KPZ fluctuations and semi-localization

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KPZ meet KPZ 2024

Directed polymer, 1+1 KPZ universality class

Statistical physics model of path in random environment

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Question: is there a relation between localization and KPZ scaling?

Plan of the talk

- questions I'd like to ask
- what we can do

Poincare inequality and replica overlap

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- localization provides an incorrect upper bound

A variance identity: Clark-Ocone formula

Stochastic calculus: let W be a BM, X be "smooth" r.v. depends on $\{W_t\}_{t\geq 0}$:

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- in spacetime setting $Z_t = \mathbb{E} e^{\int_0^t \xi(s,B_s)ds}$

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  $D_{s,y}\log Z_t$: quenched density of polymer at (s,y)

Polymer path overlap again

Continuous setting: $Z_t = \mathbb{E} e^{\int_0^t \xi(r, B_r) dr}$

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- it reduces to study how the random density $\mathbf{E}[\rho_t(s,y)|\mathcal{F}_s]$ overlap with itself

Three levels of localization

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measures the overlap of the quenched density (complete localization)
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- The KPZ fluctuation $\mathrm{Var}\log Z_t \sim t^{2/3}$ is in between

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- in the identity $\log Z_t - \mathbf{E}\log Z_t = \int_0^t \int_{\mathbb{R}} \mathbf{E}[\rho_t(s,y)|\mathcal{F}_s]\xi(s,y)dyds$, there is no correlation

zero temperature/LPP version

Quenched density

$$\rho_t(s,y) = \frac{\mathbb{E}[e^{\beta \int_0^t \xi(r,B_r)dr}\delta(B_s - y)]}{Z_t}$$

$\beta = \infty$: $\rho_t(s,y) = \delta(\pi_s - y)$ becomes the Dirac mass along the (random) geodesics

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- expect $\sum_y |\mathbf{E}[\rho_t(s,y)|\mathcal{F}_s]|^2 \sim s^{-1/3}$ (semi-localization, but why?)

Question: for each realization of random environment, the geodesic and its midpoint is given, but if we average half of random environment out, what does the midpoint look like?

Geometric questions to ask

midpoint density

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half-averaged midpoint density $\tilde{\rho}_t(s,y) := \mathbf{E}[\rho_t(s,y)|\mathcal{F}_s]$

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- I don't know if it's easier to approach these questions from geometric or analytic perspective

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- we work on a continuous model which is the KPZ equation

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- in our work we will work with white noise in $d = 1$, half of our argument works for general noise and dimensions

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- on torus, unique invariant measure $e^{B(x)}/\int e^{B(x')}dx'$ with Brownian bridge B:

$$\rho(t,x) = \frac{Z(t,x)}{\int Z(t,x')dx'} = \frac{e^{h(t,x)}}{\int e^{h(t,x')}dx'} = \frac{e^{h(t,x)-h(t,0)}}{\int e^{h(t,x')-h(t,0)}dx'}$$

What happens on a torus: CLT

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- explicit drift (using Marc Yor's density formula)

$$\gamma_L = \tfrac{1}{2}\beta^2 L \mathbb{E} \frac{1}{(\int_0^L e^{\beta B(x)} dx)^2} = \frac{\beta^2}{2L} + \frac{\beta^4}{24}$$

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  balance $t/\sqrt{L} \sim L$ leads to $1:2:3$

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- much more precise results on periodic TASEP in all regimes by Baik-Liu, Baik-Liu-Silva

Effective diffusivity for the height function

h solves KPZ with noise ξ

$$h(t,x) - \mathbf{E}h(t,x) = \int_0^t \int_{\mathbb{T}} \mathbf{E}[D_{s,y}h(t,x)|\mathcal{F}_s]\xi(s,y)dyds$$

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- midpoint distribution (for $s \gg 1$ and $t - s \gg 1$)

$$D_{s,y}h(t,x) \approx \frac{e^{B_1(y)+B_2(y)}}{\int e^{B_1(y')+B_2(y')}dy'}$$

so we have $\mathbf{E}[D_{s,y}h(t,x)|\mathcal{F}_s] \approx \mathbf{E}[\frac{e^{B_1(y)+B_2(y)}}{\int e^{B_1(y')+B_2(y')}dy'}|B_1]$

Effective diffusivity for the height function

h solves KPZ with noise ξ

$$h(t,x)-\mathbf{E}h(t,x)=\int_0^t\int_{\mathbb{T}}\mathbf{E}[D_{s,y}h(t,x)|\mathcal{F}_s]\xi(s,y)dyds$$

$D_{s,y}h(t,x)$: quenched (midpoint) density of polymer at (s,y)

- endpoint distribution (for $t \gg 1$)

$$\frac{Z(t,x)}{\int Z(t,x')dx'}=\frac{e^{h(t,x)}}{\int e^{h(t,x')}dx'}=\frac{e^{h(t,x)-h(t,0)}}{\int e^{h(t,x')-h(t,0)}dx'}\approx\frac{e^{B(x)}}{\int e^{B(x')}dx'}$$

- midpoint distribution (for $s \gg 1$ and $t-s \gg 1$)

$$D_{s,y}h(t,x)\approx\frac{e^{B_1(y)+B_2(y)}}{\int e^{B_1(y')+B_2(y')}dy'}$$

  so we have $\mathbf{E}[D_{s,y}h(t,x)|\mathcal{F}_s]\approx\mathbf{E}[\frac{e^{B_1(y)+B_2(y)}}{\int e^{B_1(y')+B_2(y')}dy'}|B_1]$

- take the rhs, square it, integrate in y, take the expectation, obtain σ^2