$\boxed{\text{Stability and chaos in last passage percolation}}$

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— joint work with

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In complex discrete systems in statistical mechanics, a probability measure μ is often written

$$\mu(\{x\}) = e^{-H(x)}, \quad x \in X$$

The Hamiltonian $H : X \to [0,\infty)$ may be viewed as an energy specified over the landscape X.

Landscape geometry

the ground state

$x \in X$ minimizes H

Three natural questions about an energy landscape ....

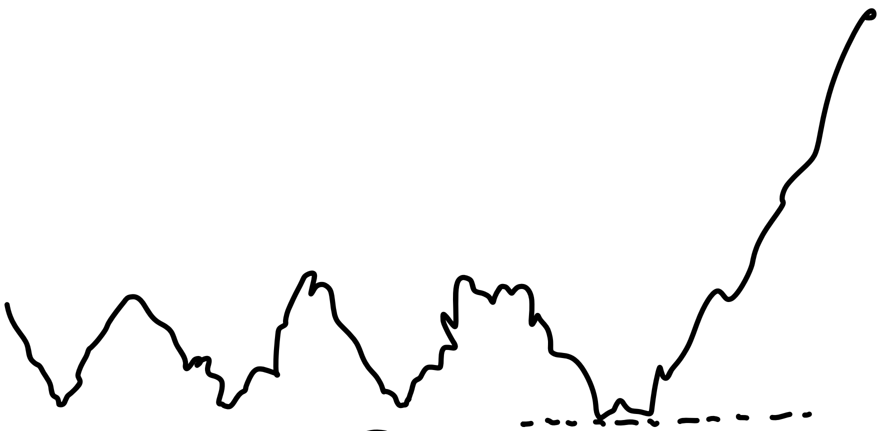
<u>Q1.</u> Is the ground state energy super-concentrated, with sublinear growth in variance, slower than

$$\mathrm{Var}\left(\sum_{i=1}^{N} \mathbb{1}_{\mathrm{HEADS}_i}\right) \asymp N$$

CLASSICAL CLT ?

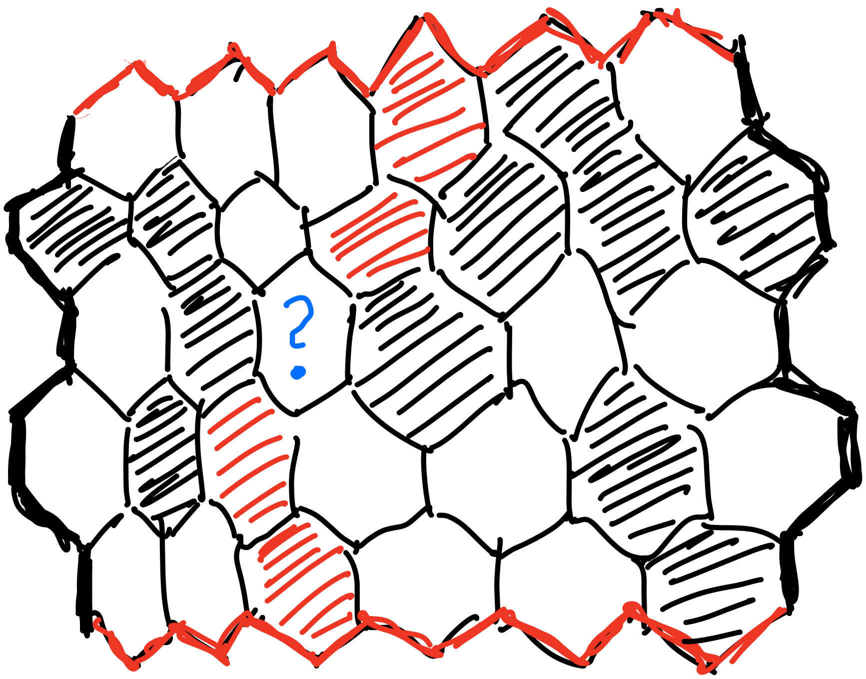
Q2. Are there

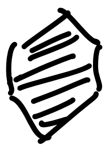
$\boxed{\text{MULTIPLE VALLEYS}}$?

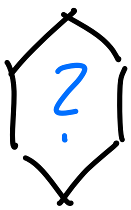
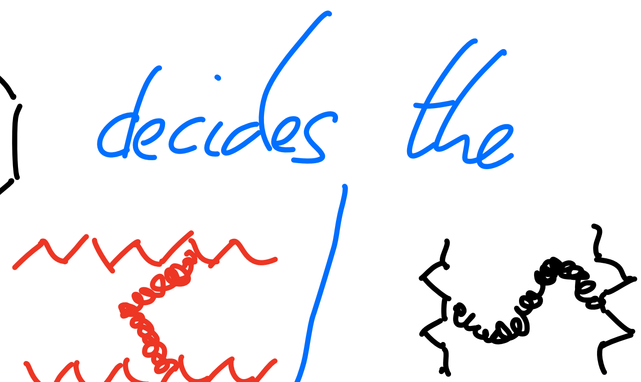


"several valleys that rival the ground state"

Q3. Does chaos reign?



HEADS  OR  TAILS

Flipping  decides the outcome 

Q3 Does chaos reign?

Is the ground state sensitive to small perturbations in the noise that specifies the Hamiltonian H.

Sourav Chatterjee's 2014 monograph

For a class of models of GAUSSIAN disorder,

Super-concentration

Chaos

Multiple valleys

GOAL: understand the order of NOISE perturbation that heralds

the onset of chaos

in a dynamical form of last passage percolation.

The three aspects at play in

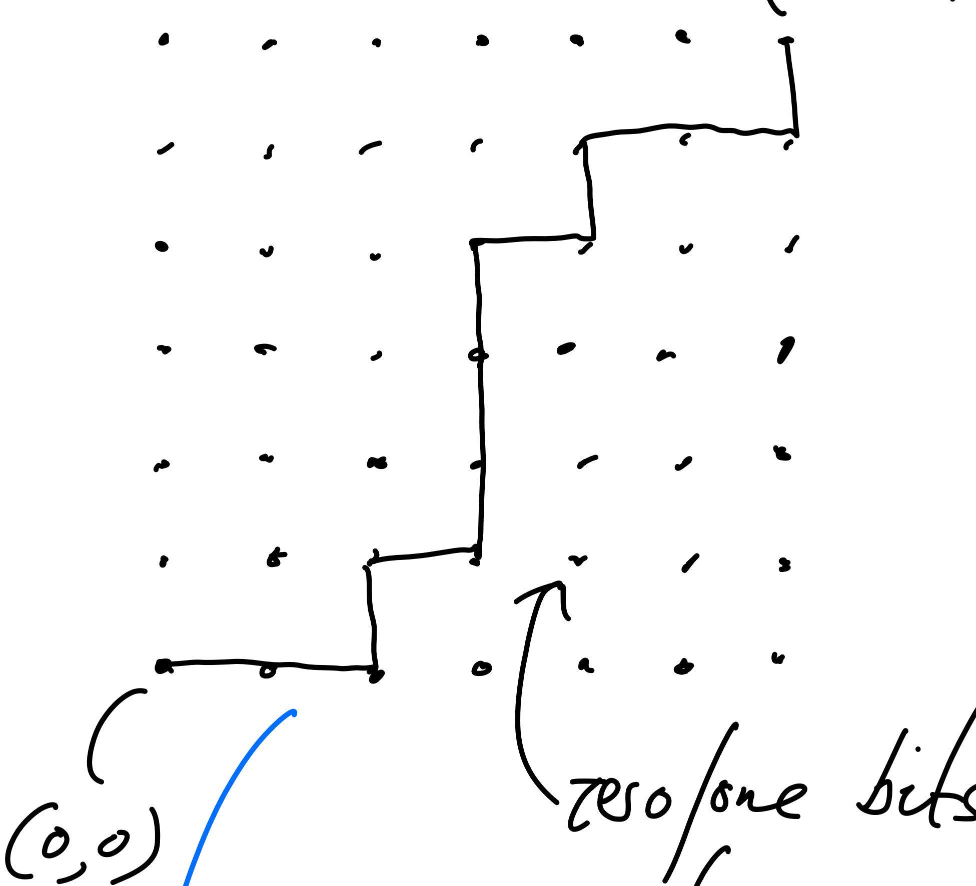
Bernoulli

last

passage

percolation

BERNOULLI LPP



zero/one bits are assigned HEADS/TAILS

any UPRIGHT path P
has energy $E(P)$
$=$ sum of values along P.

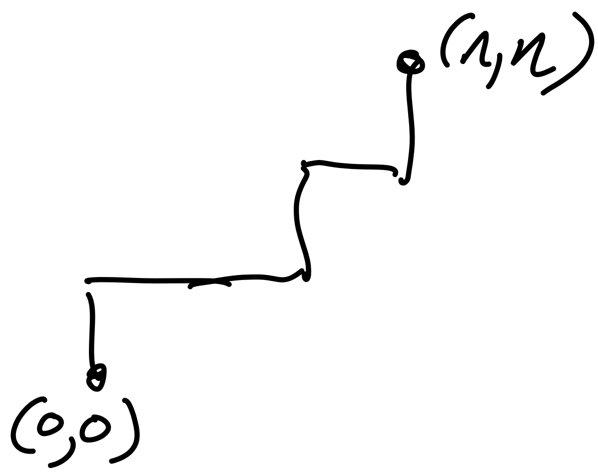
The geodesic Γ_n

is the path

$(0,0) \to (n,n)$

of maximum energy ;

set $M_n = E(\Gamma_n)$.



Super-concentration

$$M_n = an + W_n n^{1/3}$$

$a \in (1,2)$

(W_n a tight, non-degenerate, sequence of RVs)

So:

$$\mathrm{Var}(M_n) = \Theta(n^{2/3}) \ll n$$

super-concentration

The energy landscape of LPP.

Consider the correlation

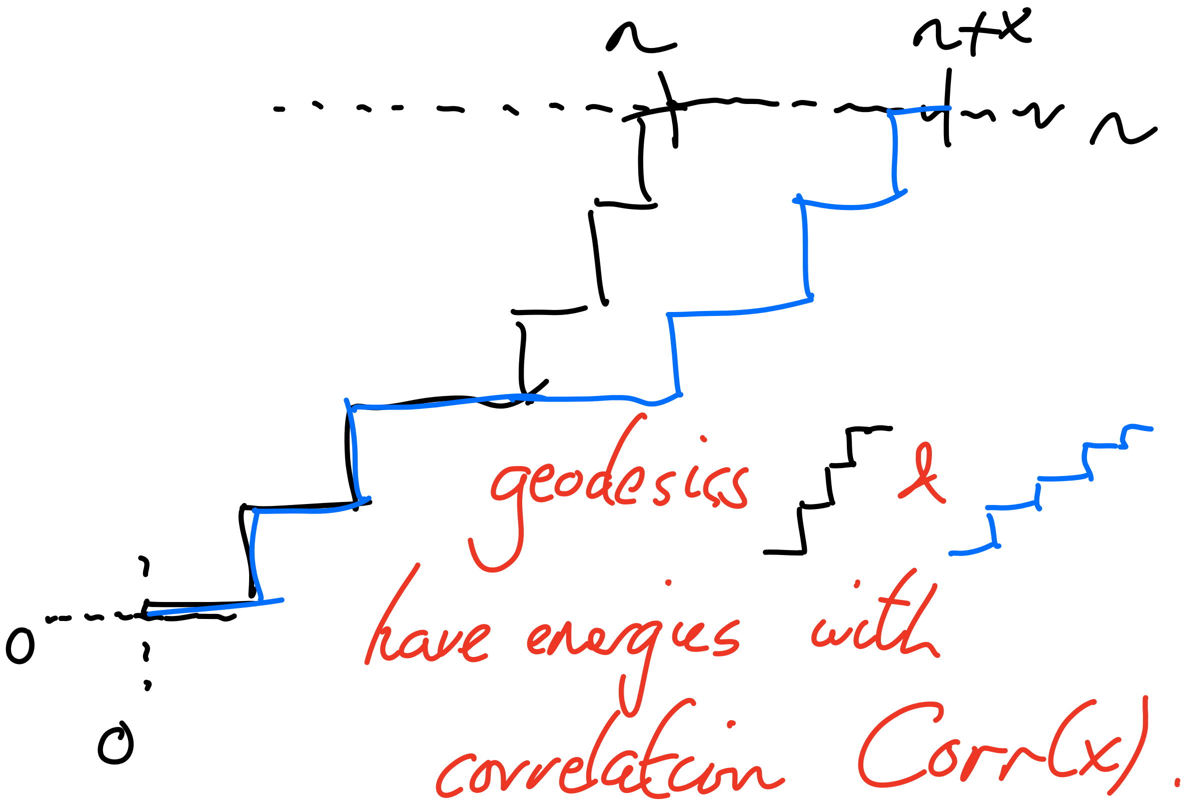
$\mathrm{Corr}(x)$ between

geodesic energies

for the routes

$(0,0) \to (n,n)$

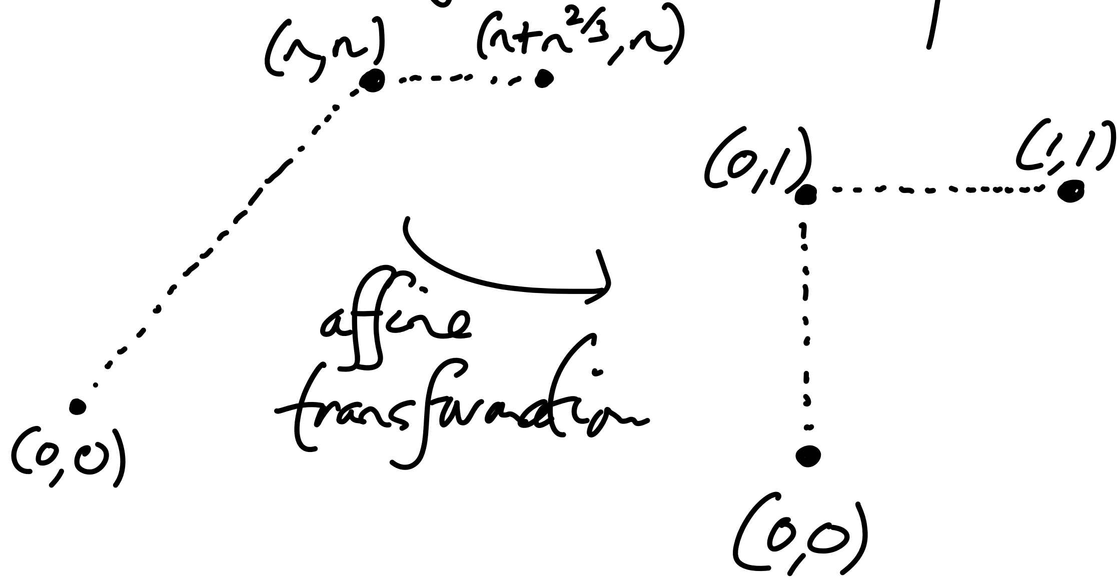
& $(0,0) \to (n+x, n)$.



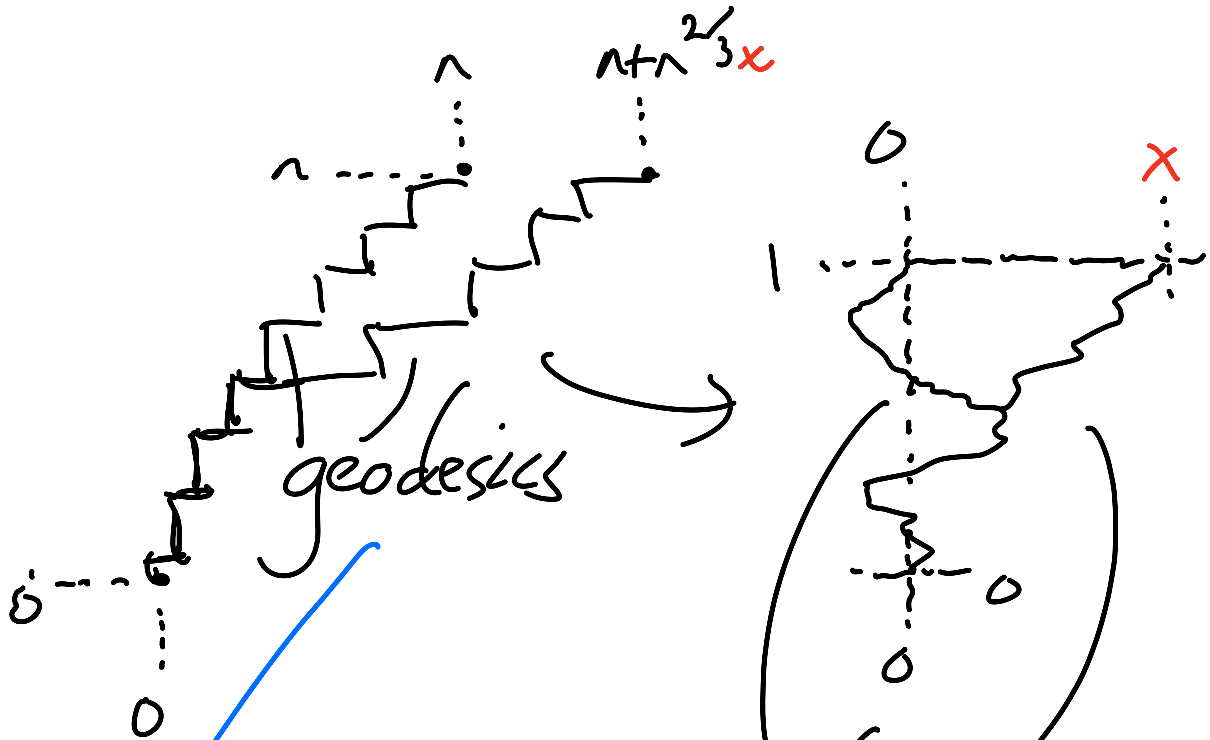
Q: what scale of x leads to $\mathrm{Corr}(x) = 50\%$?

A: $x = \Theta(n^{2/3})$.

The ANSWER suggests a SCALED PICTURE



in which geodesics may be depicted in a scaled form ....

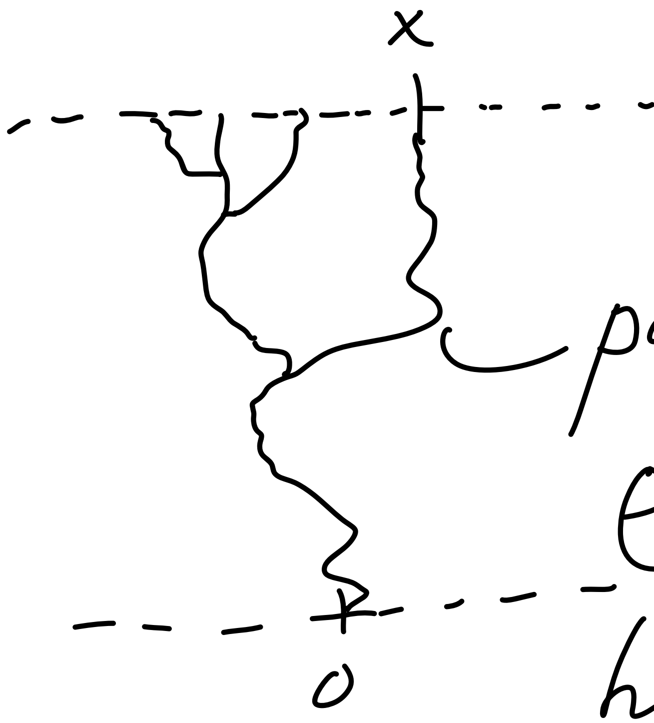


scaled geodesics, or polymers

have energy $an + W_n(x)\, n^{1/3}$

random

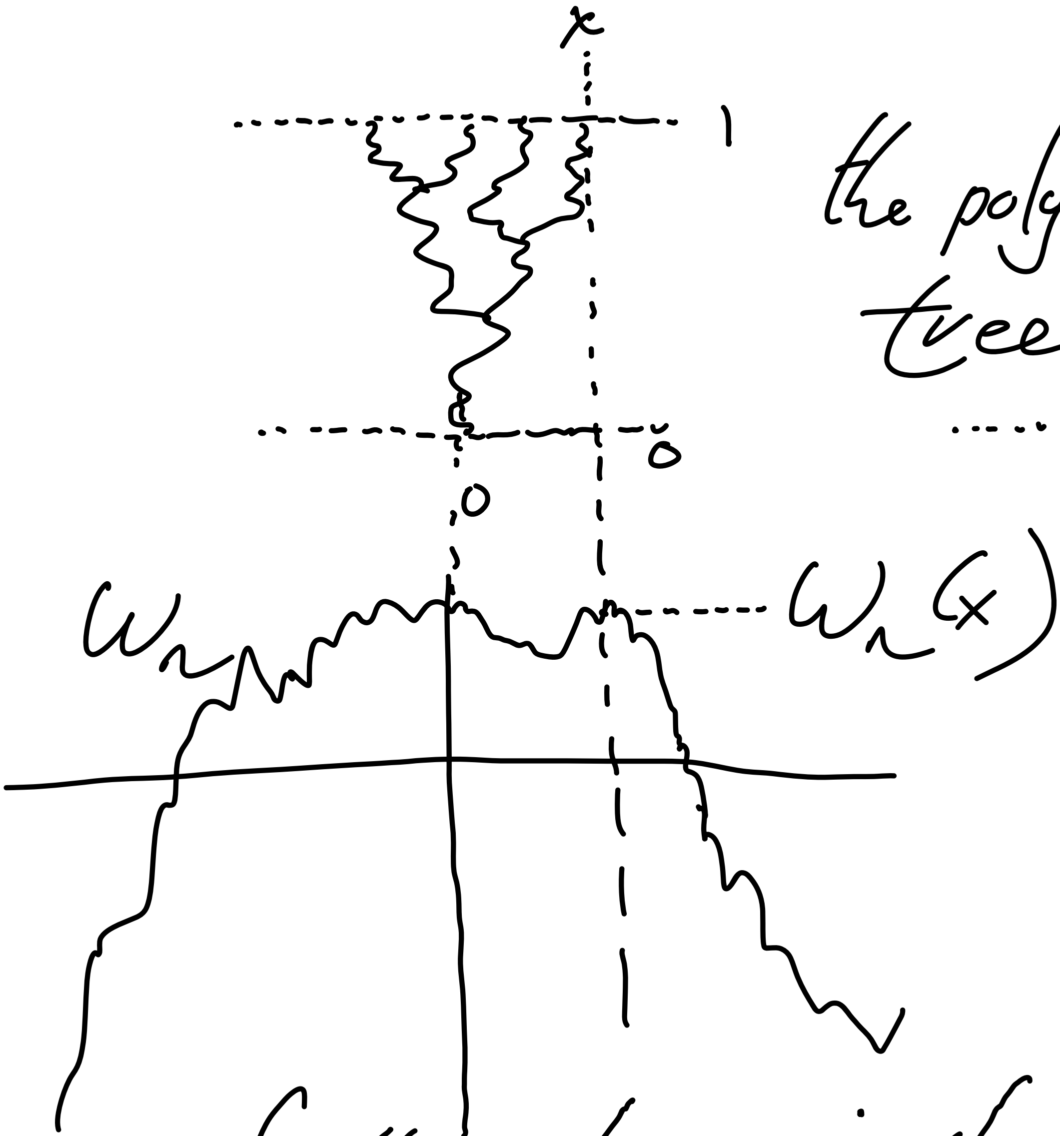
but unit-order scaled energy



polymer $\rho_n(x)$ has scaled energy, or weight, $W_n(x)$

The scaled picture:

POLYMERS & THEIR WEIGHT PROFILE



the polymer tree .....

..... and the polymer weight profile

Taking a high n limit....

$$\omega_n \xRightarrow{(d)} \omega$$

the parabolic Airy_2 process

$$x \to \mathrm{Airy}_2(x) - x^2$$

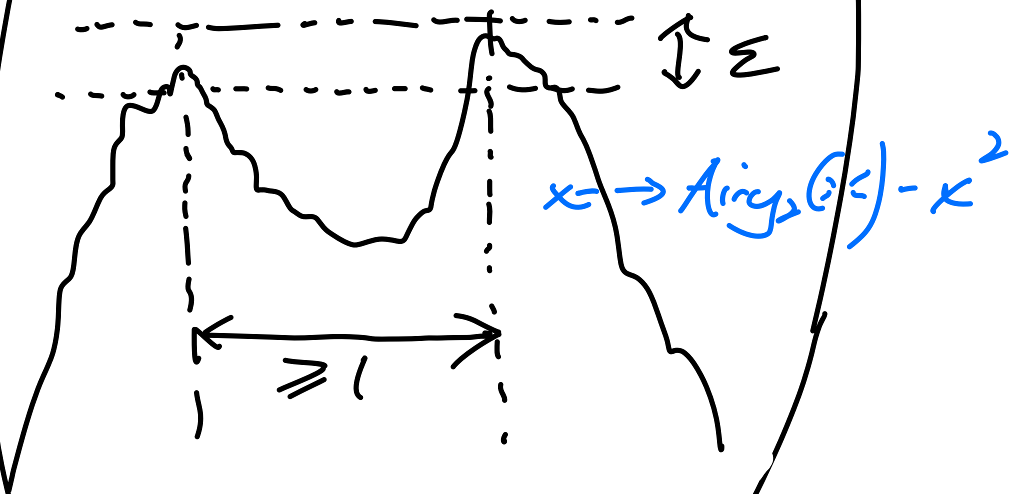
Q concerning the SCALED energy landscape:

What is the PROBABILITY of NEAR TWIN PEAKS?

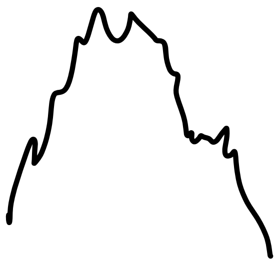
(of a rival to within ε of the energy maximum at distance at least one?

A.

$$\mathbb{P}\left(\varepsilon\text{-Twin Peaks}\right) \asymp \varepsilon.$$



Proof idea next ....

The profile 

closely resembles

Brownian motion

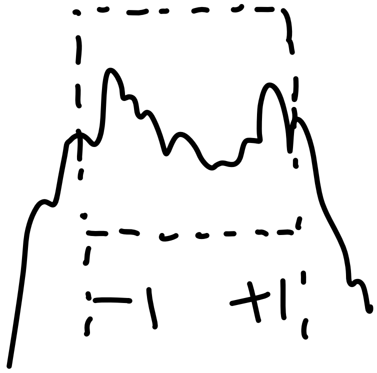
on mid-order

scales — so the

estimate is inherited

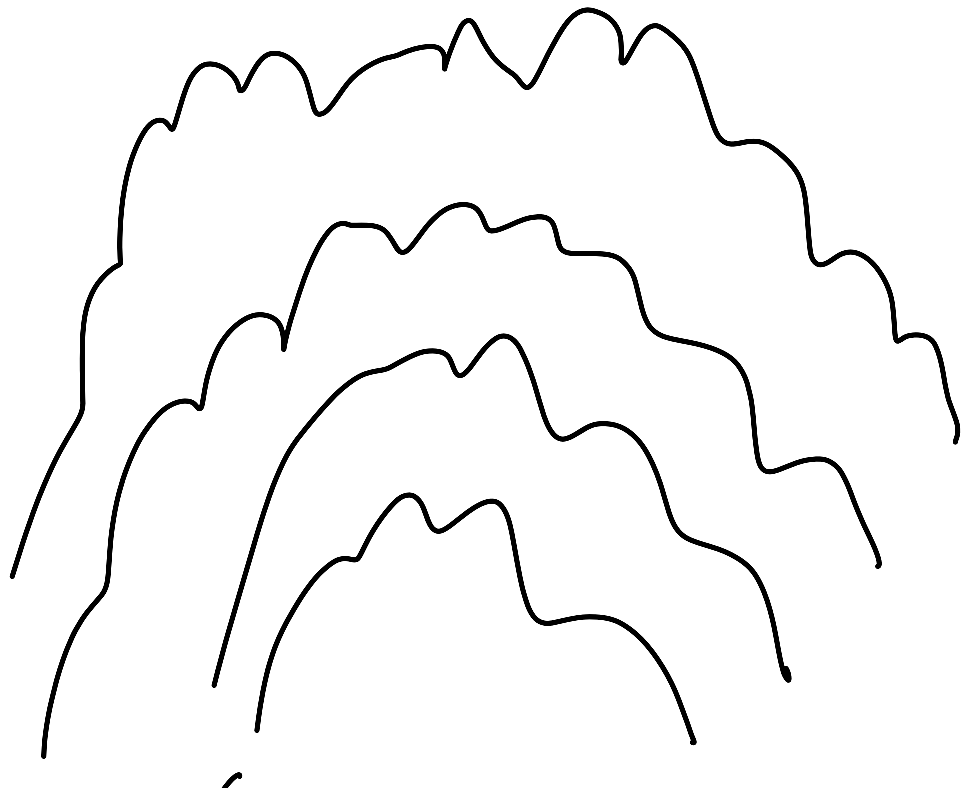
from BM.

Q: Why Brownian?



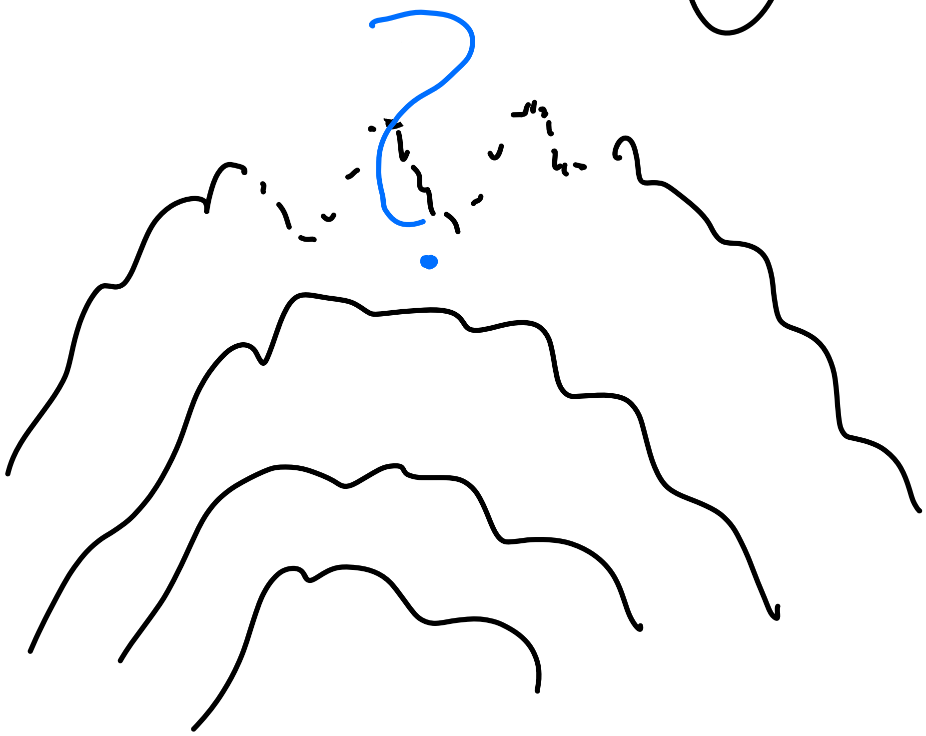
A: embed the parabolic Airy$_2$ process as the uppermost curve in an ENSEMBLE of random continuous curves ———

— the parabolic Airy line ensemble



via the RSK correspondence

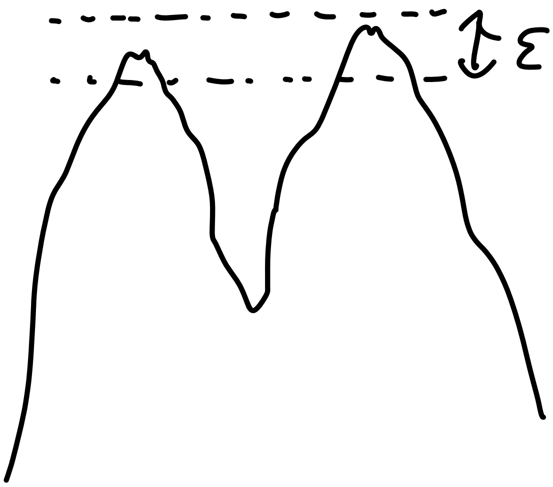
The ensemble satisfies



the Brownian Gibbs property

— a powerful tool for proving the Brownianity of the Airy_2 process.

Twin peaks' probability

$\mathbb{P}\Bigg($ 

$\Bigg) \asymp \Theta(\varepsilon)$

Calvert, Hegde, H.

— 2019

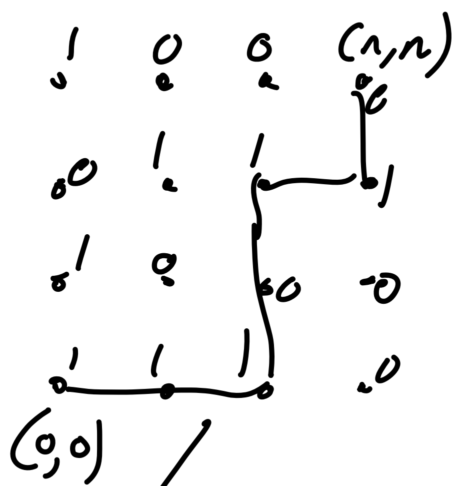
What of chaos?

$$\begin{matrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}$$

Dynamical Bernoulli LPP

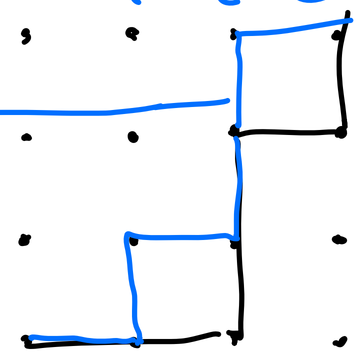
bits are independently updated at rate one Poisson times

At time zero, a copy of static Bernoulli LPP.



geodesic $(0,0) \to (n,n)$
$\Gamma_n(0)$ has energy $M_n(0)$

Equally, at any later time $t>0$:
geodesic $\Gamma_n(t)$
has energy $M_n(t)$.

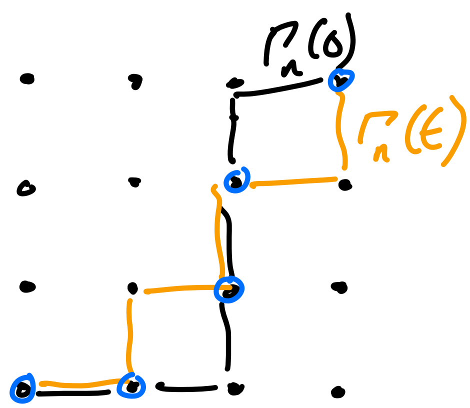


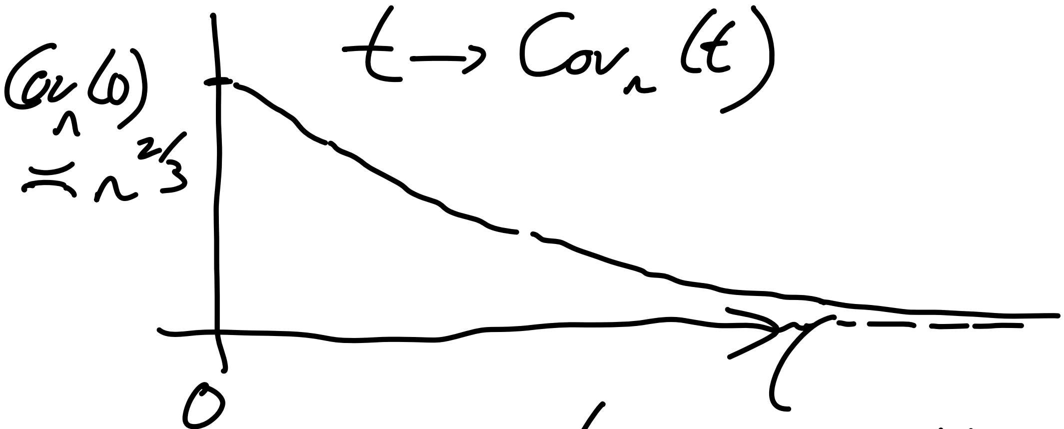
How to measure the fragility of these objects to the passage of time?

Energy: $\mathrm{Cov}_n(t) = \mathrm{Cov}(M_n(0), M_n(t))$

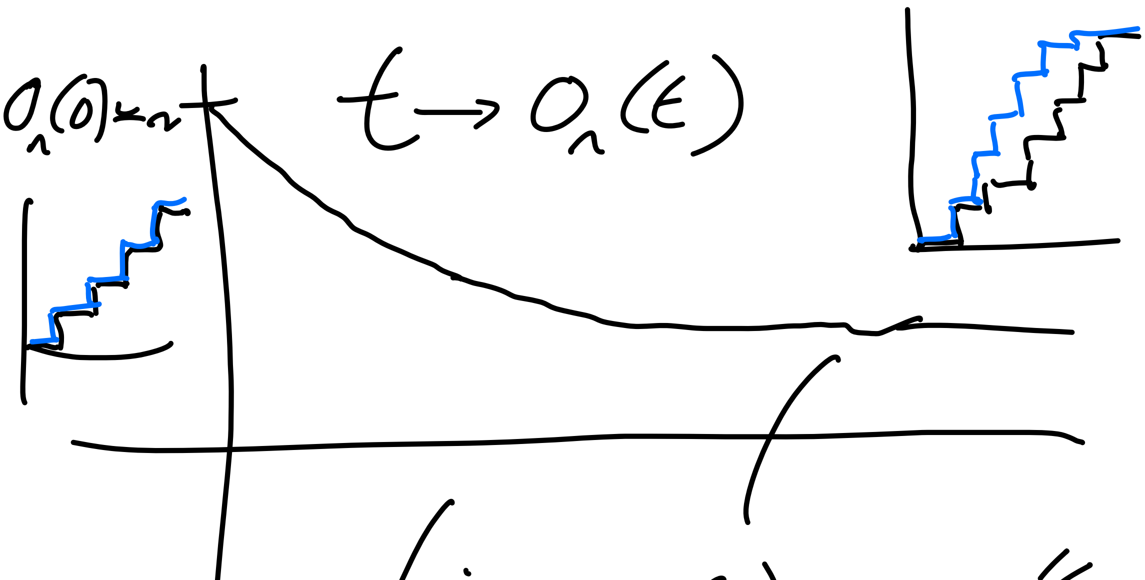
Geodesic overlap:

$O_n(t) = \#\text{vertices in } \Gamma_n(0) \cap \Gamma_n(t)$





$$\lim_{t\to\infty} Cov_n(t) = 0$$



$$\lim_{t\to\infty} O_n(t) \asymp n^{1/3}$$

Theorem [Ganguly–H.]

For dynamical BROWNIAN LPP, the TRANSITION from stability to chaos, measured by geodesic overlap, occurs on the scale $t = n^{-1/3}$.

Indeed, ...

There exists $d>0$ such that

$$\mathbb{P}\left(O_n(t) \geq dn\right) \geq 1-o(1)$$

when $t \ll n^{-1/3}$ STABILITY

&

$$\mathbb{P}\left(O_n(t) = o(n)\right) \geq 1-o(1)$$

when $t \gg n^{-1/3}$.

CHAOS

Signposts to the proof

To show the transition

stability $\to$ chaos,

we may try to prove

	Subcritical $t \ll n^{-1/3}$	Supercritical $t \gg n^{-1/3}$
Energy change	NOT MUCH: $o(n^{1/3})$	A LOT: $\Theta(n^{1/3})$
	(w/ work with S.G.)	
Geodesic overlap	HIGH ✓ $\Theta(n)$	LOW ✓ $o(n)$

Three-step game plan

	Subcritical	Supercritical
Energy change	NOT MUCH (2. ✓)	A LOT
Geodesic overlap	HIGH (3. ✓)	LOW (1. ✓)

substantial component of work with S.G.

Chatterjee's monograph / harmonic analysis

Step I:

supercritical $t \gg n^{-1/3}$

geodesic overlap is LOW.

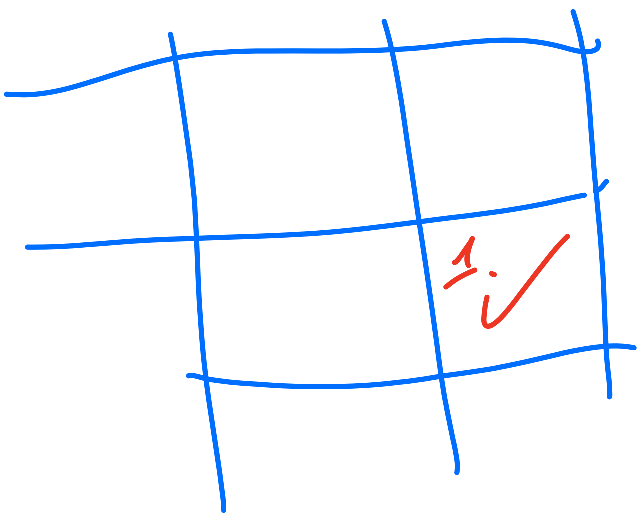
Chatterjee's theory of super-concentration and chaos concerns Gaussian polymer models.

A beautiful formula: geodesic overlap between times 0 & t.

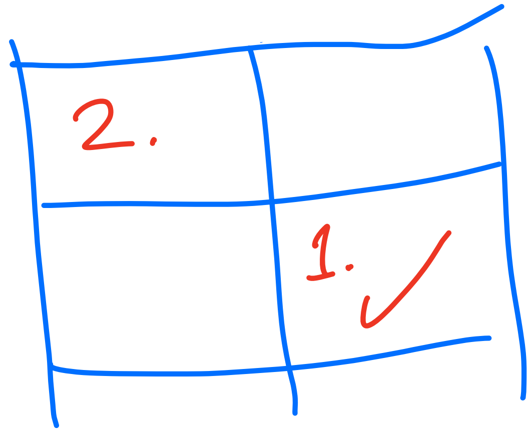
$$\mathrm{Var}\, M_n = \int_0^\infty e^{-t}\, \mathbb{E}\, O_n(t)\, dt$$

geodesic energy $(0,0) \to (n,n)$

Since $\mathrm{Var}(M_n) \asymp n^{2/3}$, $\mathbb{E}\, O_n(t)$ must fall precipitously from order n as t advances through scale $n^{-1/3}$.



Step II



Subcritical stability of geodesic energy

— via harmonic analysis of LPP.

We will argue that, for $t \geq 0$, and with

$$\mu_n^t = \text{geodesic energy } (0,0) \to (n,n) \text{ in } \textit{dynamic} \text{ Bernoulli LPP},$$

$$\mathbb{E}\left(\mu_n^t - \mu_n^0\right)^2 \leq \Theta(1)\, tn.$$

Take $t = \tau n^{-1/3}$, so that, for $\tau \ll 1$ subcritical scaled time,

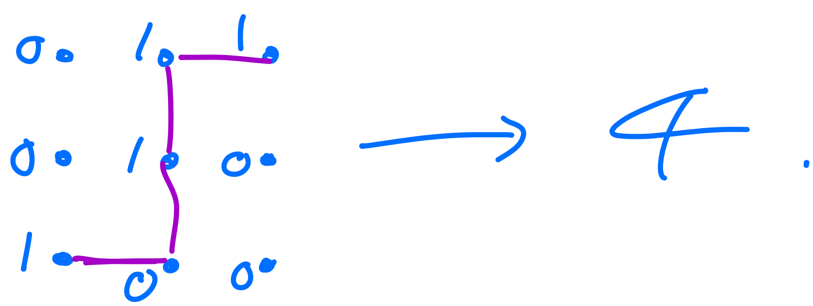
$$\mu_n^t - \mu_n^0 = o(\mu_n^0).$$

Key technique: discrete Fourier analysis.

Set $\Lambda_n = \{0,1,\dots,n\}^2$.

Geodesic energy M_n is a map

$$M_n : \{0,1\}^{\Lambda_n} \longrightarrow \mathbb{N}$$



To each subset $S \subseteq \Lambda_n$,

define

$$\chi_S : \{0,1\}^{\Lambda_n} \to \{-1,1\},$$

$$\chi_S(\omega) = \prod_{x \in S} \big(2\omega(x) - 1\big),$$

$$\chi_\emptyset = 1 .$$

The collection $\{\chi_S : S \subseteq \Lambda_n\}$ is an orthonormal basis for the L^2-space of functions mapping $\{0,1\}^{\Lambda_n}$ to $\mathbb{R}$.

As such, we may decompose

$$M_n(\omega) = \sum_{S \subseteq \Lambda_n} \alpha(S)\chi_S(\omega),$$

where $\alpha(\emptyset) = \mathbb{E}M_n$.

Parseval's formula:

$$\mathrm{Var}(\mu_n) = \sum_{S \neq \emptyset} \alpha(S)^2 .$$

This identity permits us to introduce the SPECTRAL SAMPLE, a random variable $\mathcal{S}$ under a law — call it Q — that is canonically associated to the function $\mu_n : \{0,1\}^{\Lambda_n} \longrightarrow \mathbb{R}$:

$$Q(\mathscr{S}=S) = \frac{\alpha(S)^2}{\mathrm{Var}(\mu_n)}, \quad S \subseteq \Lambda_n.$$

<u>Proposition</u> [fundamental role of the spectral sample in studying dynamics]

Let $t \geq 0$. Then

$$\mathrm{Cov}(\mu_n^0, \mu_n^t) = \sum_{S \neq \emptyset} \alpha(S)^2 e^{-t|S|}$$

and

$$\mathrm{Corr}(\mu_n^0, \mu_n^t) = \mathbb{E}_Q\left[e^{-t|\mathscr{S}|}\right].$$

The mean value of the spectral sample.

We will argue that

$$\mathbb{E}_Q |S| = \Theta(1) n^{1/3}$$

the mean value under Q.

Alongside the proposition (and Jensen's $\leq$),

we learn that ....

$$\mathbb{E}\left(\mu_n^t - \mu_n^0\right)^2 \leq 2\,\mathrm{Var}(\mu_n)\cdot \left(1 - e^{-\Theta(1) t n^{1/3}}\right).$$

When $t = o(1) n^{-1/3}$,

$$\mathrm{RHS} = \mathrm{Var}(\mu_n)\,\Theta(1)\, t n^{1/3}$$

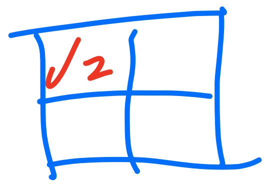
$$= \Theta(1)\, t n .$$

So ...

Subcritical energetic stability

$$\mu_n^\epsilon - \mu_n^0 = o(\mu_n^0)$$

reduces to



deriving

$$\mathbb{E}_Q |g| = \Theta(1)\, n^{1/3}.$$

three steps to this formula …

First: for $v \in \Lambda_n$,

write $M_n[v]$ for the value of M_n when the ω-value at v is flipped.

Then:

$$\mathbb{E}_Q |S| = \frac{1}{4 \mathrm{Var}(M_n)} \sum_{v \in \Lambda_n} \mathbb{E}\left(M_n - M_n[v]\right)^2.$$

Proof.

$$\mathbb{E}_Q |S| = \sum_{v \in \Lambda_n} Q(v \in \mathcal{S})$$

$$\overset{\text{def}^n \text{ of } \mathcal{S}}{=} \frac{1}{\operatorname{Var}(\mu_n)} \sum_{v \in \Lambda_n} \sum_{S : v \in S} \alpha(S)^2 .$$

To prove the sought formula, it thus suffices to show that

$$\mathbb{E}\left(\mu_n - \mu_n[v]\right)^2 = 4 \sum_{S : v \in S} \alpha(S)^2 \quad (\times)$$

To see this, note that

$$\mu_n = \sum_S \alpha(S) x_S$$

&

$$\mu_n[v] = \sum_S \alpha(S) x_S[v]$$

the value of x_S when the bit at v is flipped.

So ....

$$\mathbb{E}\left(M_n - M_n[v]\right)^2 = A - B,$$

where $A = 2\sum \alpha(S)^2$

& $B = 2\sum \alpha(S)^2 x_S x_S[v]$

$$= -2\sum_{S:v\in S} \alpha(S)^2 + 2\sum_{S:v\notin S} \alpha(S)^2$$

$$\Longrightarrow A - B$$

$$= 4\sum_{S:v\in S} \alpha(S)^2$$

so that (x) is obtained, as we sought.

We have confirmed:

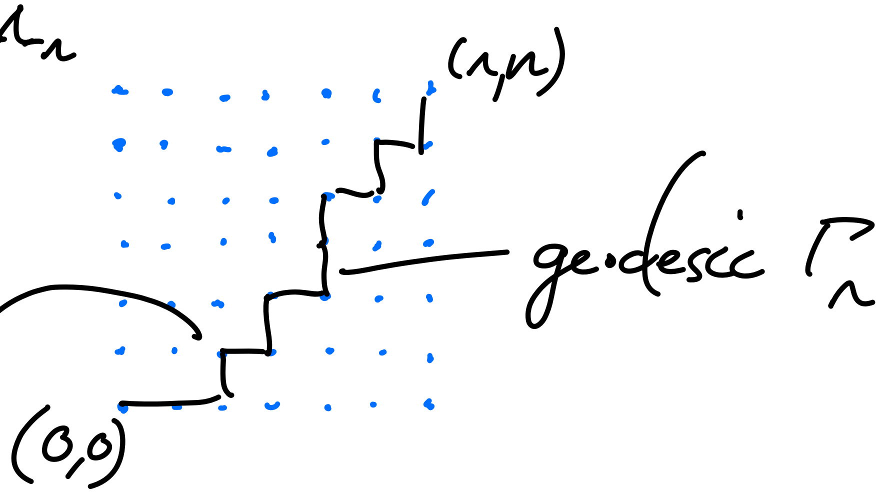
$$\mathbb{E}_Q |\rho| = \frac{1}{4\,\mathrm{Var}\,\mu_n} \sum_{v \in \Lambda_n} E(\mu_n - \mu_n[v])^2$$

is of order $n^{2/3}$ — this is LPP geodesic energy fluctuation

(will show this to be $\Theta(1)n$

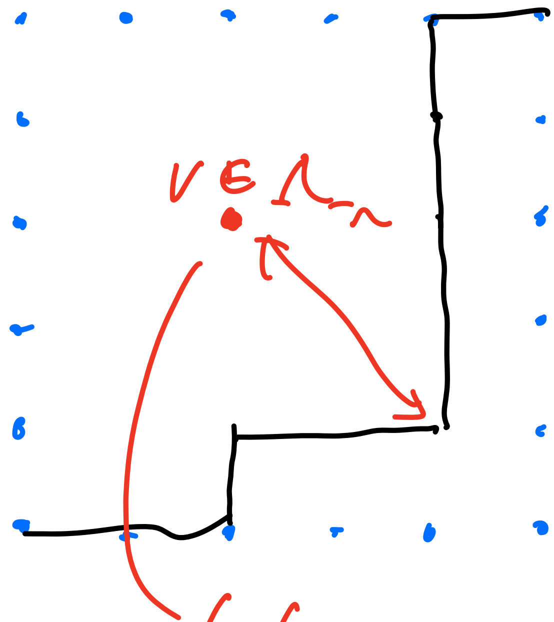
To argue that:

$$\sum_{v \in \Lambda_n} \mathbb{E}(M_n - M_n[v])^2 = \Theta(1)\, n :$$



order n elements of Γ_n have $|M_n - M_n[v]| = 1$.

Wish to argue that only a comparable order of off-geodesic vertices v have $M_n - M_n[v] \neq 0$.

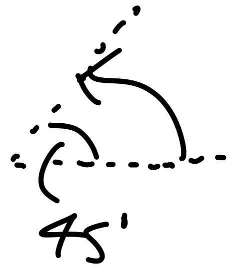


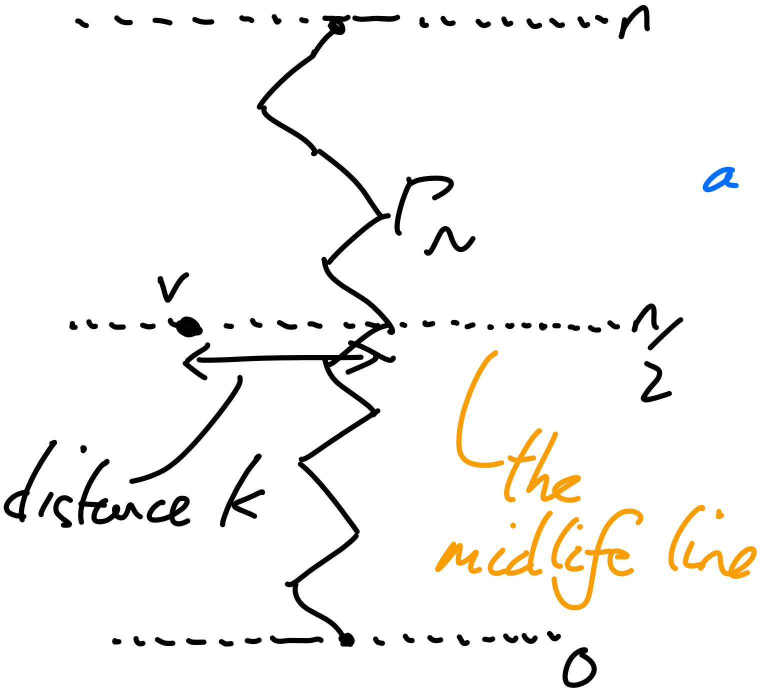
take v at anti-diagonal distance $k \in \mathbb{N}$ from Γ_n.

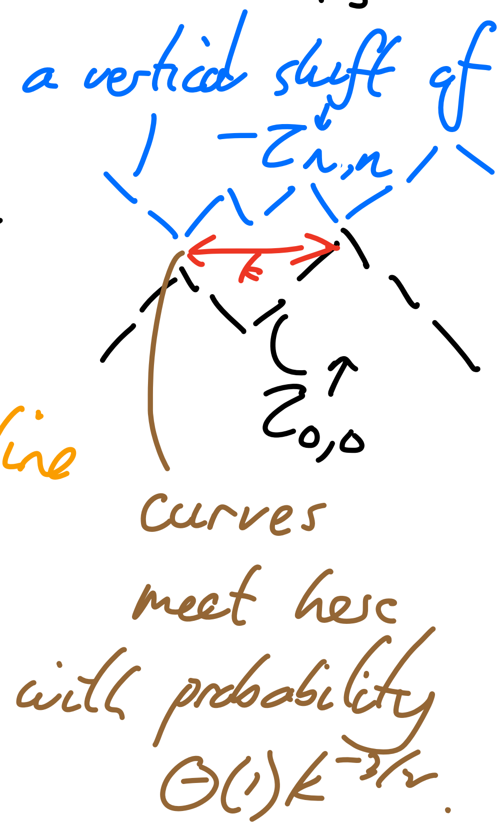
We will argue that

$$\mathbb{P}\left(\mu_n - \mu_n[v] \neq 0\right) \overset{(*)}{\leq} C k^{-3/2}.$$

ARGUING for the BOUND $(\dagger)$.







Let $Z^{\uparrow}_{0,0}$: midlife line $\longrightarrow \mathbb{N}$ denote geodesic energy from $(0,0)$ to points on the midlife line.

Let $Z^{\downarrow}_{n,n}$: midlife line $\longrightarrow \mathbb{N}$ denote geodesic energy from points on the midlife line to (n,n).

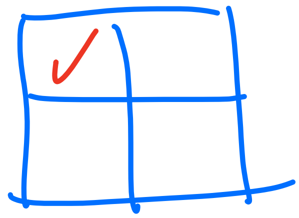
In summary, we have argued for subcritical energetic stability

$$\mu_n^t - \mu_n^0 = o(\mu_n^0)$$

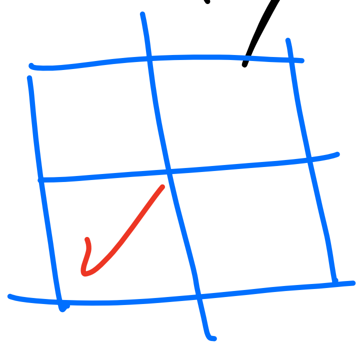
	Subcritical	Supercrit.
Energy change	✓ 2. NOT MUCH	A LOT
Geod. overlap	HIGH	LOW ✓ 1.

major technical challenge — establish this!

Harnessing subcritical energetic stability



to prove high subcritical overlap

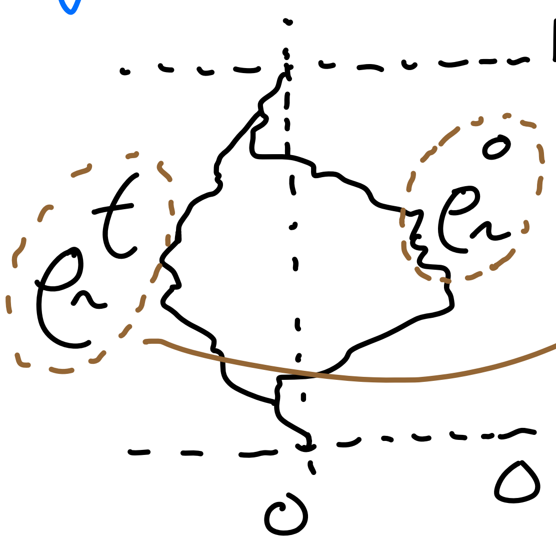


—

the proxy.

Suppose low overlap between geodesics Γ_n^0 and Γ_n^t for a subcritical time $t \ll n^{-1/3}$.

In the SCALED picture, a TYPICAL realization might resemble

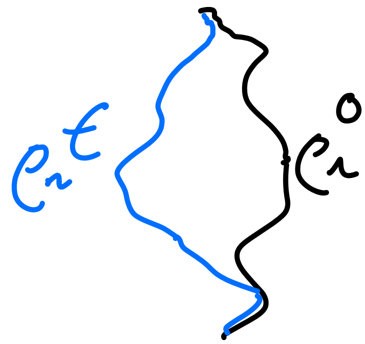


scaled geodesics with substantially different routes.

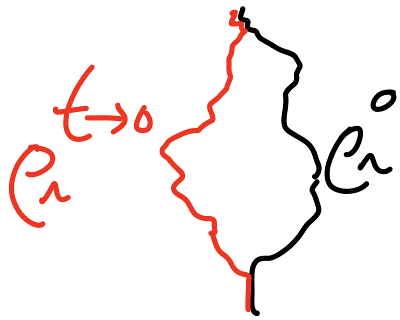
AIM: substantially transport the scaled geodesic ϱ_n^t at time t to a PROXY $\varrho_n^{t \to 0}$ — so that the proxy mimics the route of ϱ_n^t and, in its time-zero weight, mimics the time-t weight of ϱ_n^t.

the proxy is formed of interpolating time-zero local geodesics.

So the TYPICAL low overlap scenario



typically forces

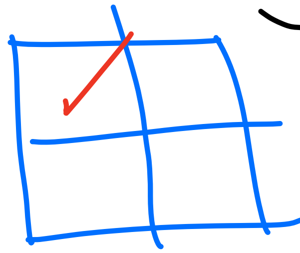


a proxy whose route is substantially different to $\mathcal{C}_n^0$'s.

But, by

subcritical weight stability,

ρ_λ^t and ρ_λ^0



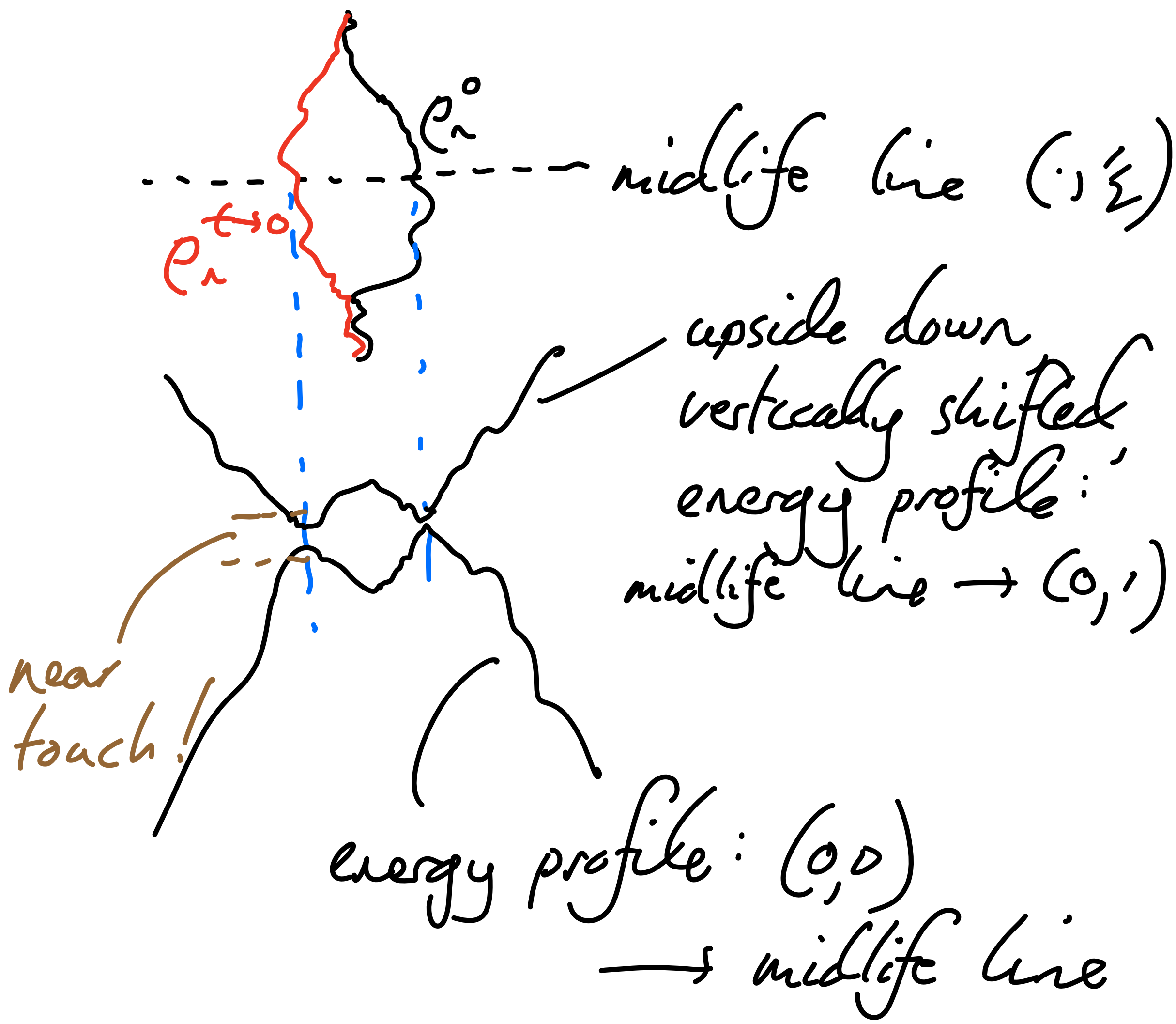
have very similar weight

— with $o(1)$ difference.

Thus, so do $\rho_\lambda^{t\to 0}$ and ρ_λ^0.

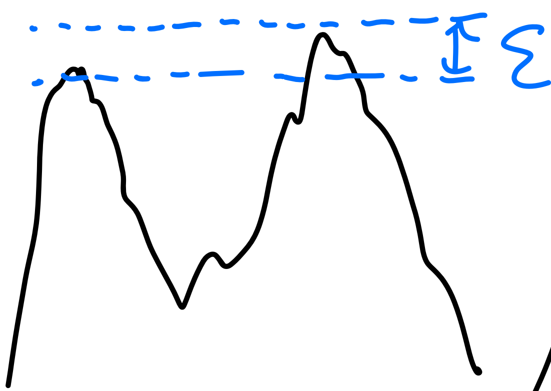
So ....

The proxy $\rho_n^{t\to 0}$ is close to ρ_n^0 in its time-zero weight.



This near-touch event is essentially the same as the 'twin peaks' event:

$\mathbb{P}\Big($



$\Big) \le \Theta(\varepsilon)$

as we discussed earlier

In summary, the PROXY transports the geometric scenario TYPICALLY forced by LOW OVERLAP between ρ_n^0 & ρ_n^t, for $t \ll n^{-1/3}$, to a provably rare TWIN PEAKS' scenario at time zero.

Thus,

	Subcrit. time	Supercrit. time
Energy change	LOW 2. ✓	HIGH
Geod. overlap	3. ✓ A LOT	✓ 1. A LITTLE

the MAIN theorem

— transition in overlap to CHAOS.