \mathrm{LF}_{\mathbb{D}}(d\phi)$.

Near the origin, field looks like GFF $- \alpha \log |z|$.

# Correlation functions of Liouville CFT

LCFT disk one-point correlation function for $\mu, \mu_B \geq 0$:

$$\langle e^{\alpha\phi(0)} \rangle_{\mathbb{D}} := \int e^{-\mu A_\phi(\mathbb{D}) - \mu_B L_\phi(\partial\mathbb{D})} \mathrm{LF}_{\mathbb{D}}^{(\alpha,0)}(d\phi).$$

# Correlation functions of Liouville CFT

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Can define Liouville field, correlation functions for general surfaces, e.g. Riemann sphere $\widehat{\mathbb{C}}$:

$$\langle \prod_{i=1}^{3} e^{\alpha_i\phi(z_i)}\rangle_{\widehat{\mathbb{C}}} := \int e^{-\mu A_\phi(\widehat{\mathbb{C}})} \mathrm{LF}_{\widehat{\mathbb{C}}}^{(\alpha_1,z_1),(\alpha_2,z_2),(\alpha_3,z_3)}(d\phi).$$

"Solving LCFT" = computing all correlation functions.