2. Partition Theory and Banach Spaces

We shall start with a sketch of another proof of the theorem proved last time.

2.1 Theorem. *(Erdös-Rado) If* $f: [\mathbb{N}]^d \to Y$*, then there exists* $A \in [\mathbb{N}]^\omega$ *such that* $f\lceil[A]^d$ *is canonical.*

Proof. By induction on d. Case $d = 1$ is trivial. Assume the statement is true for d. Fix $f: [\mathbb{N}]^{d+1} \to Y$. For $s = \{a_1, \ldots, a_{2d+2}\}_< \in [\mathbb{N}]^{2d+2}$ define an equivalence relation on $[\{1, \ldots, 2d+2\}]^d$ by $u \sim v$ iff $f(\{a_i | i \in u\}) = f(\{a_i | i \in v\})$. Define function $g: [\mathbb{N}]^{2d+2} \to \mathcal{E}$ ($\mathcal{E}$ = all equivalence relations on $[2d+2]^d$). Apply Ramsey Theorem to get $A \in [\mathbb{N}]^\omega$, which is g-homogeneous; If $f\lceil[A]^{2d+2}$ is one-to-one, then we are done ($X = d+1$). Otherwise, there are $s, t \in [A]^d$, $s \neq t$, such that $f(s) = f(t)$. Let $j < d+1$ be the first place at which s and t disagree. Prove that $f(u)$ does not depend on the j-th coordinate of u and apply the induction hypothesis. □

2.2 Lemma. *If* $d \in \mathbb{N}$, $f: [\mathbb{N}]^d \to \mathbb{R}$*, and the range of* f *is bounded, then there is an infinite subset* $A \in [\mathbb{N}]^\omega$ *such that* $\lim_{\min s \to \infty, s \in [A]^d} f(s)$ *exists.*

Proof. By induction. Case $d = 1$ is the same as the last proof. Fix $f: [\mathbb{N}]^{d+1} \to \mathbb{R}$. For $m \in \mathbb{N}$ define $g_m: [\mathbb{N}/m]^d \to \mathbb{R}$, by $g_m(s) = f(\{m\} \cup s)$. By the induction hypothesis $\mathcal{F}_m = \{B \in [\mathbb{N}]^\omega | \lim_{\min s \to \infty, s \in [B]^d} g_m(s) = r_m$ for some $r_m \in \mathbb{R}\}$ is dense. Let C be such that $C/m \in \mathcal{F}_m$ for every $m \in C$. Find $A \in [C]^\omega$ such that $\lim_{m \in A, m \to \infty} r_m = r$ for some r (case $d = 1$). Then A has the required properties. □

The basic sequences (w_i) and (v_i) are k-equivalent if $\forall (a_i)$

$$1/k|||\Sigma_{i=1}^\infty a_i w_i||| \leq |||\Sigma_{i=1}^\infty a_i v_i||| \leq k|||\Sigma_{i=1}^\infty a_i w_i|||$$

2.3 Definition. A basic sequence $\{u_n\}$ is k-spreading ($k \in [1, \infty)$) if $\forall A \subset [\mathbb{N}]^\omega$ $\{u_n\}_{n=1}^\infty$ is k-equivalent to $\{u_n\}_{n \in A}$. 1-spreading is the same as spreading

2.4 Lemma. *The usual basic sequence in* c_0, $l_p (p = 1)$ *is* 1*-spreading.*

2.5 Remark. If $\{u_i\}_{i=1}^\infty$ is equivalent (is k-equivalent for some k) to $\{u_{2i}\}_{i=1}^\infty$ and $\{u_{2i+1}\}_{i=1}^\infty$, then $X \cong X^2$.

2.6 Definition. A basic sequence $\{u_n\}_{n=1}^\infty$ in X is asymptotically spreading if there is a 1-spreading basic sequence $\{v_n\}_{n=1}^\infty$ in some Banach space Y such that $\forall k \forall \varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\{u_{n_1}, u_{n_2}, \ldots, u_{n_k}\}$ is $(1+\varepsilon)$-equivalent to $\{v_i\}_{i=1}^k$ whenever $N < n_1 < n_2 < \ldots < n_k$. Then $\{v_i\}_{i=1}^\infty$ is called a spreading model for $\{u_i\}_{i=1}^\infty$.

2.7 Lemma. *Some basic sequence in* l_1 *is spreading model for a basic sequence in Tsirelson's space.*

2.8 Theorem. *(Brunel-Sucheston)(a) If* $\{u_n\}_{n=1}^\infty$ *is a normalized basic sequence, then it has a subsequence that is asymptotically spreading.*

Proof. For every $k \in \mathbb{N}$ let D_k be a finite subset of $[0,1]^k$ that is $1/k$-dense in l_1-metric:

$$\forall (a_1, a_2, \ldots, a_k) \in [0,1]^k \exists (b_1, \ldots, b_k) \in D_k \text{ such that } \Sigma_{i=1}^k |a_i - b_i| < 1/k \ .$$

Fix k, $p \in D_k$, $p = (p_1, \ldots, p_k)$. Define $f_p: [\mathbb{N}]^k \to \mathbb{R}$ by $f_p(s) = \|\Sigma_{i=1}^k p_i u_{s(i)}\|$ (where $s = \{s(1), s(2), \ldots, s(k)\}_<$).

Note: $|f_p(s)| \leq \Sigma_{i=1}^k |p_i|$.

Let $\mathcal{F}_k = \{A \in [\mathbb{N}]^\omega | (\forall p \in D_k) \lim_{\min s \to \infty, s \in [A]^k} f_p(s) = r_p$ exists$\}$. Note that $\mathcal{F}_k$ is dense by Lemma 2.2. So, by diagonalization pick $A \in [\mathbb{N}]^\omega$ such that $A/k \in \mathcal{F}_k$ for all $k \in A$. Actually consider $\mathcal{F}'_k \subset \mathcal{F}_k$, $\mathcal{F}'_k = \{A \in [\mathbb{N}]^\omega | \lim_{\min s \to \infty, s \in [A]^k} f_p(s) = r_p$, $\forall s \in [A/k]$ $|f_p(s) - r_p| < 1/k$ $\}$ and pick A such that $A/k \in \mathcal{F}'_k$. Consider $c_{00} = \{\Sigma_{i=1}^\infty a_i e_i | (\forall^\infty i) a_i = 0\}$ and note that for $v \in c_{00}$, $v = \Sigma_{i=1}^k a_i e_i$. Define $|||\Sigma_{i=1}^k a_i e_i||| = r_p$, if $p = (a_1, \ldots, a_k)$. Note that $|||c \cdot \vec{x}||| = c|||\vec{x}|||$ for all $\vec{x}$ and rational scalars such that $|||c \cdot \vec{x}|||$ and $|||vecx|||$ are defined. Extend $||| \cdot |||$ to all of c_{00} by $|||\vec{x}||| = 1/c|||c\vec{x}|||$ for $c \in (0, \infty)$ such that $\|c\vec{x}\|_{l_\infty} < 1$. By the above this is well-defined. Then $\{e_i\}_{i=1}^\infty$ is 1-spreading.

$$|||\Sigma_{i=1}^k a_i e_i||| = |||\Sigma_{i=1}^k a_i e_{n_i}||| \ ,$$

if $n_1 < n_2 < \ldots < n_k$ and this is a spreading model for $\{u_n\}_{n \in A}$. □

2.9 Definition. A basic sequence $\{u_i\}_{i=1}^\infty$ is unconditional if

$$\forall i_0 \ \|\Sigma_{i=1}^k a_i u_i\| \geq \|\Sigma_{i=1, i \neq i_0} a_i u_i\| \ .$$

2.10 Theorem. *(Brunel-Sucheston)(b) In Theorem 2.9 if $\{u_n\}_{n=1}^\infty$ is weakly null, then the corresponding spreading model can be taken to be unconditional.*

Proof. We have $\{u_n\}_{n=1}^\infty$, $\{e_n\}_{n=1}^\infty$ spreading model and

* $\forall \varepsilon > 0, \forall k \in \mathbb{N}$ $\exists N = N_{(k,\varepsilon)} \in \mathbb{N}$ such that $\{u_{n_i}\}_{i=1}^k$ is $(1+\varepsilon)$ equivalent to $\{e_i\}_{i=1}^k$ if $N < n_1 < n_2 < \ldots < n_k$.

Since $\{u_i\}_{i=1}^\infty$ is weakly null, by a theorem of Mazur we can find convex combinations $\{z_m\}_{m=1}^\infty$ of $\{u_i\}_{i=1}^\infty$ that converge to 0 in norm. So there are $p_1 < p_2 < \ldots$ and $\gamma_i \geq 0$ such that:

$$z_i = \Sigma_{j=p_i}^{p_{i+1}-1} \gamma_j u_j \ , \quad \Sigma_{j=p_i}^{p_{i+1}-1} \gamma_j = 1 \quad \text{and} \quad \lim_{i \to \infty} \|z_i\| = 0 \ .$$

Fix $\Sigma_{i=1}^k a_i e_i$ and $1 \leq i_0 \leq k$. Pick $\delta > 0$, $\varepsilon << \delta$. Find k such that $k \geq 1/\varepsilon$, and $N(k, \varepsilon) = N$ such that * - holds. Find m so that $N + i_0 < p_m$ and $\|z_m\| < \varepsilon$. For $j \in [p_m, p_{m+1} - 1]$ define

$$w_j = \Sigma_{i=1}^{i_0-1} a_i u_{N+i} + a_{i_0} u_j + \Sigma_{i=i_0+1}^k a_i u_{p_{m+1}+i} \ .$$

Let $x = \Sigma_{i=1}^{i_0-1} a_i u_{N+i}$, $y = \Sigma_{i=i_0+1}^k a_i u_{p_{m+1}+i}$. Then $w_i = x + a_{i_0} u_j + y$, so

$$\Sigma_{j=p_m}^{p_{m+1}-1} \gamma_j w_j = x + a_{i_0} \Sigma_{j=p_m}^{p_{m+1}-1} \gamma_j u_j + y \ .$$

Since $z_m = \Sigma_{j=p_m}^{p_{m+1}-1} \gamma_j u_j$ we have

$$\|\Sigma_{j=p_m}^{p_{m+1}-1} \gamma_j w_j\| \geq \big| \|x + y\| - a_{i_o} \|z_m\| \big| \geq \|x + y\| - a_{i_0} \varepsilon \ .$$

Also

$$| \, \|w_j\| - |||\Sigma_{i=1}^k a_i e_i||| \, | < \varepsilon \ ,$$

using the fact that $\{e_i\}$ is a spreading model. But

$$\Sigma_{j=p_m}^{p_{m+1}-1} \gamma_j \|u_j\| \geq \|\Sigma_{j=p_m}^{p_{m+1}-1} \gamma_j w_j\|$$

and so

$$\|x+y\| - |a_{i_0}|\varepsilon \leq \Sigma_{j=p_m}^{p_{m+1}-1} \gamma_j (\|\Sigma_{i=1}^k a_i e_i\| + \varepsilon) = \|\Sigma_{i=1}^k a_i e_i\| + \varepsilon \ .$$

Choose $\delta = \varepsilon(1 + a_{i_0})$. □

A result of Casazza, Johnson and Tzafriri implies the following:

2.11 Lemma. *If $\{e_i\}$ is the standard basis for Tsirelson's space, then $\{e_i\}_{i\in\omega}$ and $\{e_{2i}\}_{i\in\omega}$ are equivalent.*

2.12 Theorem. *(Bellenot) If $\{e_i\}$ is the standard basis for Tsirelson's space $\{e_i\} \sim \{e_{f(i)}\}$ if and only if there is a primitive recursive function $g \geq f$.*