The Markov property of the Gaussian free field

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Universidad de Chile /CMM

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Contents

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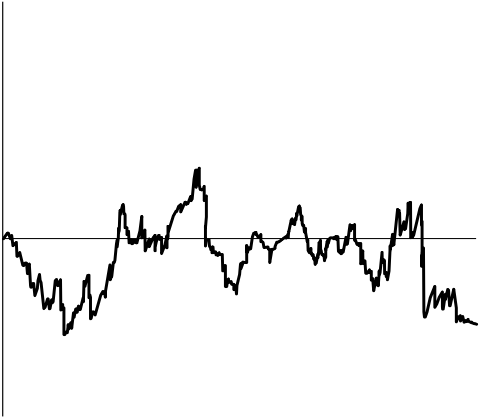
The Brownian motion



$$\tau_{-a,b} := \inf\{t \geqslant 0 : -a \leqslant B_t \leqslant b\},$$
$$\tau_{-a} := \inf\{t \geqslant 0 : B_t \geqslant -a\}.$$

Plan

The Gaussian free field

Take $D \subseteq \mathbb{C}$, the (continuum) Gaussian free field (GFF) is a centred Gaussian process with covariance given by

$$\mathbb{E}\left[\Phi(x)\Phi(y)\right] = G_D(x,y) \overset{x \to y}{\sim} -\log(\|x-y\|).$$

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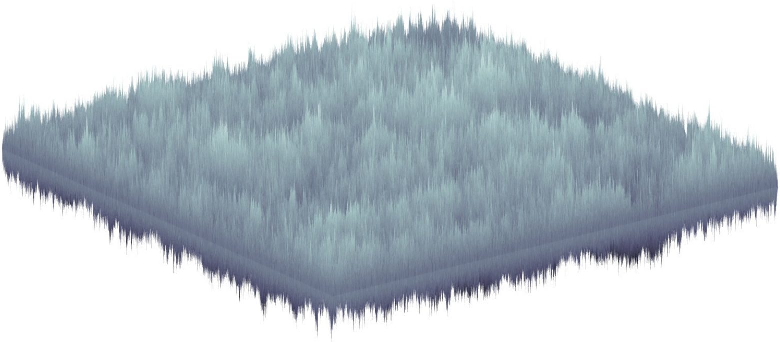
The Gaussian free field, is defined as a random "generalised function" such that $(\Phi, f)_{f \text{ smooth}}$ is a centred Gaussian process with

$$\mathbb{E}\left[(\Phi, f)(\Phi, g)\right] = \iint_{D\times D} f(x) G_D(x,y) g(y) dx dy.$$

Approximation of a continuum Gaussian free field



Weak Markov property

Let A be a closed set of $D \subseteq \mathbb{C}$. Then there exist two independent "generalized functions" Φ_A and Φ^A such that

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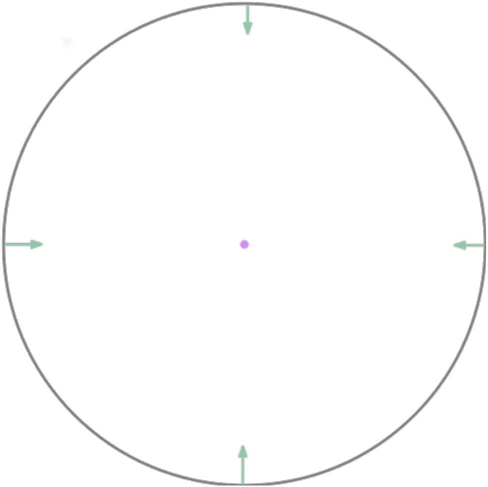
3. Φ^A is a GFF in $D \backslash A$.

Example of a stopping set



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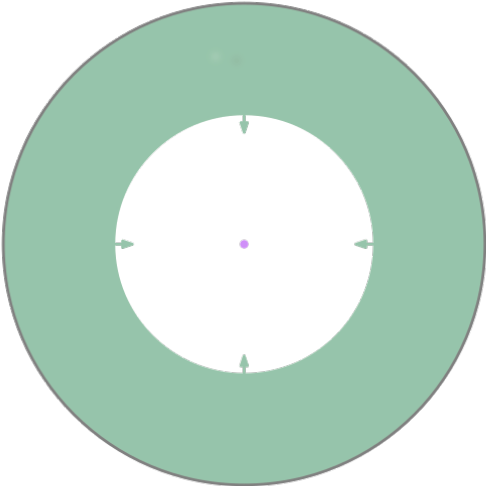

Strong Markov property

Let A be a stopping set of Φ. Then, conditionally on A, there exist two conditionally independent "generalised functions" Φ_A and Φ^A such that

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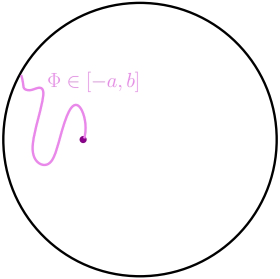
Two-valued sets



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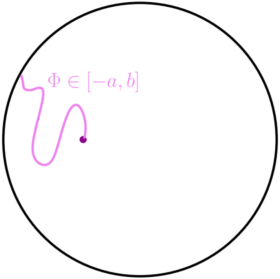
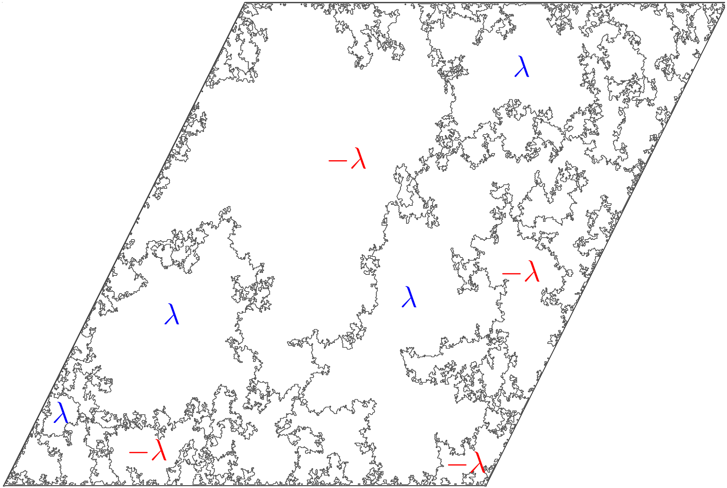


Theorem (Aru-S.-Werner '17)

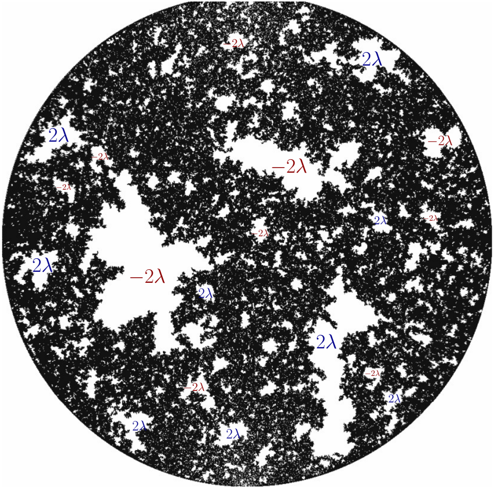
Take $a, b \geqslant 0$, such that $a + b \geqslant 2\lambda := \pi$. There exists a unique stopping set $\mathbb{A}_{-a,b}$ such that $\Phi_{\mathbb{A}_{-a,b}}$ is a harmonic function constant in each connected component taking values in $\{-a, b\}$.

$\mathbb{A}_{-\lambda,\lambda}$



Simulation by B. Werness.

$\mathbb{A}_{-2\lambda,2\lambda}$ =CLE$_4$ (Miller-Sheffield)



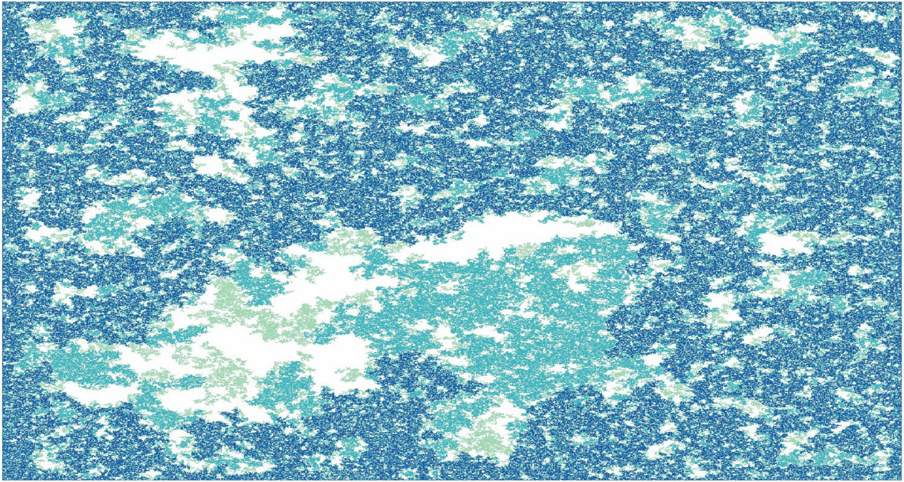
Simulation by D. Wilson.

Monotonicity

Proposition (Aru-S.-Werner '17)

Let $a, b \geqslant 0$ *with* $a + b \geqslant 2\lambda$ *and* $a', b' \geqslant 0$ *such that* $[-a, b] \subseteq [-a', b']$ *then* $\mathbb{A}_{-a,b} \subseteq \mathbb{A}_{-a',b'}$.

Simulation for monotonicity



Non-existence

Proposition (Aru-S.-Werner '17)

Let $a, b \geqslant 0$ with $a + b < 2\lambda$. There is no stopping set $\mathbb{A}_{-a,b}$ with the property that $\Phi_{\mathbb{A}_{-a,b}}$ is a harmonic function with values in $\{-a, b\}$.

Explicit laws

Theorem (Aru-S.-Werner'17)

The law of $-\log(CR(0, \mathbb{D} \backslash \mathbb{A}_{-a,b}))$ *is equal to the law of the first time a Brownian motion exits* $[-a, b]$.

Multifractal spectrum

Theorem (Schoug-S.-Viklund '19)

A.s. for any $a, b > 0$ *with* $a, b \geqslant 2\lambda$ *the Haussdorf dimension of* $\mathbb{A}_{-a,b}$ *is equal to*

$$2 - \frac{2\lambda^2}{(a+b)^2}.$$

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$$2 - \frac{2\lambda^2}{(a+b)^2}.$$

This is related to the imaginary chaos

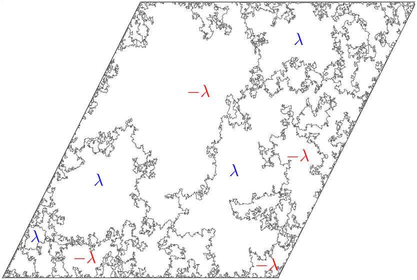
$$: \cos(i\alpha\Phi) :|_{\mathbb{A}_{-a,a}} = \mu_{\mathbb{A}_{-a,a}}$$

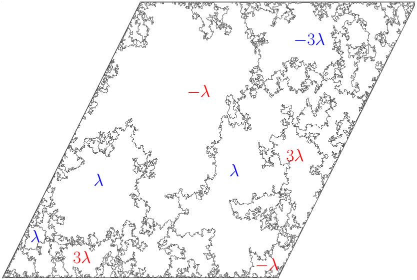
when $\alpha a = \pi/2$.

Coupling between 0-boundary and free-boundary GFF

Theorem (Qian-Werner '18)

One can couple a free-boundary GFF and a 0-boundary GFF using $\mathbb{A}_{-\lambda,\lambda}$.





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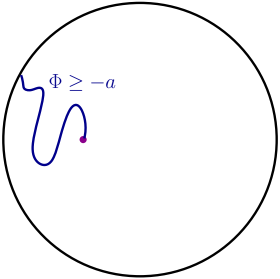
Definition



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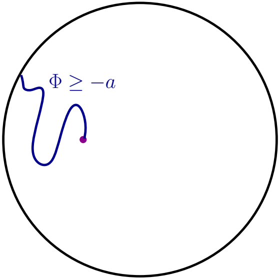
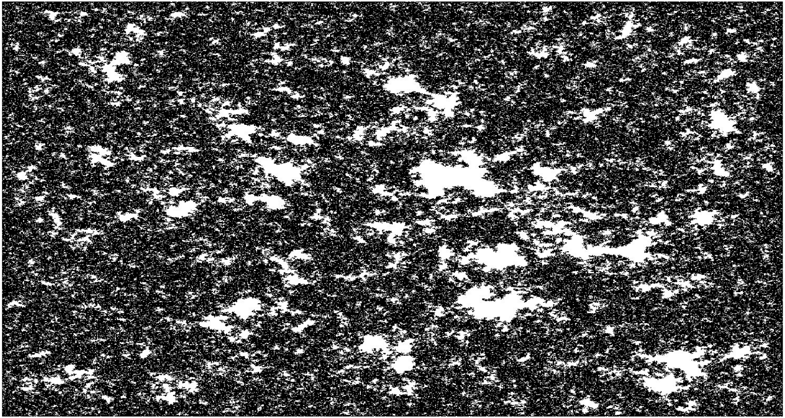


Theorem (Aru-Lupu-S.'19)

Take $a \geqslant 0$. There exists a unique stopping set $\mathbb{A}_{-a}$ such that

1. *Restricted to $D \backslash \mathbb{A}_{-a}$, $\Phi_{\mathbb{A}_{-a}}$ is strictly equal to $-a$.*
2. *$\Phi_{\mathbb{A}_{-a}} + a$ is a positive measure supported in $\mathbb{A}_{-a}$.*

Simulation: $\mathbb{A}_{-2\lambda}$



Monotonicity and relationship with TVS

Proposition (Aru-Lupu-S. '19)

Take $0 \leqslant a \leqslant a'$, *then* $\mathbb{A}_{-a} \subseteq \mathbb{A}_{-a'}$.

Monotonicity and relationship with TVS

Proposition (Aru-Lupu-S. '19)

Take $0 \leqslant a \leqslant a'$*, then* $\mathbb{A}_{-a} \subseteq \mathbb{A}_{-a'}$.

Theorem (Aru-Lupu-S. '19)

Take $a, b \geqslant 0$ *such that* $a + b \geqslant 2\lambda$*, then*

$$\mathbb{A}_{-a,b}(\Phi) = \mathbb{A}_{-a}(\Phi) \cap \mathbb{A}_{-b}(-\Phi)$$

Dimension

Theorem (Aru-Lupu-S. '18)

$\mathbb{A}_{-a}$ *has fractal dimension equal to 2. Furthermore, the non-trivial measure* $\Phi_{\mathbb{A}_{-a}} + a$ *correspond to a Minkowski content measure of the gauge* $r \mapsto r^2 |\log r|^{1/2}$.

Dimension

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$\mathbb{A}_{-a}$ has fractal dimension equal to 2. Furthermore, the non-trivial measure $\Phi_{\mathbb{A}_{-a}} + a$ correspond to a Minkowski content measure of the gauge $r \mapsto r^2 |\log r|^{1/2}$.

This is strongly related with the real exponential of the GFF : $e^{\gamma\phi}$:

End

Gracias!