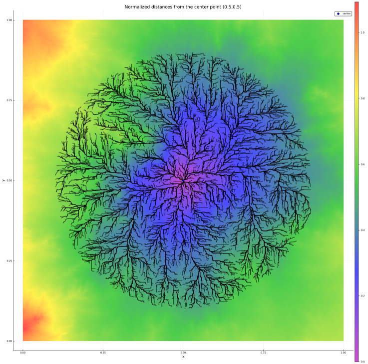
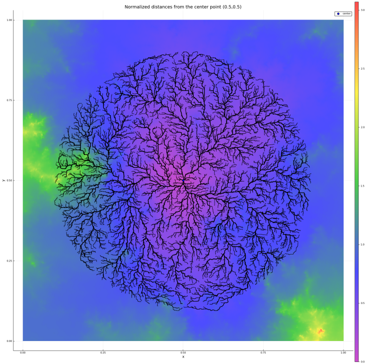
Subsequential scaling limits for Liouville graph distance

March 5, 2024

Liouville first-passage percolation

- “First passage percolation on the exponential of Gaussian free field”: Liouville first-passage percolation (LFPP) is first-passage percolation on $e^{\xi h}$, where h is a (discrete or circle-averaged continuous) Gaussian free field and ξ is an inverse-temperature parameter.





Distance exponent

- How does the metric (left–right crossing distance, point-to-point distance, diameter...) scale with the box size S (equivalently the mollification scale δ) and the temperature ξ?
  - Y. Watabiki (1993): non-rigorous prediction for the distance exponent.
  - J. Ding–F. Zhang (PTRF, 2018): the exponent of the distance is *non-universal* among log-correlated Gaussian fields. Therefore, answering this question must use rather fine information about the Gaussian free field.
  - Ding–S. Goswami (CPAM, 2019): upper bounds on the exponent for high temperature $\xi \ll 1$ (seems to contradict Watabiki).
  - Ding–E. Gwynne (Preprint, 2018): bounds for general (subcritical) temperature.
    - Includes precise value only at one special value of ξ, which corresponds to expected equivalence between limiting metric and the Brownian map (J.P. Miller–S. Sheffield 2015–'16).

Limiting metric

- Does there exist a scaling limit of the metric as the box size goes to ∞ (equivalently, as the mollification scale goes to 0)?
  - Need to rescale the metric by the typical distance, which is so far unknown.
  - No non-universality expected here: a limiting metric should exist for a wide class of log-correlated Gaussian fields.
- Uniqueness of the limiting metric done via axiomatic characterization by Gwynne and Miller.

Liouville quantum gravity

- *Liouville quantum gravity* is supposed to be a random geometry given by reweighting the Lebesgue measure, or the Euclidean metric, by $e^{\gamma h}$, where h is a (continuum) Gaussian free field.
- Problem: h is not a function but rather a distribution, and so it cannot be exponentiated.
- Construction of LQG measure (B. Duplantier–S. Sheffield 2010):
  - Consider $\mu_\varepsilon(dx) = e^{\gamma h_\varepsilon(x) + \frac{1}{2}\gamma^2 \log \varepsilon} dx$, where h_ε is the circle average process of the Gaussian free field with circles of radius ε.
    - Note the *renormalization* by $\varepsilon^{\gamma^2/2}$.
  - If $\gamma < 2$ (critical temperature), then there is a random measure μ so that, with probability 1, we have

$$\lim_{k \to \infty} \mu_{2^{-k}} = \mu$$

weakly.

Liouville graph distance

- Liouville FPP is intuitive to define, but there is another notion of discretized metric related to Liouville quantum gravity.
- Let $\Omega \subset \mathbf{R}^2$. Consider the γ-LQG measure μ on some domain $\Omega' \supset \Omega$.
- For $x, y \in \Omega$, define $d^{LGD}_{\Omega',\Omega,\delta}(x, y)$ to be the minimum size of a set of *Euclidean* balls with centers in Ω, each having *LQG measure* at most δ, whose union forms a connected set that includes x and y.

Main result

Notation: if $\mathfrak{R}$ is a box, let $\mathfrak{R}^*$ be a box with three times the side lengths, centered around $\mathfrak{R}$.

Theorem

Suppose that $\gamma < 2$. Let $\mathfrak{R}$ be a rectangular subset of $\mathbf{R}^2$. For any sequence $\delta_n \downarrow 0$, there is a subsequence (δ_{n_k}) and a nontrivial random metric $d : \mathfrak{R} \times \mathfrak{R} \to \mathbf{R}$ so that

$$Q_{\delta_{n_k}}^{-1} d^{LGD}_{\mathfrak{R}^*, \mathfrak{R}, \delta_{n_k}} \to d$$

in distribution with respect to the uniform topology of functions $\mathfrak{R} \times \mathfrak{R} \to \mathbf{R}$. Here Q_δ is the median Liouville graph distance between and left- and right-hand sides of $\mathfrak{R}$.

- No assumption of high temperature beyond subcriticality.
- Don't need to know the scaling exponent (size of Q).

Multiscale analysis: the effect of the coarse field

- We would like to understand the LGD metric using multiscale analysis.
  - If $\mathfrak{S}$ is a subbox of $\mathfrak{R}$, we would like to be able to relate $d^{LGD}_{\mathfrak{R}^*,\mathfrak{S},\delta}$ and $d^{LGD}_{\mathfrak{S}^*,\mathfrak{S},\delta}$.
- There is a relationship between $h_{\mathfrak{R}^*}$ and $h_{\mathfrak{S}^*}$: we can couple the GFFs so that $h_{\mathfrak{R}^*}\restriction_{\mathfrak{S}^*} = h_{\mathfrak{S}^*} + h_{\mathfrak{R}^*:\mathfrak{S}^*}$, where $h_{\mathfrak{R}^*:\mathfrak{S}^*}$ ("coarse field") is the harmonic interpolation of $h_{\mathfrak{R}^*}$ on $h_{\mathfrak{S}^*}$ (hence smooth). Moreover, $h_{\mathfrak{R}^*:\mathfrak{S}^*}$ and $h_{\mathfrak{S}^*}$ ("fine field") are independent.
  - Compare this to the branching random walk picture.
  - This means that

$$d^{LGD}_{\mathfrak{R}^*,\mathfrak{S},\delta''} \le d^{LGD}_{\mathfrak{R}^*,\mathfrak{S},\delta} \le d^{LGD}_{\mathfrak{R}^*,\mathfrak{S},\delta'},$$

where

$$\delta'' = \exp\left\{-\gamma \min_{x\in\mathfrak{S}} h_{\mathfrak{R}^*:\mathfrak{S}^*}(x)\right\}, \quad \delta' = \exp\left\{-\gamma \max_{x\in\mathfrak{S}} h_{\mathfrak{R}^*:\mathfrak{S}^*}(x)\right\}.$$

Multiscale analysis: the effect of the coarse field

- $d^{LGD}_{\mathfrak{R}^*,\mathfrak{S},\delta''} \le d^{LGD}_{\mathfrak{R}^*,\mathfrak{S},\delta} \le d^{LGD}_{\mathfrak{R}^*,\mathfrak{S},\delta'}$, where
  $\delta'' = \exp\{-\gamma \min_{x\in\mathfrak{S}} h_{\mathfrak{R}^*:\mathfrak{S}^*}(x)\}\,\delta$,
  $\delta' = \exp\{-\gamma \max_{x\in\mathfrak{S}} h_{\mathfrak{R}^*:\mathfrak{S}^*}(x)\}\,\delta$.
- Also have *scaling property of LQG*:

$$d^{LGD}_{\alpha\mathfrak{R}^*,\alpha\mathfrak{S},\alpha^{2+\gamma^2/2}\delta} \overset{\text{law}}{=} d^{LGD}_{\mathfrak{R}^*,\mathfrak{S},\delta}.$$

- Maximum of the Gaussian free field: if $\mathfrak{S}_1,\ldots,\mathfrak{S}_{K^2}$ is a set of translates of $\frac{1}{K}\mathfrak{R}$, all contained in $\mathfrak{R}$, then we have

$$\min_{i=1}^{K^2} \min_{x\in\mathfrak{S}} h_{\mathfrak{R}^*:\mathfrak{S}^*}(x) \gtrsim -2\log K; \quad \max_{i=1}^{K^2} \max_{x\in\mathfrak{S}} h_{\mathfrak{R}^*:\mathfrak{S}^*}(x) \lesssim 2\log K$$

  with Gaussian concentration.
- Putting this all together: we have that

$$d^{LGD}_{\alpha_i\mathfrak{S}_i^*,\alpha_i\mathfrak{S}_i,\delta} \lesssim d^{LGD}_{\mathfrak{R}^*,\mathfrak{S}_i,\delta} \lesssim d^{LGD}_{\beta_i\mathfrak{S}^*,\beta_i\mathfrak{S}_i,\delta},$$

  where (since $\gamma < 2$)

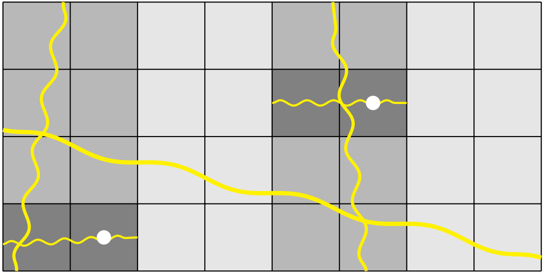
$$\max_i \beta_i \le K^{\frac{2\gamma}{2+\gamma^2/2}} < K.$$

Subcriticality leads to averaging

- Recall that $\mathfrak{S}_1, \ldots, \mathfrak{S}_{K^2}$ is a set of translates of $\frac{1}{K}\mathfrak{R}$.
- $d^{LGD}_{\alpha_i \mathfrak{S}_i^*, \alpha_i \mathfrak{S}_i, \delta} \leq d^{LGD}_{\mathfrak{R}^*, \mathfrak{S}_i, \delta} \lesssim d^{LGD}_{\beta_i \mathfrak{S}^*, \beta_i \mathfrak{S}_i, \delta}$, where $\max_i \beta_i \leq K^{\frac{\gamma}{2+\gamma^2/2}} < K$.
- "Subboxes look smaller than the overall box, even after considering the coarse field."
- Idea: the crossing weight across a large box should feel the weight from many subboxes, and thus should be doing some kind of averaging, leading to concentration.

Chaining argument

- Mechanism to show tightness: show that the metric is Hölder continuous.
- Use a chaining argument:



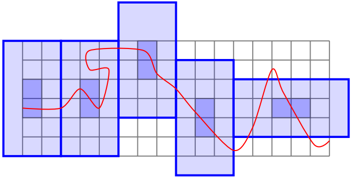
- Ingredients:
  - Concentration for the left–right crossing distance at each scale.
  - Crossing distances grow at least polynomially in the scale:

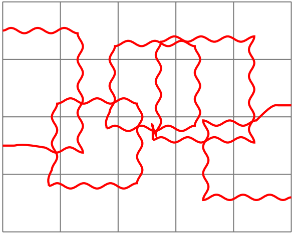
$$\text{median}(d^{LGD}_{K\mathfrak{R}^*,K\mathfrak{R},\delta}) \gtrsim K^c \text{median}(d^{LGD}_{\mathfrak{R}^*,\mathfrak{R},\delta})$$

    - $\Longrightarrow$ summability.

Percolation arguments

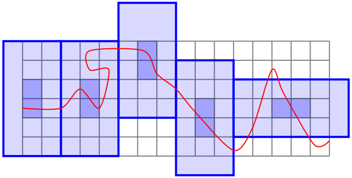
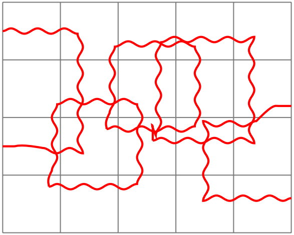
- Left–tails and right–tails of the crossings, as well as relationships between crossing distances at different scales, can be obtained through “percolation arguments”:



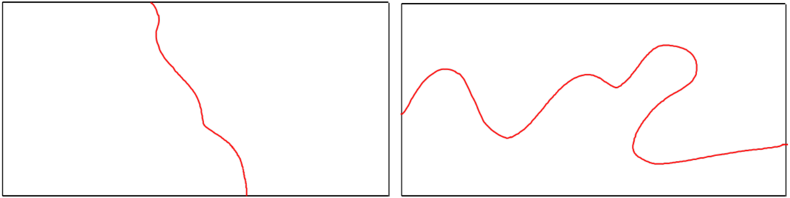


- A left–right crossing of the large box must include many left–right crossings of smaller boxes.
- A left–right crossing of the large box can be assembled from left–right circuits around smaller annuli.
- Similar to strategies used extensively before, most closely in Ding–D. high-temperature result and Dubédat–Falconet result for *-scale invariant fields.

Concentration through percolation

-  
- Problem: the left and right tails aren't in terms of quite the same thing.
  - Left tail in terms of *small* quantiles of *"easy"* crossings at smaller scales.
  - Right tail in terms of *large* quantiles of *"hard"* crossings at smaller scales.
- So... need to relate easy and hard crossings, and need to relate small and large (but fixed) quantiles.

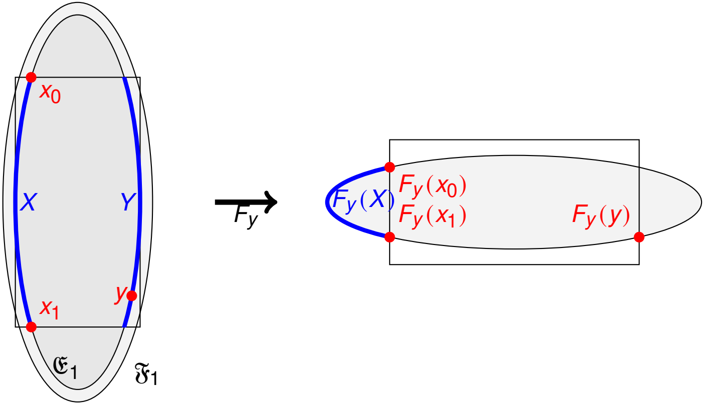
RSW result: relate easy and hard crossing quantiles



- First results: Russo, Seymour, Welsh for Bernoulli percolation in 1979–'80.
- Ding–D. high-temperature result for LFPP: adapted intricate inductive RSW result due to V. Tassion (2016).
- Dubédat–Falconet: introduced very simple RSW proof based on conformal invariance.
  - We adapt the Dubédat–Falconet proof to Liouville graph distance.

RSW strategy

- Dubédat–Falconet RSW strategy: map



- A positive fraction of easy–crossing geodesics can be mapped to hard–crossing geodesics if y is chosen correctly.
- Uses approximate conformal invariance of LGD (coming from conformal invariance of GFF).

Relating small and large quantiles

- In order to tie together the percolation arguments and the RSW arguments, we need to relate different quantiles of the crossing distances.
- To do this, we seek to bound $\mathrm{Var}\left(\log d_{\mathfrak{R}^*,\mathfrak{R},\delta}(\text{left–right})\right)$.
- Then $p < q$ quantiles of $d_{\mathfrak{R}^*,\mathfrak{R},\delta}(\text{left–right})$ are related by a multiplicative factor of at most

$$\exp\left\{\left(q^{-1/2} + (1-q)^{-1/2}\right)\sqrt{\mathrm{Var}\left(\log d_{\mathfrak{R}^*,\mathfrak{R},\delta}(\text{left–right})\right)}\right\}.$$

Efron–Stein argument

- The field can be subdivided into a sum of "fine" and "coarse" pieces using the "white-noise decomposition."
- "Fine" here means concentrated on boxes a factor of K smaller, where K is very large but fixed.
- Efron–Stein inequality:

$$\mathrm{Var}\left(\log d_{\mathfrak{R}^*,\mathfrak{R},\delta}(\text{left–right})\right) \le \sum_i \mathbf{E}\left(\log \frac{\tilde{d}^{(i)}_{\mathfrak{R}^*,\mathfrak{R},\delta}(\text{left–right})}{d_{\mathfrak{R}^*,\mathfrak{R},\delta}(\text{left–right})}\right)^2,$$

  where $\tilde{d}^{(i)}_{\mathfrak{R}^*,\mathfrak{R},\delta}$ is the metric where piece i has been resampled.
- So the goal becomes to control the ratio between the resampled and original distances.

Resampling the coarse field

- The coarse field is smooth, and when it is resampled it has a Lipschitz effect on the crossing distance.
- Gaussian concentration and a variance bound on the coarse field implies that the variance coming from resampling the coarse field is bounded by $\log K$.

Resampling the fine field

- When we resample the fine field in a box, the weight coming from most of the geodesic (the part far away from where the field is resampled) doesn't change much.
- On the other hand, we can bound the change in the weight of the geodesic close to the resampled location by replacing the path with an annular crossing in that region.
- So the variance coming from resampling the small boxes is like

$$\sum_i \mathbf{E}\left(\frac{\tilde{d}^{(i)}_{\mathfrak{R}^*,\mathfrak{S}_i,\delta}(\text{circuit})}{d_{\mathfrak{R}^*,\mathfrak{R},\delta}(\text{left--right })}\right)^2 \leq \left(\sum_i \frac{\tilde{d}^{(i)}_{\mathfrak{R}^*,\mathfrak{S}_i,\delta}(\text{circuit})}{d_{\mathfrak{R}^*,\mathfrak{R},\delta}(\text{left--right })}\right)\left(\max_i \frac{\tilde{d}^{(i)}_{\mathfrak{R}^*,\mathfrak{S}_i,\delta}(\text{circuit})}{d_{\mathfrak{R}^*,\mathfrak{R},\delta}(\text{left--right })}\right).$$

- Here, $\mathfrak{C}_i$ is a box containing the area where the field is resampled, and $\tilde{d}^{(i)}_{\mathfrak{R}^*,\mathfrak{S}_i,\delta}(\text{circuit})$ is the weight of an annular circuit around $\mathfrak{C}_i$.

The effect of subcriticality

- In the inequality

$$\sum_i \mathbf{E}\left(\frac{\tilde{d}^{(i)}_{\mathfrak{R}^*,\mathfrak{S}_i,\delta}(\text{circuit})}{d_{\mathfrak{R}^*,\mathfrak{R},\delta}(\text{left–right })}\right)^2 \leq \left(\sum_i \frac{\tilde{d}^{(i)}_{\mathfrak{R}^*,\mathfrak{S}_i,\delta}(\text{circuit})}{d_{\mathfrak{R}^*,\mathfrak{R},\delta}(\text{left–right })}\right)\left(\max_i \frac{\tilde{d}^{(i)}_{\mathfrak{R}^*,\mathfrak{S}_i,\delta}(\text{circuit})}{d_{\mathfrak{R}^*,\mathfrak{R},\delta}(\text{left–right })}\right),$$

  the first factor is on the order of $\exp\left\{C\sqrt{V}\right\}$, where V is the variance bound inductively attained at the smaller scale.
  - That is, by the concentration at the smaller scale, size of first factor does not depend on K, but it does depend on the previously-attained variance bound which is necessary for the concentration at the smaller scale.
- The second factor is order $\exp\left\{C\sqrt{V}\right\} K^{-c}$, because, as previously discussed, boxes at the smaller scale “look smaller,” even after considering the coarse field.

The bound on the variance of the logarithm

- Combining fine- and coarse-field contributions, obtain

$$V' := \mathrm{Var}\left(\log d_{\mathfrak{R}^*,\mathfrak{R},\delta}(\text{left–right}\ )\right) \le C\left(\log K + e^{C\sqrt{V}}K^{-c}\right).$$

- In order to close the induction, need a V so that

$$C\left(\log K + e^{C\sqrt{V}}K^{-c}\right) \le V.$$

- V can be taken to be a small constant times $\log K$.
- Actually, there is an additional term in the variance bound, which goes to 0 as $S \to \infty$ but to ∞ as $K \to \infty$.
    - Comes from the inability of the path to be subdivided in the presence of LGD balls with large Euclidean diameter.
    - Thus the base case of the induction imposes additional challenges: use an *a priori* bound of J. Ding–O. Zeitouni–F. Zhang (preprint, 2018).

Further questions

- Can the result be extended toto other similar discrete approximations? To other models?
  - What are good techniques for RSW results in the absence of conformal invariance?
- What are the scaling exponents?
- Any question you want to ask about the limiting metrics.