Free Curves, Eigenschemes and Pencils of Curves

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Tangent derivations to a divisor

- $\mathbb{K}$: algebraically closed field with characteristic 0
- $R = \oplus_{k \geq 0} R_k = \mathbb{K}[x_0, \ldots, x_n]$: $\mathbb{Z}$-graded ring in $n+1$ variables
- $\mathbb{P}^n = \mathbb{P}(R)$: projective space of dimension n

Definition

For R as above, let $d \in \mathbb{N}$, the module of $\mathbb{K}$-derivations

$$\mathrm{Der}_{\mathbb{K}}(R) = \{\sum_{j=1}^{n} P_{ij} \frac{\partial}{\partial x_j} | P_{ij} \in R_d\}$$

is free of rank $n+1$, with basis $\{\partial_{x_0}, \ldots, \partial_{x_n}\}$.

Let $D = V(f)$ be a reduced (but not necessarily irreducible) divisor on X. The module of derivations tangent to D is

$$Der(f) = T_X(-\log D)) = \{\delta \in \mathrm{Der}_{\mathbb{K}}(R) | \delta(f) \in \langle f \rangle\}.$$

Freeness

Definition

The divisor $V(f)$ is *free* if $\mathrm{Der}(f)$ is a free R-module.

The Euler derivation

$$\delta_E = \sum_{i=0}^{n} x_i \frac{\partial}{\partial x_i}$$

satisfies $\delta_E(f) = d \cdot f$. Hence, for any $\delta \in \mathrm{Der}(f)$ we have the decomposition

$$\delta = \delta' + \frac{\delta(f)}{d \cdot f}\delta_E, \text{ with } \delta' = \delta - \frac{\delta(f)}{d \cdot f}\delta_E \text{ and } \delta'(f) = 0,$$

Freeness, continued

$$\mathrm{Der}(f) = R\delta_E \oplus \mathrm{Der}_0(f)$$

where

$$\mathrm{Der}_0(f) = \{\delta \in \mathrm{Der}_{\mathbb{K}}(R) \mid \delta(f) = 0\}.$$

$$\delta \in Der_0(f) \Leftrightarrow \delta \in Syz(\nabla f)$$

Definition

$V(f)$ is free with exponents $(a_1, \ldots, a_n)$

$$\Updownarrow$$

$$Der_0(f) = R(-a_1) \oplus \cdots \oplus R(-a_n)$$

$$\Sigma_i a_i = \deg(V(f)) - 1$$

Freeness, continued

Let $\nabla(f) = (\partial_{x_0} f, \ldots, \partial_{x_n} f)$ be the vector of partial derivatives, which generate the Jacobian ideal. So $\mathrm{Der}_0(f)$ is simply the kernel of the Jacobian map, aka the *syzygies* on $\nabla(f)$:

$$0 \to Der_0(f) \to R^{n+1} \stackrel{\nabla(f)}{\to} R(d-1)$$

WHO CARES: In a 1980 *Inventiones* paper, Orlik-Solomon determined $H^*(\mathbb{P}^n \setminus A, \mathbb{Z})$ for an arrangement A, and in a 1981 *Inventiones* paper, Terao showed that if A is free, then the betti numbers are determined by the exponents.

Conjecture

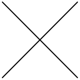
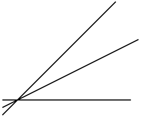
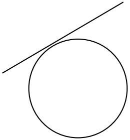
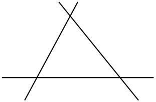
[Terao(1981)]. *Freeness depends only on the intersection lattice.*

Curves in $\mathbb{P}^2$

Some combinatorics:
Let $\deg f = d$ and $Der_0(f) = R(-a) \oplus R(-b)$ with $a, b \geq 0$

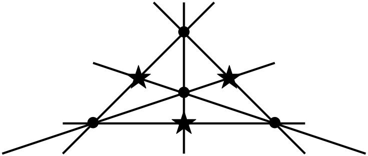
- $$a + b = d - 1$$
- $$ab = \binom{d-1}{2} - \Sigma_{p \in Sing(f)} \binom{m_p - 1}{2}$$
  where m_p is the multiplicity of p on $V(f)$. For a line arrangement, the m_p can be determined from the Möbius function.

First examples in $\mathbb{P}^2$

-  is free $Der(f) = R \oplus R(-1)$
-  is free $Der(f) = R \oplus R(-2)$
-   are free $Der(f) = R^2(-1)$
- Not free: smooth irreducible curve of degree ≥ 3 (e.g. $f = x^3 + y^3 + z^3$), or only nodes/cusps (Dimca-Sernesi 2014).

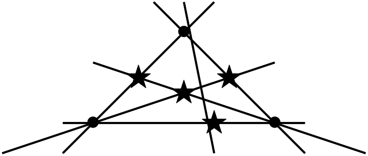
Freeness for some line arrangements

The divisor D consisting in the 6 lines $xyz(x-y)(x-z)(y-z)$, four triple points and 3 double points is free with exponent $(2,3)$



The divisor consisting in the 6 lines $xyz(x-y)(x-z)(y-2z)$ with 3 triple points and 6 double point cannot be free (it should have

$a+b=5$ and $ab=\binom{5}{2}-3=7$



Saito's criterion

Theorem

A reduced curve $C = V(f)$ *of degree* d *is free with exponents* (a, b)

$$\Updownarrow$$

there exist two derivations $\delta = P_1\partial_x + P_2\partial y + P_3\partial z \in Der(f)_a$ *and* $\mu = Q_1\partial x + Q_2\partial y + Q_3\partial z \in Der(f)_b$ *such that*

$$\det \begin{pmatrix} x & P_1 & Q_1 \\ y & P_2 & Q_2 \\ z & P_3 & Q_3 \end{pmatrix} = cf \;\; c \in \mathbb{K}^*,$$

This is the Hilbert-Burch theorem for codimension two Cohen-Macaulay ideals, and Euler's relation.

History

Even for hyperplane arrangements, freeness is a subtle property which is generally nontrivial to verify.

- 1970's: Saito: Grothendieck algebraic de Rham, Deligne: Reflection arrangements.
- 1980's: Orlik-Solomon cohomology ring, Saito LCS formula
- 1980's: Brylawski, Reiner, Stanley, Ziegler combinatorics, supersolvability
- 1980's: Saito, Falk-Randell LCS formula, Terao Freeness.
- $\vdots$
- Abe, D.&F.Cohen, DeConcini-Procesi, Dimca, Denham, Feichtner, Huh, Libgober, Mattei, Mustata, Papadima, Suciu, Yoshinaga, Yuzvinsky......
- Question: what about *hypersurfaces*?

Curves and Quasihomogenity

Recent interest in *hypersurface* arrangements:

- -, Tohaneanu: line-conic arrangements: freeness is not combinatorial.
- -,Terao, Yoshinaga: deletion/restriction theorems for curves.
- Valles: pencils of curves.

Most results on curve arrangements require constraints on the singularities: the singularities must be *quasihomogeneous*.

Definition

The point $(0,0)$ is a quasihomogeneous singularity of $V(f)$ iff f is weighted homogeneous: if $f(x;y) = \sum c_{ij}x^i y^j$ then there exist $\alpha, \beta \in \mathbb{Q}$ such that $\sum c_{ij}x^{i\alpha}y^{j\beta}$ is homogeneous.

Milnor and Tjurina numbers

Definition

Let $p = (0,0)$ be a singular point of a plane curve, and $C\{x,y\}$ the ring of convergent power series ; then $deg(J) = \sum_{P \in Sing(D)} \tau_P$ where

$$\text{Tjurina number: } \tau_p(C) = \dim \frac{\mathbb{K}\{x,y\}}{(f, \partial_x(f), \partial_y(f))}$$

and

$$\text{Milnor number: } \mu_p(C) = \dim \frac{\mathbb{K}\{x,y\}}{(\partial_x(f), \partial_y(f))}$$

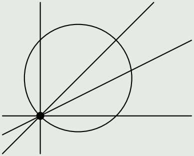
So $\tau_p(C) \leq \mu_p(C)$, and $p \in C$ smooth $\Rightarrow \quad \tau_p(C) = 0 = \mu_p(C)$

Reiffen (1967), Saito(1971) $p \in Sing(C)$ is quasihomogeneous $\Leftrightarrow \quad \tau_p(C) = \mu_p(C)$

A non quasihomogeneous singularity

Example

Let $C = V(xy(x-y)(x-2y)(x^2 - xz + y^2 - yz))$ be as below:



C has five singular points, all ordinary. When p is an ordinary singularity and C has n distinct branches at p, then $\mu_p(C) = \binom{n-1}{2}$, so the sum of the Milnor numbers is 20. However, $deg(J) = 19$; at $(0:0:1)$ we have $\mu = 16$ but $\tau = 15$.

Free line arangements

- The Ceva-Braid arrangement

$$xyz(x-y)(x-z)(y-z)=0$$

  is free with exponents $(2,3)$.

- The Hesse arrangement

$$\prod_{\epsilon=\infty,1,j,j^2}(x^3+y^3+z^3-3\epsilon xyz)=0$$

  is free with exponents $(4,7)$.

- The Fermat arrangement

$$(x^n-y^n)(x^n-z^n)(y^n-z^n)=0$$

  is free with exponents $(n+1,2n-2)$.

Artal and Cogolludo observed that each of the three divisors above is the union of all the singular members of suitable pencils

Theorem (Valles)

Let f, g two reduced polynomials in R_d such that $B = V(f) \cap V(g)$ is smooth (so consists of d^2 distinct points).

Denote by D the union of all the singular curves of the pencil (a $\mathbb{P}^1$ with coordinates $(s : t)$)of curves $sf + tg$
and
D_k the union of $k \geq 2$ curves in the pencil, then

$$D_k \text{ is free with exponents } (2d-2, d(k-2)+1)$$

$$\Updownarrow$$

$$f\ D \subset D_k \text{ and the singularities of } D \text{ are quasihomogeneous.}$$

Dropping the quasihomogeneous condition: eigenschemes

Definition

The eigenscheme associated to three homogeneous polynomials $\mathcal{P} = (P_1, P_2, P_3)$ of the same degree $n \geq 1$ is the closed subscheme $\Gamma_{\mathcal{P}} \subset \mathbb{P}^2$ defined by the 2×2 minors of the matrix

$$M = \begin{pmatrix} x & P_1 \\ y & P_2 \\ z & P_3 \end{pmatrix}.$$

- C a curve $\subseteq V(P_1, P_2, P_3) \Rightarrow C$ is in the eigenscheme $\Gamma_{\mathcal{P}}$.
- Even if $V(P_1, P_2, P_3)$ is a finite set of points or empty, the eigenscheme $\Gamma_{\mathcal{P}}$ may be of codimension one: for example given f, g, Q_1, Q_2, Q_3 homogeneous forms such that

  $$P_1 = xf + gQ_1, P_2 = yf + gQ_2, P_3 = zf + gQ_3,$$

  then the eigenscheme of (P_1, P_2, P_3) clearly contains $V(g)$.

Finite eigenscheme

If $\Gamma_{\mathcal{P}}$ is a finite scheme, then $\Gamma_{\mathcal{P}}$ is defined by

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-n) \to \mathcal{O}_{\mathbb{P}^2}^3 \to \mathcal{J}_\Gamma(n+1) \to 0,$$

and $length(\Gamma) = c_2(\mathcal{J}_\Gamma(n+1)) = 1 + n + n^2$.

Lemma

Assume that Γ is a finite scheme. Let $f \in R_N$ with $N \geq n+1$. Then $f \in H^0(\mathcal{J}_\Gamma(N))$ if and only if there exists

$$(Q_1, Q_2, Q_3) \in R^3_{N-(n+1)}$$

such that

$$\det \begin{pmatrix} x & P_1 & Q_1 \\ y & P_2 & Q_2 \\ z & P_3 & Q_3 \end{pmatrix} = cf, \;\; c \in \mathbb{K}^*$$

First eigenscheme theorem

Theorem

Let $\delta = P_1\partial_x + P_2\partial y + P_3\partial z$ be a non zero irreducible derivation of degree $a \geq 1$ such that it's eigenscheme Γ_δ is a finite scheme. Let $f \in R_{d \geq a+1}$, such that $\delta(f) = 0$ then

$$V(f) \text{ is free with exponents } (a, d-a-1) \Leftrightarrow f \in I_{\Gamma_\delta}.$$

Fixing the exponents is necessary: Suppose $V(f)$ is a free curve of degree 5 with exponents $(2, 2)$, with μ and ν generators for $Der(f)_2$; $\delta = x\mu + y\nu$ is a derivation of degree 3.

$V(f)$ is free but its exponents are not $(3, 1)$ then $f \notin I_{\Gamma_\delta}$.

In addition, if $\delta(f) = 0$ with δ of degree d with $d \leq a$ then f can be free but δ will not be a generator of $Der(f)$, for degree reasons.

Pencils, canonical derivations, and eigenschemes

Let $f \in R_n$ and $g \in R_m$ with no common factor. A key point in Vallé's results on pencils is the existence of a *canonical derivation* associated to the pencil generated by f and g:

$$\delta_{f,g} := [\nabla f \wedge \nabla g].\nabla = \det \begin{pmatrix} \partial_x f & \partial_x g & \partial_x \\ \partial_y f & \partial_y g & \partial_y \\ \partial_z f & \partial_z g & \partial_z \end{pmatrix}.$$

Let $\delta_{f,g}$ denote the eigenscheme associated to the canonical derivation, and $\mathcal{B} = V(f) \cap V(g)$. Hence

$$\delta_{f,g} = \mathcal{B} \cup V(\nabla f \wedge \nabla g)$$

set theoretically, where $\mathcal{B}$ is a finite set. But $V(\nabla f \wedge \nabla g)$ may not be finite. When $n = m$, $V(\nabla f \wedge \nabla g)$ contains a curve if and only if the pencil contains non-reduced curve.

Pencils, cont.

Proposition

Let $f \in R_n$ and $g \in R_m$ be two reduced polynomials without common factors such that $V(\nabla f \wedge \nabla g)$ is a finite scheme. Let a and b integers such that $\operatorname{lcm}(n, m) = a \times n = b \times m$. Let Γ be the eigenscheme associated to $\delta_{f,g}$.

1. *The number of curves different from $V(g^b)$ and $V(f^a)$ in the pencil $\mathcal{C} = (f^a, g^b)$ that are singular outside the base locus $\mathcal{B} = V(f) \cap V(g)$ is finite and bounded by*

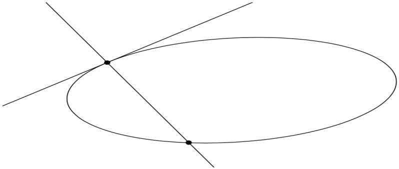
$$(n-1)^2 + (n-1)(m-1) + (m-1)^2,$$

   which is the length of the scheme $\mathcal{Z} = V(\nabla f \wedge \nabla g)$.

2. *The scheme Γ is the union of the schemes $\mathcal{B}$ and $\mathcal{Z}$.*

Example

Consider a pencil of osculating conics. Up to a linear transformation, these conics can be defined by $f = xz$ and $g = z^2 - xy$. The canonical derivation δ_{fg} has degree 2 and the associated eigenscheme Γ has length 7 and consists of one smooth point (the intersection point where there is no tangency) and a subscheme of length 6 supported at the point of tangency.



The ideal defining the eigenscheme is

$$I_\Gamma = \langle x(z^2 + xy), x^2 z, z^3 \rangle, \text{ which we write as } (u, v, w).$$

The equation of the curve $fg = xz^3 - x^2yz = xw - yv$ belongs to I_Γ proving that $fg = 0$ is free with exponents $(1, 2)$.

Example, continued

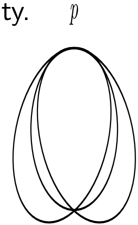
The union of 3 smooth curves of the pencil is also free with exponents $(2, 3)$.

$g = 0$, $f + g = 0$ and $f - g = 0$. Then

$$g(f+g)(f-g) = w^2 - v^2 - yz(x - 4y + 3z)v - xy^2u \in I_\Gamma,$$

proving that $g(f+g)(f-g) = 0$ is free with exponents $(2, 3)$.

We note that two smooth osculating curves are not free: they meet in degree 4 along the singular point instead of degree 6. Adding a third smooth curve allows us to reach degree 6. At the singular point p, $\tau_p = 15$ and $\mu_p = 16$; this shows that p is not a quasihomogeneous singularity.



Main theorem

Theorem

Let $f \in R_n$ and $g \in R_m$ be two reduced polynomials without common factors such that $V(\nabla f \wedge \nabla g)$ is a finite scheme. Let $V(F_k)$ be the union of $k \geq 2$ curves in the pencil generated by (f^a, g^b) where $\mathrm{lcm}(n, m) = a \times n = b \times m$ and let F be a polynomial of degree $N > n + m - 1$ such that $F \mid F_k$. Then

$$V(F) \text{ is free with exponents } (n + m - 2, N - n - m + 1)$$

$$\Updownarrow$$

$$F \in (I_{\Gamma_{\delta_{f,g}}})_N.$$

Application: intersection of two conics

- $n = m = 2$
  A union of conics and lines coming from the pencil will be free

  $$\Updownarrow$$

  it contains the eigenscheme.
- when such a union is free, adding a conic or a line from the pencil to this union yields another free arrangement.
- More generally, when a union of curves from a pencil is free, it will remain free by adding smooth curves from the pencil.

Two conics, cont.

As a consequence, it follows that the following types of unions of smooth conics C, D are free:

- If $C \cap D$ consists of 2 points, one simple and one triple point, three smooth members of the pencil are needed to contain the eigenscheme

$$\implies \text{ the union is free with exponents } (2, 3).$$

- If $C \cap D$ meets in a quadruple point, then the union of two smooth members of the pencil contains the eigenscheme, which is of length 3

$$\implies \text{ the union is free with exponents } (1, 2).$$