4. Descriptive Set Theory

Recall the notion of a topological group: $(G, \cdot, \mathcal{T})$, where $(G, \cdot)$ is a group, $(G, \mathcal{T})$ is a topological space and the operations $\cdot$ and $^{-1}$ are continuous.

Definition. A locally compact group is Polish if and only if it is second countable.

Definition. An action of G on X is a homomorphism from G into the group of permutations of X. An action of G on X can also be defined as a function $a: G \times X \to X$, where $a(g, x)$ is written as $g \cdot x$ such that

(i) $e \cdot x = x$ for all x

(ii) $g \cdot (h \cdot x) = (gh) \cdot x$

Remark. Whenever X is a topological space, we have continuous actions and Borel actions.

Definition. A G-space is an action of G

Definition. A Polish G-space: X is Polish and a continuous;

Definition. A Borel G-space: X is Polish (or a Borel set in a Polish space), a is Borel measurable.

Definition. Orbit equivalence relation on X: $xE_a y \iff \exists g(g \cdot x = y)$ and the equivalence classes are orbits;

Examples.
- G = group of rotations of the plane $\mathbb{R}^2$ about $(0, 0)$; G acts on $\mathbb{R}^2$ in an obvious way: $\varphi \cdot (x, y) = (x, y)$ rotated by φ about the origin;
- Countable discrete groups are Polish and locally compact.
- $(\mathbb{R}, +)$ - usually considered as time; Then $(\mathbb{R}, +)$ acts on X is usually interpreted as - something changes in X with time.

Note that the notion **dynamics** is used in two main meanings:
- 1st meaning: something changes with time
- 2nd meaning: actions of any group

- $(\mathbb{R}^n, +)$
- $\mathsf{GL}_n(\mathbb{C})$ = group of $n \times n$ nonsingular matrices with complex entries;

Claim. A closed subgroup of a Polish (or a locally compact) group is Polish (locally complact) group.

- H a Polish group, G closed subgroup, G acts on H:

(i) left-multiplication: $g \cdot h = gh$

(ii) right-multiplication: $g \cdot h = hg^{-1}$

(iii) conjugation: $g \cdot h = ghg^{-1}$

- U_n = group of unitary $n \times n$ matrices; U_n acts on $\mathsf{GL}_n(\mathbb{C})$ by conjugation - equivalence relation is unitary equivalence of matrices or operators.
- $\mathbb{Z}_2$ acts on $\mathbb{R}$ by $0 \cdot r = r$, $1 \cdot 0 = 0$, $1 \cdot r = 1/r$ (if $r \neq 0$).

Claim. A countable product of Polish groups is a Polish group.

Examples.

- $(\mathbb{Z}_2)^\omega$= Cantor group (compact)
- $(\mathbb{R},+)^\omega$ ($\mathbb{R}^\omega$) - non-locally complact
- A separable Banach space with +, norm topology;
- H is an infinite dimensional separable Hilbert space;

L = the set of bounded linear oprators from H to H

$U_\infty \subset L$ - the set of unitary oprators, which is a group

U_∞ with the weak operator topology (= strong operator topology) is a Polish group; U_∞ acts on L by conjugation - orbit equivalence relation is unitary equivalence of operators; It is not a Polish G-space in any natural way, but it is a Borel G-space; Chooses an orthonormal bases, then the oprations are $\omega \times \omega$ matrices so they are Borel sets in $\mathbb{C}^{\omega\times\omega}$; and clearly the action is Borel;

- S_∞ - the group of permutations of ω - basis: b - a bijection between finite subsets of ω; $N_b = \{g \in S_\omega : g \text{ extends } b\}$ - basis; S_∞ is a G_δ subspace of ω^ω;

- Let L= language with 1 binary relation symbol. Space $X_L = 2^{(\omega\times\omega)}$, $x \in X_L$ encodes $\mathcal{A}_x =< \omega, R_x >$ where $R_x(m,n) \iff x(< m,n >) = 1$. S_∞ acts on X_L so that the orbit equivalence relation is isomorphism.

Logic action:

$$J_l\colon S_\infty \times X_L \to X_L$$

given by $J_L(g,x) = y$ if and only if

$$(\forall m,n)[y(m,n) = 1 \iff x(g^{-1}(m), g^{-1}(n)) = 1]\ .$$

For any countable language, there is a logic action for L - a continuous action by S_∞ on a space homeomrphic to 2^ω.

- K is compact Polish; $H(K)$ = the group of homomorphisms of K - a G_δ subspace of $C(K,K)$

4.1 Theorem. *Every Polish group is a closed subgroup of* $H([0,1]^\omega)$.

4.2 Theorem. *(Open Mapping Theorem) If $f\colon G_1 \to G_2$ is a continuous homomorphism onto G_2, then f is open.*

Remark. There are no finer topologies on G: a group G admits at most one definable "Polish topology".

Definition. A metric d on G is called left-invariant if: $d(gh_1, gh_2) = d(h_1, h_2)$ for all $g, h_1, h_2 \in G$.

4.3 Theorem. *Any Polish group admits a left-invariant metric.*

Definition. A *tsi* group is one with 2-sided invariant metric. A *cli* group is one with a complete left-invariant metric.

Remark. If d_1 and d_2 are left invariant, a sequence is d_1 - Cauchy if and only if it is d_2 - Cauchy.

Remark. Left-Cauchy: If d is left -invariant and $d^*(g,h) = d(g^{-1},h^{-1})$, then d^* is right-invariant and vice-versa.
• $H(K)$ - supnorm metric - right-invariant
• S_∞; $d(x,y) = (1 + \text{least n s.t. } x(n) \neq y(n))^{-1}$ - left invariant; Note that left-Cauchy means convergent to a one-to-one (not necessarily onto) function. S_∞ is not cli. A closed subgroup of S_∞ is cli $\iff$ it is closed in ω^ω.

4.4 Remark. Note that we have the following inclusions between classes of Polish groups, where each class is contained in the class right of it:

compact locally compact

discrete tsi cli

Furthermore the class of all abelian groups is contained in the class of all tsi groups; the class of all nilpotent groups (which contains the class of all abelian groups) is contained in the class of all solvable groups, which itself is contained in the class of all cli groups.

Claim. tsi groups and cli groups are closed under:
(i) closed subgroups
(ii) countable products
(iii) continuous homomorphic images

Claim. Suppose G is a Polish group and K compact; Then $C(K,G)$ (consisting of all continuous functions from K to G) with binary opration $(f_1 f_2)(x) = (f_1(x))(f_2(x))$ is a Polish group. If G is cli, so is $C(K,G)$.

Example. Gauge group: G a Lie group of physical symmetries; K a physical space;

Remark. Recall Corollary 1.5 from the 1st class:
If X-Polish, $Y \subset X$. Then TFAE:
(i) Y is Borel
(ii) There is a Polish topology $\mathcal{T}$ on X, finer than the usual one, with Y $\mathcal{T}$ - open;

Now let $a\colon G \times X \to X$ be an action and Y a Borel subset of X; Can you refine the topology and keep a continuous? Not if Y is 1-point in an uncountable orbit!

4.5 Theorem. *Let G be a Polish group; X - a Polish space and $a\colon G \times X \to X$ a continuous action; Let $Y \subset X$ be Borel and a - invariant. Then there is a topology $\mathcal{T}$ on X such that:*
(i) *$\mathcal{T}$ is Polish*
(ii) *$\mathcal{T}$ is finer than the usual one*
(iii) *Y is $\mathcal{T}$ - open*
(iv) *a is continuous with respect to $\mathcal{T}$.*

Remark. L is a language and F is a countable fragment of $L_{\omega_1\omega}$ (collection of formulas closed under $\neg$, $\wedge$, $\exists$ and subformulas). If $\sigma(k_1,\dots,k_n) \in L_{\omega_1\omega}$, then

$$\mathrm{Mod}(\sigma, n_1, \dots, n_k) = \{x \in X_L : \mathrm{A}_x \models \sigma(n_1,\dots,n_k)\} .$$

Topology on X_L with bases $\mathrm{mod}\{(\sigma, n_1, \dots, n_k) : \sigma \in F, n_i \in \mathbb{N}\}$. Note that the topology is Polish and the action is continuous.

4.6 Theorem. *Let G be a Polish group; X - a Polish space and $a: G \times X \to X$ a continuous action; Let $Y \subset X$ be Σ_1^1 and a - invariant. Then there is a topology $\mathcal{T}$ on X such that:*

(i) *$\mathcal{T}$ is second countable and Strong Choquet.*
(ii) *$\mathcal{T}$ is finer than the usual one.*
(iii) *Y is $\mathcal{T}$ - open.*
(iv) *a is continuous with respect to $\mathcal{T}$.*

Definition. Suppose we are given G-spaces (X, a) , (Y, b) (the same group G). A G-space isomorphism is a bijection $\Pi: X \to Y$ such that $\Pi(a(g, x)) = b(g, \Pi(x))$.

4.7 Theorem. *For all Polish G, there exists a Polish G-space U_G (i.e. there exists a continuous action $a: G \times U_G \to U_G$) such that for every Borel G-space X, there is a Borel invariant subset B of U_G such that X is Borel isomorphic to $a\lceil B$.*

Remark. Note that the topological analog is false! Group S_∞ - the logical action of L is universal S_∞-space if and only if L has symbols of unbounded arity.

4.8 Corollary. *A Borel G-space is Borel isomorphic to a Polish G-space.*

Proof. By Theorem 4.4 and 4.6. □

• Orbit equivalence relation: clearly $\mathbf{\Sigma_1^1}$. Not in general Borel - in fact it may violate Theorem 3.7 (Glimm-Effros dichotomy).

• Let L=language of groups; T=theory of abelian p-groups (p is a fixed prime). Consider the logical action restricted to the G_δ set $Mod(T)$. Now p-groups are determined by Ulm-invariance. So p-group correspond to an ordinal $\alpha < \omega_1$ and a function $f: \alpha \to (\omega \cup \{\infty\})$.

• A separating family of ω_1 measurable sets, so violates Glimm-Effros.

• A continuous action by a locally complact group has an F_σ - equivalence relation. By Corollary Borel action by locally compact groups do have Borel equivalence relations.

• Open question: In any Polish G-space at least one of the following holds for the orbit equivalence relation:

(i) smooth
(ii) contains E_0
(iii) There is a $\mathbf{\Sigma_2^1}$-set of $\aleph_1$ (not perfectly many) orbits.

Topological Vaught's Conjecture: Any Polish G-space contains countably many or perfectly many orbits.

Definition. Call G big if there is a continuous homomorphism from a closed subgroup of G onto S_∞. Otherwise call G small.

Remark. cli groups are small

4.9 Theorem.

(i) *(Hjorth) Small groups satisfy the topological Vaught's conjecture.*
(ii) *(Knight) Big groups violate it ?*

4.10 Theorem.

(i) *cli groups satisfy the Glimm-Effros dichotomy (2nd dichotomy)*
(ii) *Big groups violate it.*
(iii) *For all other groups it is an open question.*

4.11 Theorem. *Let X be a Borel G-space ($a: G \times X \to X$). There is a function $f: X \to \omega_1$ such that*
(i) *For all $\alpha < \omega_1$ $f^{-1}([0,\alpha))$ is Borel-invariant.*
(ii) *For all $\alpha < \omega_1$, $E_\alpha \lceil f^{-1}([0,\alpha))$ is a Borel equivalence relation.*
(iii) *The asociated prewellordering of X is $\mathbf{\Delta^1_2}$, universaly measurable and with the Baire property.*

Remark. For logic action $f(x)$= Scott height of A_x.

4.12 Corollary. *In a Borel G-space all orbits are Borel.*

Quesiton: Characterize those $\mathbf{\Sigma^1_1}$ (or Borel) equivalence relations which are (or are up to a mutual $\leq_B$) the orbit equivalence relation of a Borel action of a Polish group.

Remark. E_1 on $(2^\omega)^\omega$: $xE_1y \iff \exists m(\forall n > m)(x(m) = y(n))$.

4.13 Fact. E_1 is not $\leq_B$ to an orbit equivalence relation.

Conjecture: Let E be a Borel equivalence relation. Either

(i) $E_1 \leq_B E$, or

(ii) $E \leq_B$ an orbit equivalence relation.