Upper tail scaling limit of continuum path measures in KPZ

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(based on joint works with Shirshendu Ganguly and Lingfu Zhang)

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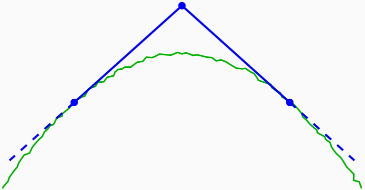
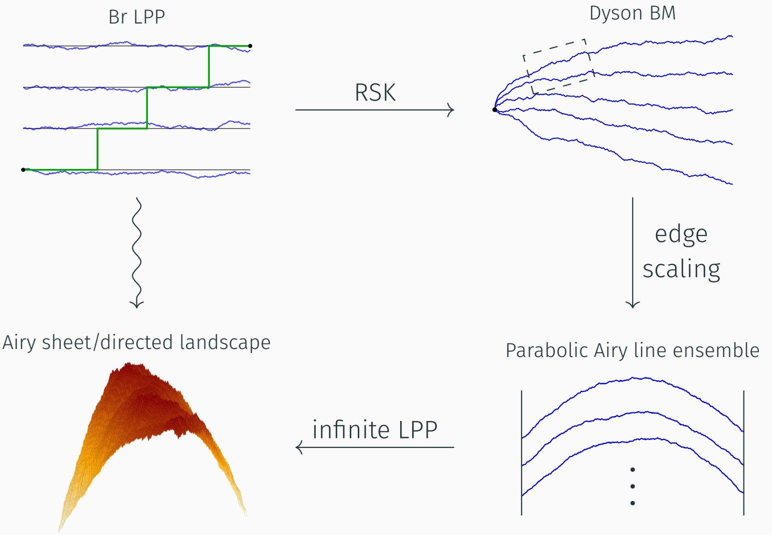
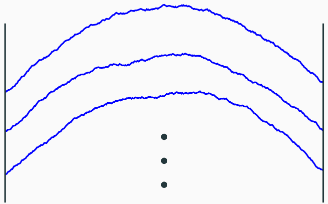
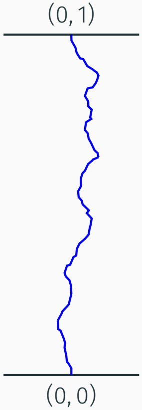



Br LPP
RSK
Dyson BM
edge
scaling
Airy sheet/directed landscape
infinite LPP
Parabolic Airy line ensemble

How are the geometry of the landscape and Airy line ensemble related?

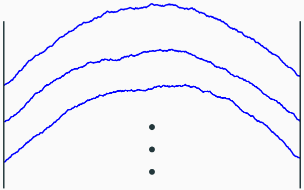




We will focus on the geometry of both when we condition the top curve $\mathcal{P}_1$ to be large at various points.

The Brownian Gibbs property

Why embed $\mathcal{P}_1(\cdot) = \mathcal{L}(0,0;\cdot,1)$ as the top/lowest-indexed curve in $\mathcal{P}$?



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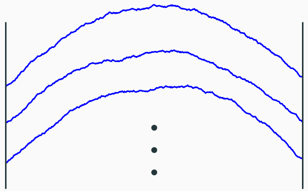

$\mathcal{P}$ has a resampling property, the Brownian Gibbs property.

It says the conditional distribution of $\mathcal{P}_1$ on an interval is a non-intersecting Brownian bridge (of rate 2).

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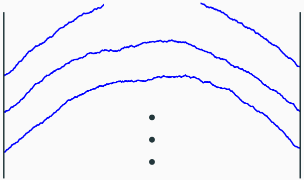

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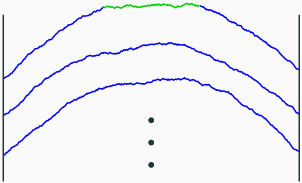

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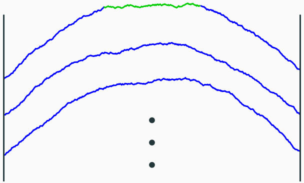


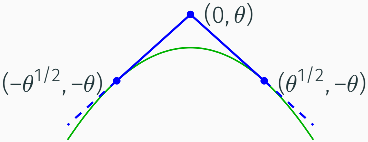
A useful heuristic to keep in mind:

$\mathcal{P}_1$ is like a Brownian bridge conditioned to stay above a parabola $-x^2$ with which it shares endpoints.

The geometry of the line ensemble

Main results: geometry under one-point conditioning

Define Triangle$_\theta : [-\theta^{1/2}, \theta^{1/2}]$ to be



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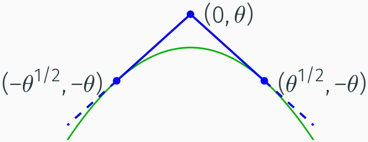

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Theorem (Ganguly-H.)

There exist θ_0 and $c > 0$ such that, for all $t \geq 1$, $\theta > \theta_0$, and $M > 0$,

$$\mathbb{P}\left(\sup_{x\in[-\theta^{1/2},\theta^{1/2}]} |\mathcal{P}_1(x) - \text{Triangle}_\theta(x)| > M\theta^{1/4} \,\middle|\, \mathcal{P}_1(0) = \theta \right) \leq \exp(-cM^2).$$

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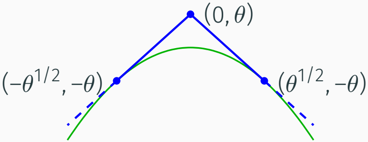

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- $\theta^{1/4}$ is the Brownian fluctuation scale on an interval of size $\theta^{1/2}$ and is optimal.

A quantitative consequence: one-point upper tail asymptotics

From the limit shape, one can obtain sharp asymptotics for the upper tail:

Theorem (Ganguly-H.)

There exist $C < \infty$ and θ_0 such that, for all $\theta > \theta_0$,

$$\exp\left(-\frac{4}{3}\theta^{3/2} - C\theta^{3/4}\right) \leq \frac{1}{\mathrm{d}\theta}\mathbb{P}\left(\mathcal{P}_1(0) \in \mathrm{d}\theta\right) \leq \exp\left(-\frac{4}{3}\theta^{3/2} + C\theta^{3/4}\right).$$

As an immediate consequence, the same bounds also hold for $\mathbb{P}(\mathcal{P}_1(0) > \theta)$.

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As an immediate consequence, the same bounds also hold for $\mathbb{P}(\mathcal{P}_1(0) > \theta)$.

By more refined coupling arguments, we also get a comparison statement:

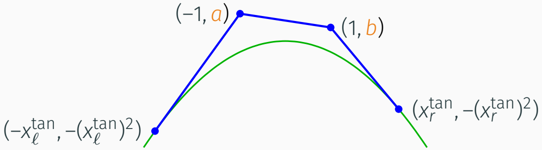
Theorem (Ganguly-H.-Zhang)

There exist $C < \infty$ and θ_0 such that, for all $\delta > 0$ and $\theta > \theta_0$,

$$\frac{\mathbb{P}\left(\mathcal{P}_1(0) \ge \theta + \delta\right)}{\mathbb{P}\left(\mathcal{P}_1(0) \ge \theta\right)} = \exp\left(-2\delta\theta^{1/2} + O(\delta L^{-1/4})\right).$$

Main results: geometry under two-point conditioning

Define Quad$_{a,b}$: $[-x_\ell^{\tan}, x_r^{\tan}]$ to be



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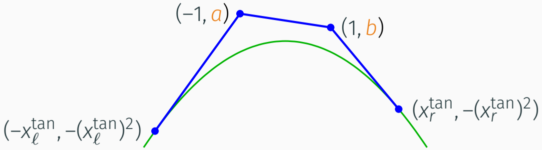

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Theorem (Ganguly-H.)

Assuming some non-degeneracy conditions on a and b, there exists $c > 0$ such that, for all $t \geq 1$, $M > 0$, and large enough a, b,

$$\mathbb{P}\left(\sup_{x\in[-x_\ell^{\tan}, x_r^{\tan}]} \mathcal{P}_1(x) - \mathrm{Quad}_{a,b}(x) > M(a^{1/4} + b^{1/4}) \;\middle|\; \mathcal{P}_1(-1) = a, \mathcal{P}_1(1) = b\right) \leq \exp(-cM^2).$$

A quantitative consequence: Two-point upper tail asymptotics

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For $t \geq 1$ and if a, b are large enough and satisfy the non-degeneracy condition, then

$$\mathbb{P}\left(\mathcal{P}_1(-1) \geq a, \mathcal{P}_1(1) \geq b\right)$$
$$= \exp\left(-\frac{1}{24}\left[16\left((1+a)^{3/2} + (1+b)^{3/2}\right) + 3(a-b)^2 + 24(a+b) + 32\right] + \text{error}\right).$$

The error term has explicit upper and lower bounds.

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The error term has explicit upper and lower bounds.

- A similar bound also holds at $\pm K$ in place of ± 1, or in general any two points (using stationarity).

A multi-point question: Sharpness of FKG

Because of connections to statistical mechanics models, it is known that $\mathcal{P}_1$ enjoys the FKG inequality, so that, for all a and b,

$$\mathbb{P}\left(\mathcal{P}_1(-K^{1/2}) > a, \mathcal{P}_1(K^{1/2}) > b\right) \geq \mathbb{P}\left(\mathcal{P}_1(-K^{1/2}) > a\right) \cdot \mathbb{P}\left(\mathcal{P}_1(K^{1/2}) > b\right).$$

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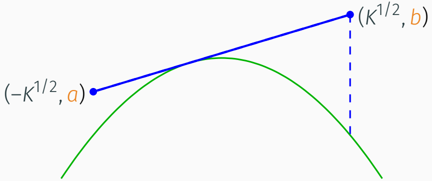
But in many applications the inequality is suboptimal.

Is FKG sharp for any values of a and b, and, if so, which ones?

Corollary: Geometric condition for sharpness of FKG



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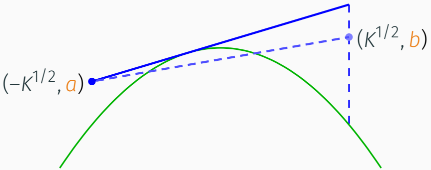

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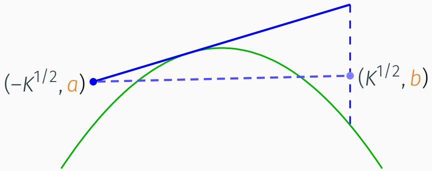

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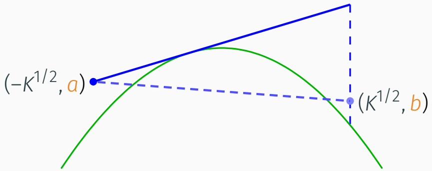

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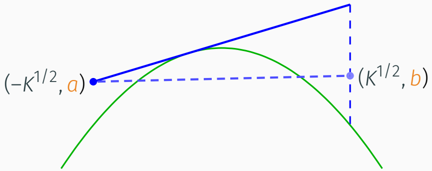
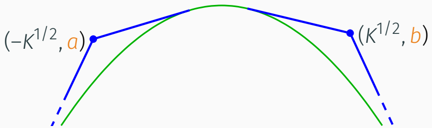


Corollary (Ganguly-H.)

Let K be fixed. If the line joining $(-K^{1/2}, a)$ and $(K^{1/2}, b)$ is tangent to or intersects $-x^2$ inside $[-K^{1/2}, K^{1/2}]$, then

$$\begin{aligned}\mathbb{P}\left(\mathcal{P}_1(-K^{1/2}) > a, \mathcal{P}_1(K^{1/2}) > b\right) &= \exp\left(-\frac{4}{3}[(K+a)^{3/2} + (K+b)^{3/2}] + \text{error}\right)\\ &\approx \mathbb{P}(\mathcal{P}_1(-K^{1/2}) > a) \cdot \mathbb{P}(\mathcal{P}_1(K^{1/2}) > b).\end{aligned}$$



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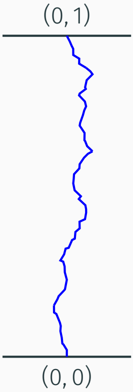
Geometry of the landscape and geodesic

How does the landscape change under the conditioning $\mathcal{P}_1(0) = \mathcal{L}(0,0;0,1) > \theta$?



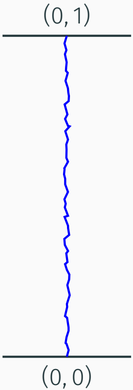
Qualitative features under the upper tail conditioning

How does the landscape change under the conditioning $\mathcal{P}_1(0) = \mathcal{L}(0,0;0,1) > \theta$?

An energy-entropy tradeoff occurs: larger fluctuations give the geodesic more choice of paths, but the cost grows with θ.

So the path measure will become more rigid, i.e., have much smaller transversal fluctuations. (It also becomes a "highway" for geodesics to nearby points.)

Heuristically, a uniformly (on some scale) random path is chosen and made to be the geodesic.



The scaling limit of the geodesic under upper tail conditioning

Let $\Gamma_\theta : [0,1] \to \mathbb{R}$ be the geodesic in the directed landscape from $(0,0)$ to $(0,1)$, conditioned on $\mathcal{L}(0,0;0,1) > \theta$.

Theorem (Ganguly-H.-Zhang)

$\theta^{1/4}\Gamma_\theta \xrightarrow{d} \frac{1}{2}B$ in the uniform topology with B = standard Brownian bridge.

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This result had been conjectured by Zhipeng Liu, who proved the one-point scale and one-point convergence using exact formulas.

A similar result had earlier been conjectured by Basu-Ganguly for the geodesic in exponential LPP under a large deviation conditioning.

Heuristics and proof ideas

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- So the geodesic fluctuating by ε means it suffers a loss of $O(\varepsilon^2)$.
- Under the conditioning of being $> \theta$, this loss has to be made up; akin to $\mathcal{P}_1(0) > \theta + O(\varepsilon^2)$ (by stationarity).
- But $\dfrac{\mathbb{P}(\mathcal{P}_1(0) > \theta + O(\varepsilon^2))}{\mathbb{P}(\mathcal{P}_1(0) > \theta)} \approx \exp(-C\varepsilon^2\theta^{1/2})$.
- This is $O(1)$ exactly when $\varepsilon = O(\theta^{-1/4})$.

The source of the Brownian bridge

Recall that

$$\Gamma_\theta(s) = \operatorname*{argmax}_y \big(\mathcal{L}(0,0;y,s) + \mathcal{L}(y,s;0,1)\big)$$

under the conditioning $\mathcal{L}(0,0;0,1) = \max_y(\mathcal{L}(0,0;y,s) + \mathcal{L}(y,s;0,1)) > \theta$.

The tent picture suggests that $\mathcal{L}(0,0;y,s)$ and $\mathcal{L}(y,s;0,1)$ are like independent Brownian bridges on $[-\theta^{1/2}, \theta^{1/2}]$.

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The variance at $y = L^{-1/2}(L^{1/2}+y)(L^{1/2}-y) = L^{1/2} - y^2L^{-1/2}$, so density at y is proportional to

$$\exp\left(-c\frac{L^2}{L^{1/2} - y^2L^{-1/2}}\right) = \exp\left(-cL^{3/2} - cy^2L^{1/2} + O(L^{-1/2})\right).$$

This is Gaussian on scale $L^{-1/4}$!

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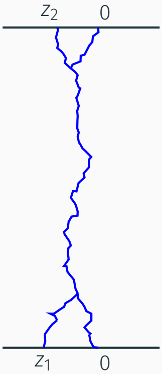

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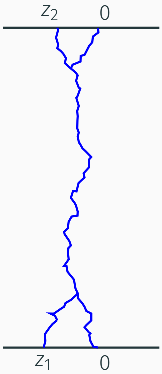
- $(\Gamma(s), \Gamma(t)) = \underset{z_1, z_2}{\operatorname{argmax}}\, \mathcal{L}(0,0; z_1, s) + \mathcal{L}(z_1, s; z_2, t) + \mathcal{L}(z_2, t; 0, 1)$



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- Coalescence gives quadrangle equality:
  $\mathcal{L}(z_1,s;z_2,t) + \mathcal{L}(0,s;0,t) = \mathcal{L}(z_1,s;0,t) + \mathcal{L}(0,s;z_2,t)$
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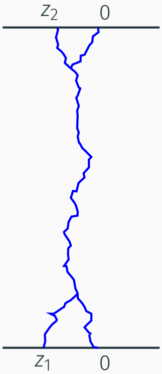

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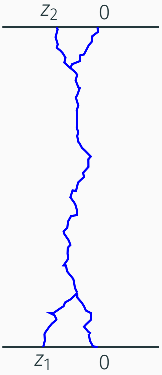
- The double argmax separates into two single argmaxes.
- Heuristically, coalescence also implies the two process on the RHS are (approximately) independent.
- The proof of independence relies crucially on shift invariance of $\mathcal{L}$ or free energy fields.



Let $\Gamma_\theta^{\text{ann}} : [0,1] \to \mathbb{R}$ be a sample from the annealed polymer measure from $(0,0)$ to $(0,1)$ in the CDRP, under the conditioning that the free energy $> \theta$.

Theorem (Ganguly-H.-Zhang)

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The scaling limit of the CDRP polymer measure under upper tail conditioning

Let $\Gamma_\theta^{\mathrm{ann}} : [0,1] \to \mathbb{R}$ be a sample from the annealed polymer measure from $(0,0)$ to $(0,1)$ in the CDRP, under the conditioning that the free energy $> \theta$.

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What about the quenched situation? The polymer measure concentrates in a $O(\theta^{-1/2})$ window around a random "backbone" $\Gamma_\theta^{\mathrm{back}}$, and $\theta^{-1/4}\Gamma_\theta^{\mathrm{back}} \xrightarrow{d} \frac{1}{2}B$.

Summary

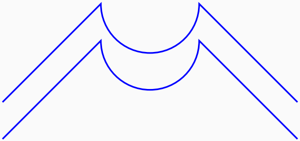
- Using geometric methods + Brownian Gibbs properties, we can obtain the shape of the weight profile of $\mathcal{L}$ under upper tail events.
- These also give sharp upper tail asymptotics and probability comparison statements.
- With these + "tent" picture, can prove that geodesic/polymer measure rescaled by $\theta^{-1/4}$ converges to a Brownian bridge, under upper tail.
- Further, the polymer measure fluctuates on scale $\theta^{-1/2}$ around a random "backbone" curve.

Thank you!

A convex consequence

A key idea is that the resampling in terms of Brownian bridges implies that the limit shapes should be convex.

Indeed, suppose the limit shape of the top curve is not convex in some neighbourhood. This pushes the second curve down on the interval.



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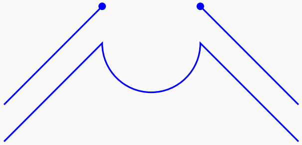
Then resample the top curve on that interval. *Since the non-convexity means the second curve is far away, Brownian bridge naturally avoids it.*



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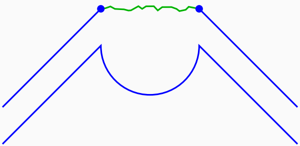
Unconditioned Brownian bridge approximately follows a straight line, so can't recreate the earlier non-convexity. A contradiction!



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