Domino tilings in 3D
KPZ meets KPZ, Fields Institute

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Based on joint work with Nishant Chandgotia and Scott Sheffield
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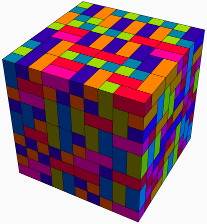
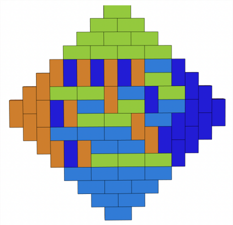
What is the dimer model and what is this talk about?

A dimer tiling is a *perfect matching* of a graph, namely a collection of edges such that every vertex is contained in exactly one edge. In $\mathbb{Z}^2$ or $\mathbb{Z}^3$ these can be drawn as *domino tilings* like this:





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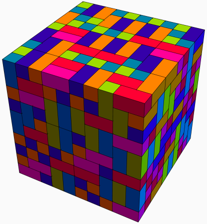

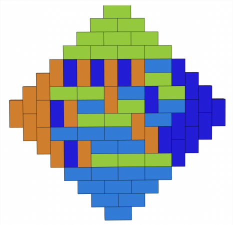

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This talk: In the hope that this more relevant to this conference, I am going to focus more on our methodology than our result, and try to explain some of the tools and ideas we use instead of exact solvability.

Why does the 3D model seem not exactly solvable?

Example 1: Kasteleyn determinant formula fails in 3D

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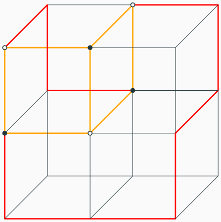
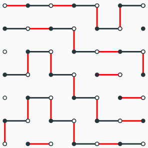
$\mathbb{Z}^3$ contains $K_{3,3}$ given only four lattice cubes.



Example 2: non-intersecting paths and (non)-solvability?

Another way to compute the partition function for dimers in 2D is via the bijection with *non-intersecting paths* in $\mathbb{Z}^2$ by overlaying a tiling (red) with a brickwork tiling (black).



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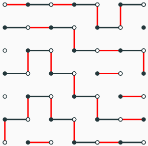
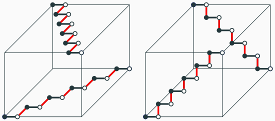


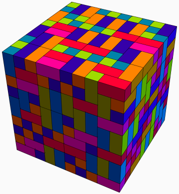
There is an analogous bijection between dimer tilings of $\mathbb{Z}^3$ and non-intersecting paths in $\mathbb{Z}^3$. But these paths are not ordered, they can be braided in various ways, etc.

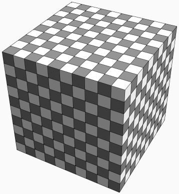


Main question: scaling limits of random tilings?

Dimers and vector fields

For any d, $\mathbb{Z}^d$ is a bipartite lattice, with fixed underlying black and white checkerboard.





The colors of the dimers represent the cardinal direction of the dimer (north, south, east, west, up, down for $d = 3$), viewed as a vector from white to black.

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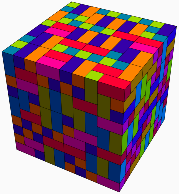

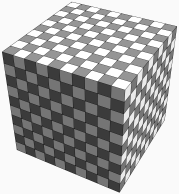

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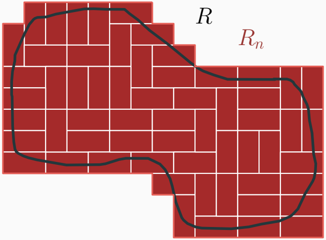
There is a correspondence between 1) a dimer tiling τ of $\mathbb{Z}^d$ and 2) a *discrete vector field* v_τ defined by: for each edge e of $\mathbb{Z}^d$ oriented from white to black,

$$v_\tau(e) = \begin{cases} 1 & e \in \tau \\ 0 & e \notin \tau \end{cases}$$

Scaling limits of random tilings

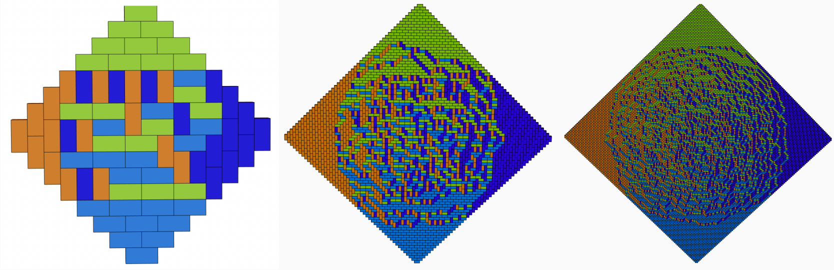
As $n \to \infty$, the scaling limits of flows v_τ corresponding to tilings of $\frac{1}{n}\mathbb{Z}^d$ are divergence-free measurable vector fields on $\mathbb{R}^d$ that have L^1 norm less than 1.



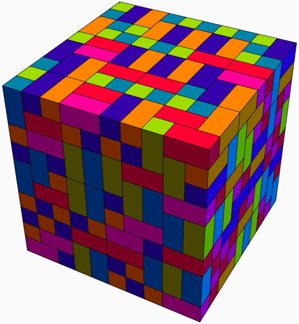
Question: Fix a region and boundary condition (R, b) in $\mathbb{R}^d$. Let $R_n \subset \frac{1}{n}\mathbb{Z}^d$ be a sequence of lattice regions approximating (R, b). What does the flow corresponding to a uniformly random dimer tiling of R_n look like as $n \to \infty$?

Versions of this question: is there a law of large numbers (yes!)? Large deviation principle? What is the expected limiting flow? Fluctuations?

2D example: Aztec diamond

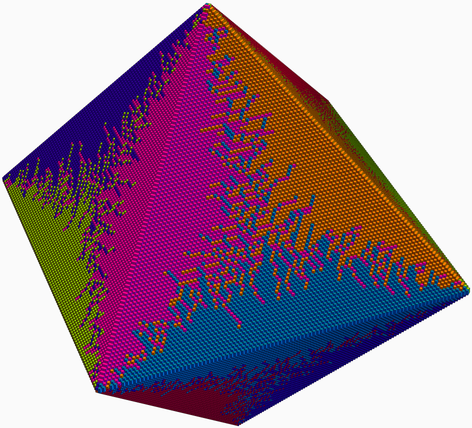


A note: boundary conditions have a big effect



In 2D or 3D (so the square $[1, n]^2$ or the cube $[1, n]^3$), the limit shape is just the zero flow. In the tiling picture, this means you should see approximately equal proportions of all the tile colors. (This is the only limit shape in 3D that know explicitly right now!)

3D example: Aztec octahedron



Chain swapping