Gaussian curvature for LQG surfaces and random planar maps

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Toronto, Canada
March 8, 2024.

Preliminaries

Gaussian free field: Centered Gaussian field $\Phi^{\mathbb{C}}$ with covariance given by

$$\mathbb{E}(\Phi^{\mathbb{C}}(z)\Phi^{\mathbb{C}}(w)) = \log|z|_{+} + \log|w|_{+} + \left(\frac{1}{|z-w|}\right).$$

$\Phi^{\mathbb{C}}$ is a generalized function, called the whole-plane Gaussian free field (GFF).

Quantum cone field: Another important field is the γ-quantum cone field Φ,

$$\Phi = \Phi^{\mathbb{C}} - \gamma \log|\cdot|.$$

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Notation: Let d_γ be the fractal dimension of γ-LQG. Let $\xi := \frac{\gamma}{d_\gamma}$, and $Q := \frac{\gamma}{2} + \frac{2}{\gamma}$.

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• D_Φ is a length metric; D_Φ is local; and D_Φ satisfies Weyl scaling: for any $u, v \in \mathbb{C}$, a.s.

$$e^{\xi f} \cdot D_\Phi(u, v) = D_{\Phi+f}(u, v)$$

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- Coordinate change (for scaling and rotation): If $r > 0$ and $z \in \mathbb{C}$, then for all $u, v \in \mathbb{C}$, a.s.

$$D_\Phi(ru+z, rv+z) = D_{\Phi(r\cdot+z)+Q\log r}(u,v).$$

Curvature in LQG surfaces

If S is a smooth Riemannian surface with metric $e^f(dx^2 + dy^2)$, then the Gaussian curvature is given by

$$K_S(z) = -\frac{\Delta f}{2} e^{-f(z)}.$$

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- Because of mismatch between LQG measure and metric, "right" notion should be

$$K_\Phi(z) := \frac{\gamma\Delta\Phi}{2} e^{-\gamma\Phi}.$$

Definition of curvature on an LQG surface

$K_\Phi(z)$ is defined weakly: for any smooth compactly supported test function f, we define

$$\int_{\mathbb{C}} f(z)K_\Phi(z)d\mu_\Phi = \int_{\mathbb{C}} \frac{\gamma}{2}\Delta f(z)\Phi(z)dz.$$

Note: $K_\Phi(z)$ is invariant under LQG coordinate change.

Discrete curvature

Suppose $\mathcal{G}$ is a triangulation.

Discrete curvature: The discrete curvature is given by

$$K_{\mathcal{G}}(v) = 6 - \deg(v).$$

Conjecture: If M is a model of infinite random planar maps believed to be in the universality class of the γ quantum cone (e.g. uniform infinite triangulations), and we embed the map in the plane via any "reasonable embedding", then the scaling limit of K_M is $K_\Phi(z)$ for $\gamma = \sqrt{\frac{8}{3}}$.

Poisson Mated CRT map

Poisson point process: Take a Poisson point process on $\mathbb{R}$ with intensity ε^{-1}, $\Lambda = \{y_j\}_{j\in\mathbb{Z}}$.

Poisson mated CRT map: Suppose that $(L, R) : \mathbb{R} \to \mathbb{R}^2$ is a pair of correlated two sided standard linear Brownian motions, normalized such that $L_0 = R_0 = 0$ and such that $\operatorname{corr}(L_t, R_t) = -\cos\left(\frac{\pi\gamma^2}{4}\right)$ for $t \neq 0$. The mated CRT map $\mathcal{G}_\varepsilon$ is defined to be the random planar map obtained by mating discretized versions of the continuum random trees constructed from L and R, that is, two vertices $y_j, y_k \in \Lambda$ such that $j < k$ are connected if either

$$\left(\inf_{t\in[y_{j-1},y_j]} L_t\right) \vee \left(\inf_{t\in[y_{k-1},y_k]} L_t\right) \leq \left(\inf_{t\in[y_j,y_{k-1}]} L_t\right) \tag{1}$$

or the same holds with L replaced by R. If $|j-k| \geq 2$ and the inequality above holds for both L and R, then y_j, y_k are connected by two edges.

CRT map cells

Let η be a whole-plane space-filling SLEκ' from ∞ to ∞ sampled independently from Φ and then parameterized by γ-quantum mass with respect to Φ, where $\kappa' = 16/\gamma^2 > 4$.

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Alternate construction of Poisson CRT maps: One can define the graph $\bar{\mathcal{G}}_\varepsilon$ with vertex set Λ where two vertices y_i, y_j are connected if the CRT map cells $\eta([y_{i-1}, y_i])$ and $\eta([y_{j-1}, y_j])$ share a nontrivial boundary arc.

Fact: $\bar{\mathcal{G}}_\varepsilon$ and $\mathcal{G}_\varepsilon$ have the same law (Duplantier-Miller-Sheffield '14).

Curvature against smooth functions

Theorem (CH-Gwynne)

Let $\mathcal{G}_\varepsilon$ be the ε Poisson mated CRT map with vertex set $\mathcal{V}\mathcal{G}_\varepsilon$. For any smooth compactly supported $f \in C_c^\infty(\mathbb{C})$ on $\mathbb{C}$ we have with probability going to 1 as $\varepsilon \to 0$, that

$$\sum_{v \in \mathcal{V}\mathcal{G}_\varepsilon} f(v) K_{\mathcal{G}_\varepsilon}(v) = \varepsilon^{o(1)}.$$

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Note: Since the sum is over ε^{-1} vertices, intuitively one could expect the sum to be of order $\varepsilon^{-\frac{1}{2}}$. The theorem above tells us there is much more cancellation.

Total curvature on CRT cells

From now on, we use the imaginary geometry field Ψ and SLEκ', sampled independently from the γ-quantum cone field Φ.

We define the total curvature on a CRT map cell C as

$$K^{C}_{\mathcal{G}_\varepsilon} := \sum_{v \in \mathcal{V}\mathcal{G}_\varepsilon \cap \mathcal{C}} K_{\mathcal{G}_\varepsilon}(v).$$

where $\mathcal{C}$ is the set of vertices in $\mathcal{G}_\varepsilon$ contained in the CRT map cell C.

Total curvature on CRT cells

Theorem (CH-Gwynne)

We have that

$$\frac{K_{\mathcal{G}_\varepsilon}(C)}{\varepsilon^{-\frac{1}{4}}} \to \mathcal{B},$$

where the law of $\mathcal{B}$ can be described as follows. Sample $\mathcal{L}$ according to the law of the boundary length of the mated CRT cell C. Now let Θ be sampled according to a Gaussian distribution with mean 0 and variance $\mathcal{L}$. Then $\mathcal{B}$ and Θ have the same law.

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Remark: The previous theorem tells us the right scaling for the curvature against a smooth function is $\varepsilon^{o(1)}$, while this theorem tells us that the scaling for the total curvature on a CRT map cell is $\varepsilon^{-1/4}$. This suggests it is impossible to define both Gaussian curvature and geodesic curvature simultaneously.

Outline of proofs: curvature against a test function

First ingredient: Convenient cancellation when computing

$$\sum_{v \in \mathcal{V}\mathcal{G}_\varepsilon} f(v) K_\Phi(v).$$

Notation: If $\overrightarrow{e}$ has starting vertex v_1 and end vertex v_2, let $\mathcal{D}f(e) = f(v_2) - f(v_1)$.

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Proposition

There is an orientation on edges of $\mathcal{G}_\varepsilon$ such that for any smooth compactly supported test function f,

$$\sum_{v \in \mathcal{V}\mathcal{G}_\varepsilon} f(v) K_{\mathcal{G}_\varepsilon}(v) = \sum_{e \in \mathcal{E}\mathcal{G}_\varepsilon} \mathcal{D}f(\overrightarrow{e}).$$

Cancellations

First we split our sum $\sum_{v\in\mathcal{V}\mathcal{G}_\varepsilon} f(v)K_{\mathcal{G}_\varepsilon}(v)$ into the edges corresponding to when (1) holds for L and those for R.

Second ingredient: There is an injection $v \mapsto e_v$ from $\mathcal{V}\mathcal{G}_\varepsilon$ to the set of edges $e \in \mathcal{E}\mathcal{G}_\varepsilon$ defined by taking the "rightmost past" edge. Hence

$$\sum_{e\in\mathcal{E}\mathcal{G}_\varepsilon} \mathcal{D}f(\overrightarrow{e}) = \sum_{v\in\mathcal{V}\mathcal{G}_\varepsilon} \mathcal{D}f(\overrightarrow{e}_v).$$

Cancellations

Now we split e_v at the first point it exists the CRT map cell containing v, called m_v.

Grouping all pieces adjacent to v, we obtain

$$\sum_{v \in \mathcal{V}\mathcal{G}_\varepsilon} f(v) K_{\mathcal{G}_\varepsilon}(v) = \sum_{v \in \mathcal{V}\mathcal{G}_\varepsilon} G_f(v).$$

Goal: Control $\mathbb{E}\left(\left(\sum_{v \in \mathcal{V}\mathcal{G}_\varepsilon} G_f(v)\right)^2\right)$.

Rewriting as an integral

Next: Rewrite the sum above as an integral:

$$\sum_{v \in \mathcal{V}\mathcal{G}_\varepsilon} G_f(v) = \int_{\mathbb{C}} \frac{G_f(x_z)}{\mathrm{Area}(H_z^\varepsilon)} dz$$

where H_z^ε is the CRT map cell containing z, and x_z is the vertex in $\mathcal{V}\mathcal{G}_\varepsilon$ contained in H_z^ε.

Notation: Let $\hat{H}_z^\varepsilon$ be the union of all CRT map cells adjacent to H_z^ε together with H_z^ε itself.

Fixing CRT map cell size

Splitting into events: Let E_j be the event

$$E_j := \{z \in \mathbb{C} : \mathrm{diam}(\widehat{H}_z^\varepsilon) \in [\varepsilon^{\alpha_{j+1}}, \varepsilon^{\alpha_j}],\ \mathrm{area}(H_z^\varepsilon) \geq \varepsilon^\beta\}$$

where the α_j's are a partition of a sufficiently large interval, with $|\alpha_{j+1} - \alpha_j|$ small.

Now we focus on bounding

$$\mathbb{E}\left(\left(\int_{\mathbb{C}} 1_{E_j} \frac{G_f(x_z)}{\mathrm{Area}(H_z^\varepsilon)}\right)^2\right).$$

Splitting

We split our expression as

$$\mathbb{E}\left(\left(\int_{\mathbb{C}} 1_{E_j} \frac{G_f(x_z)}{\mathrm{Area}(H_z^\varepsilon)}\right)^2\right) = \mathbb{E}\left(\int\int_{|z-w|\leq \varepsilon^{\alpha_j - \zeta}} X_z^\varepsilon X_w^\varepsilon dzdw\right) + \mathbb{E}\left(\int\int_{|z-w|\geq \varepsilon^{\alpha_j - \zeta}} X_z^\varepsilon X_w^\varepsilon dzdw\right)$$

where $X_z^\varepsilon = \frac{G_f(x_z)}{\mathrm{Area}(H_z^\varepsilon)}$. Essentially, the first term is small since we are integrating on a small measure set, while the second is small because of the long range properties of G_f and H_z^ε.

Outline of proofs: Total curvature on a CRT map cell

First ingredient: Combinatorial graph identity relating edges, vertices, and perimeter of a triangulation. With it, one obtains

$$K_{\mathcal{G}_\varepsilon}^{\mathcal{C}} = \mathrm{Perim}(\mathcal{C}) - 6 - \sum_{v \in \mathcal{C}} \deg_{ext}^{\mathcal{C}}(v)$$

This can be rewritten as

$$K_{\mathcal{G}_\varepsilon}^{\mathcal{C}} = \sum_{v \in \partial_{\mathcal{G}_\varepsilon} \mathcal{C}} K_g^{\mathcal{C}}(v) - 6$$

where $K_{\mathcal{G}_\varepsilon}$ is the "discrete geodesic curvature".

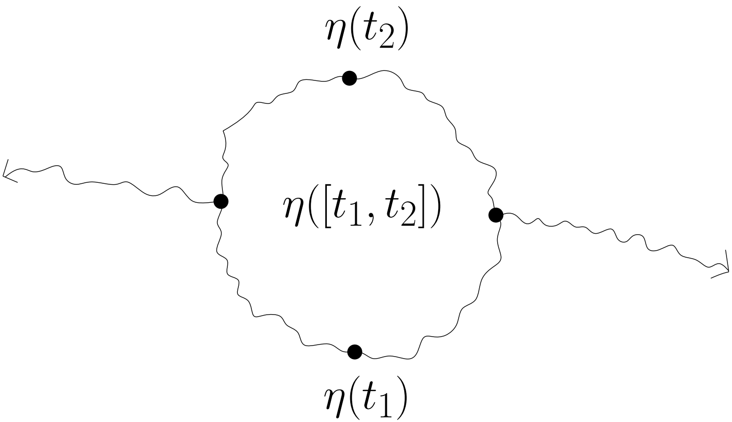
Next: We split this sum into four parts, corresponding to the past-left, past-right, future-left, future-right components of the boundary of $\mathcal{C}$.

We have

$$\begin{aligned} K^C_{\mathcal{G}_\varepsilon} &= \sum_{v\in \bar{C}_{PR}} K^C_g(v) + \sum_{v\in \bar{C}_{PL}} K^C_g(v) \\ &+ \sum_{v\in \bar{C}_{FR}} K^C_g(v) + \sum_{v\in \bar{C}_{PL}} K^C_g(v). \end{aligned}$$

Each sum is treated the same way.

Second ingredient: Central limit theorem together with an independence property for $K^C_g(v)$ along the boundary.


$\eta(t_2)$
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Open problems

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- Is the scaling limit of $K_{\mathcal{G}_\varepsilon}$ equal to K_Φ? What is the scaling factor?
- Is K_Φ a universal limit?
- Is there an observable on CRT maps converging to the underlying field?

Thank you!