Boundaries of Random Surfaces and KPZ Models

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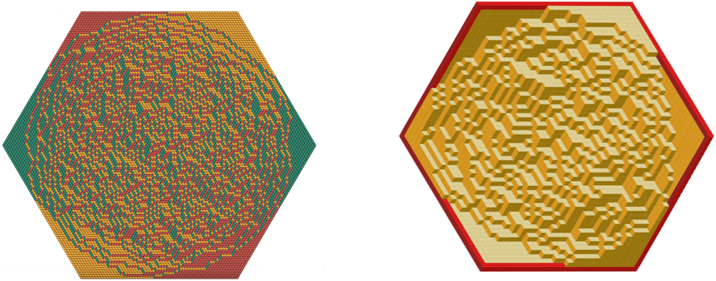
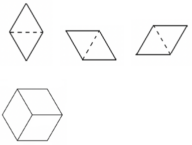
Random Tilings as Random Surfaces



Figures from `https://storage.lpetrov.cc/img/blog/hex120_uniform.png` and `http://math.mit.edu/~borodin/hexagon.html`.



- Uniformly random tiling of domain R by “lozenges”
- Also view as random stepped surface made of cubes
  - Gives rise to height function $H : R \to \mathbb{Z}$

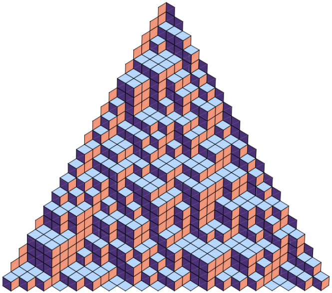
Question: How does the tiling behave as $\mathrm{diam}(R) \to \infty$?

Gaussian Free Field Under Planar Boundary Conditions

- **Planar boundary conditions**: Take $R \approx N \cdot \mathcal{D}$ so $H|_{\partial R}$ is nearly linear

Belief: Fluctuations
of H "look like" Gaussian free field (GFF)

- Weak
  convergence / level lines to CLE(4) / e^H
  to Liouville quantum gravity (LQG) / ...



Weak convergence: If $\varphi \in \mathcal{C}_c^\infty(\mathcal{D})$,

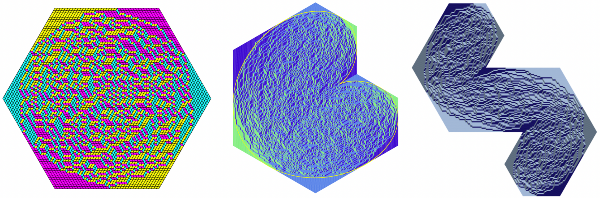
$$\int_{\mathcal{D}} \varphi(z) H_N(Nz) dz - \mathbb{E}\left[\int_{\mathcal{D}} \varphi(z) H_N(Nz) dz \right] \xrightarrow{N \to \infty} \int_{\mathcal{D}} \varphi(z) \mathrm{GFF}(z) dz$$

- Inverse Kasteleyn matrix: Kenyon (1997) / Kenyon–Okounkov–Sheffield (2003)
- Discrete complex analysis: Kenyon (1999) / Russkikh (2018) / Chelkak–Laslier–Russkikh (2020)
- Imaginary geometry: Berestycki–Laslier–Ray (2016)

Convergence does not see level lines / LQG

Nonplanar Boundary Conditions

Non-planar domains: Can exist **arctic curve** separating facets / rough regions



Conjecture (Kenyon–Okounkov, 2005)

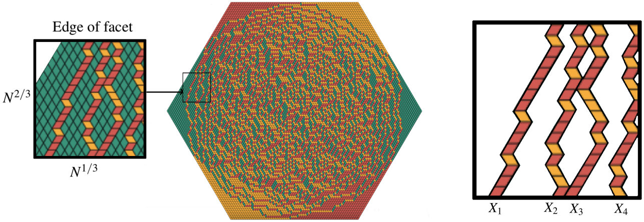
In rough region, fluctuations of H (weakly) converge to GFF under suitable coordinate change

Proven in some nonplanar cases (where faceted regions appear)

- Borodin–Ferrari (2008) / Petrov (2012) / Ahn–Russkikh–Van Peski (2021): Determinantal point process
- Bufetov–Gorin (2017): Schur generating function / loop equations
- Huang (2020): Analytical comparison to system satisfying loop equations

Edge Limits

Question: How does model transition from faceted to rough regions?



- Red and orange tiles form paths $X_1, X_2, \ldots$, where $X_i = \big(X_i(t)\big)$

Edge statistics question: What is the joint scaling limit of these paths?

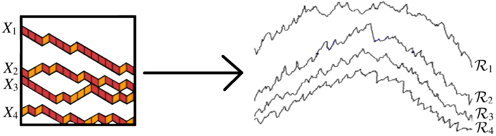
- Scaling: Fluctuations $X_i(T) - \mathfrak{c}N \sim N^{1/3}$; time $T \sim tN^{2/3}$

Baik–Kriecherbauer–McLaughlin–Miller (2007), Petrov (2012): Exact calculation on hexagon, exist $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ so that $\mathfrak{a}N^{-1/3} \cdot \big(X_i(\mathfrak{b}tN^{2/3}) - \mathfrak{c}N\big) \xrightarrow{N \to \infty} \mathcal{R}(t)$

- A.–Huang (2021): Also holds on generic polygons

Parabolic Airy Line Ensemble

Edge limit: $\mathfrak{a}N^{-1/3} \cdot \left(X_i(\mathfrak{b}tN^{2/3}) - \mathfrak{c}N\right) \xrightarrow{N\to\infty} \boldsymbol{\mathcal{R}}(t)$



- Limit $\boldsymbol{\mathcal{R}}(t) = \left(\mathcal{R}_1(t) > \mathcal{R}_2(t) > \cdots\right)$: *Parabolic Airy line ensemble* (originally introduced by Prähofer–Spohn, 2001)
- *Airy line ensemble* $\boldsymbol{\mathcal{A}} = \left(\mathcal{R}_j(t) + t^2\right)$ is translation-invariant

Determinantal point process: $\mathbb{P}\left[\bigcap_{j=1}^{m} \{(t_j, x_j)\} \in \mathcal{A}\right] \sim \det\left[\mathcal{K}(t_i, x_i; t_j, x_j)\right]$

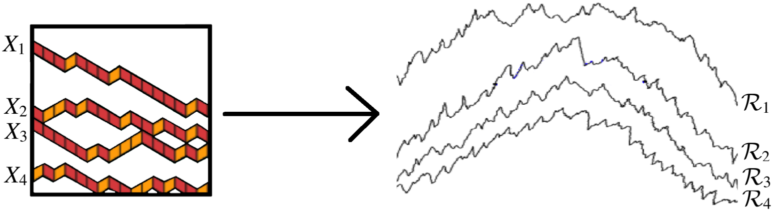
- Extended Airy kernel $\mathcal{K}(s,x;t,y) = \begin{cases} \int_0^\infty e^{u(t-s)}\mathrm{Ai}(x+u)\mathrm{Ai}(y+u)du & \text{if } s \geq t \\ -\int_{-\infty}^0 e^{u(t-s)}\mathrm{Ai}(x+u)\mathrm{Ai}(y+u)du & \text{if } s < t \end{cases}$

Convergence to Airy line ensemble is open for most surface models

- More complicated tilings / restricted solid-on-solid models / low-temperature three-dimensional Ising model / ...

Examining the Airy Line Ensemble

Question: Axiomatically describe the Airy line ensemble, in a way useful for proving convergence to it



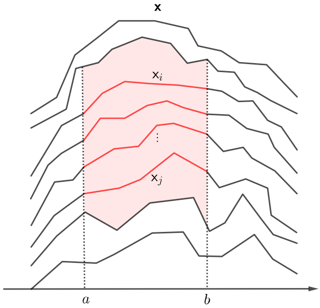
- Walks X_i do not intersect
- Walks should converge to Brownian motions

Limit should look like “infinitely many non-intersecting Brownian motions”

Brownian Line Ensembles

Brownian line ensemble: Sequence $\mathbf{x} = (\mathsf{x}_1, \mathsf{x}_2, \dots)$ of random functions $\mathsf{x}_k : \mathbb{R} \to \mathbb{R}$, with $\mathsf{x}_1 > \mathsf{x}_2 > \cdots$, satisfying *Brownian Gibbs property*

- Informally, $\mathbf{x}$ acts as infinitely many non-intersecting Brownian motions



Specifically, for any $i < j$ and $a < b$, the below hold

- Condition on $\mathsf{x}_k(s)$ for $(k, s) \notin [i, j] \times (a, b)$
- Then, the law of $\big(\mathsf{x}_k(s)\big)$ for $(k, s) \in [i, j] \times (a, b)$ are Brownian bridges starting at $\mathbf{x}(a)$, ending at $\mathbf{x}(b)$, and conditioned to not intersect

Exact Formulas in the Airy Line Ensemble

- Corwin–Hammond (2011): Parabolic Airy line ensemble $\boldsymbol{\mathcal{R}}$ is a Brownian line ensemble

Question (Okounkov, Sheffield, 2006): Can we characterize the Airy line ensemble as the unique Brownian line ensemble satisfying certain properties?

- Random surfaces satisfy discrete analogs of Brownian Gibbs property
  - Try showing discrete Gibbs property converges to Brownian one
    - Barraquand, Corwin, Dimitrov, Serio, Wu, . . .: Results towards convergence to Brownian line ensemble for some surface models
- Falls under the subfield statistical mechanics concerning of classification of Gibbs measures

Theorem (A.–Huang, 2023; Informal statement)

Fix $\sigma > 0$. If $\boldsymbol{\mathcal{L}} = (\mathcal{L}_1, \mathcal{L}_2, \ldots)$ is a Brownian line ensemble such that $\mathcal{L}_1(t) = -\sigma t^2 + o(t^2)$ likely holds, then $\boldsymbol{\mathcal{L}}$ is a parabolic Airy line ensemble, up to scaling and an affine shift.

- Exist other Brownian line ensembles with $\mathcal{L}_1(t) \sim -t$

Characterization Result

- Fix $\sigma > 0$ and a Brownian line ensemble $\boldsymbol{\mathcal{L}} = (\mathcal{L}_1, \mathcal{L}_2, \dots)$

Theorem (A.–Huang, 2023)

Assume for any $\varepsilon, \delta > 0$ that there exists $\mathfrak{K} > 0$ such that

$$\mathbb{P}\Big[\big| \mathcal{L}_1(t) + \sigma t^2 \big| > \varepsilon t^2 + \mathfrak{K} \Big] < \delta, \qquad \textit{for all } t \in \mathbb{R}.$$

Then there exist a parabolic Airy line ensemble $\boldsymbol{\mathcal{R}}$ and an independent pair of random variables $(\mathfrak{l}, \mathfrak{c}) \in \mathbb{R}^2$, such that $\boldsymbol{\mathcal{L}}(t) = \sigma \cdot \boldsymbol{\mathcal{R}}(t/2\sigma^2) + \mathfrak{l}t + \mathfrak{c}$.

Corollary

If $\boldsymbol{\mathcal{L}}(t) + \sigma t^2$ is translation-invariant and extremal, then exists $c \in \mathbb{R}$ with $\boldsymbol{\mathcal{L}}(t) = \sigma \cdot \boldsymbol{\mathcal{R}}(t/2\sigma^2) + c$.

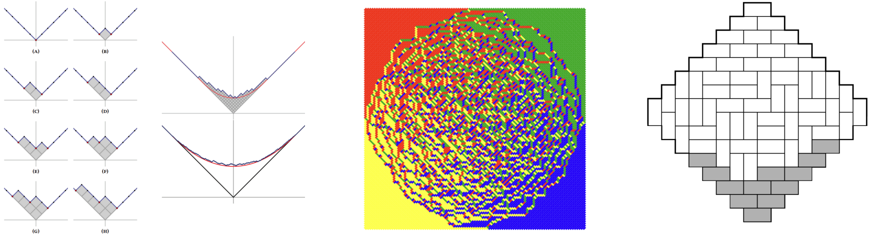
Predicted by Okounkov and Sheffield (2006)

- General theorem does not need any information about $\mathcal{L}_j$ for $j > 1$
- Also only needs $\mathcal{L}_1$ close to $-\sigma t^2$, not translation-invariance of $\mathcal{L}_1 + \sigma t^2$
  - Useful in showing convergence of discrete models

Corner Growth Models

Corner growth model: At any time $t \in \mathbb{Z}_{\geq 0}$, corners flip up with probability $\frac{1}{2}$

- Denote interface after time N by F_N
- Basic example of stochastic growth model



Figures by Corwin / Chhita–Young / Jockusch–Propp–Shor

Johansson (2003): $\mathfrak{a}N^{-1/3} \cdot \left(F_N(\mathfrak{b}N^{2/3}t) - \mathfrak{c}N\right) \xrightarrow{N\to\infty} \mathcal{R}_1(t)$

- Jockusch–Propp–Shor (1998): F_N equal in law to facet edge (extreme path) of random domino tiling of Aztec diamond

Stochastic Growth Models

Convergence to $\mathcal{R}_1$ believed for many other stochastic growth models

- Last passage percolation models
- Exclusion processes and interacting particle systems
- Directed polymers in random media
- Stochastic vertex models
- . . .

A.–Borodin (2024): Using Yang-Baxter equation, some of these models also can be mapped to facet edge of random surface

- Borodin–Bufetov–Wheeler (2016), Bufetov–Mucciconi–Petrov (2017): Special cases
- Apply characterization to show surface facet edge converges to Airy line ensemble, and deduce $\mathcal{R}_1$ convergence of growth models

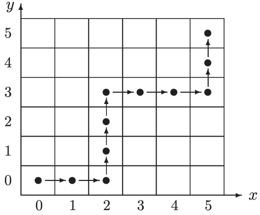
Colored / multi-species growth models get mapped to a family of many line ensembles, all interacting with each other

- A.–Corwin–Hegde (2024): Uncovers structure that reduces analysis of all of them to just one, and relate to Airy sheet / directed landscape

Application to Directed Polymers in Random Media

Random variable $R_{ij} \sim R$ in cell (i, j)

- Weight path $\mathcal{Q}$ by $w(\mathcal{Q}) = \prod_{(i,j)\in\mathcal{Q}} X_{ij}$
- Partition function: $Z(X, Y) = \sum_{\mathcal{Q}:(0,0)\to(X,Y)} w(\mathcal{Q})$
  - Sample path $\mathcal{Q} : (0, 0) \to (X, Y)$ with $\mathbb{P}[\mathcal{Q}] = w(\mathcal{Q}) \cdot Z(X, Y)^{-1}$



Model governed by free energy $F(X, Y) = \log Z(X, Y)$

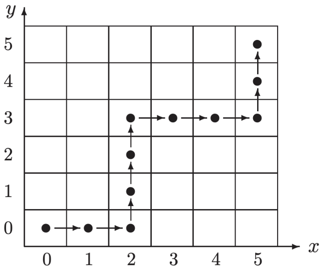
- Should have KPZ fluctuations if one looks at $F(X, Y)$ in tube of width $N^{2/3}$ around line $X = Y \sim N$
- If $R_{ij} = U_{ij}^{\beta}$ and $\beta \to \infty$, polymer $\to$ last passage percolation (LPP)

Example: Log-gamma polymer (Seppäläinen, 2009) $R \sim \Gamma(x)^{-1}x^{\theta-1}e^{-x}dx$

- Example of a polymer model that can be mapped to random surface

Correlated Noise

Assumed in LPP / polymers that noise R_{ij} was independent over (i, j)



- Believed that KPZ statistics should still hold if noise decorrelates sufficiently quickly in all directions
- Not clear if noise does not decorrelate quickly enough

Question

What happens if R_{ij} come from (discretization of) Gaussian free field?