# 3. Descriptive Set Theory

Recall Theorem 2.10 from the second lecture:

Theorem 2.10

(i) For all $\mathbf{\Pi}^1_1$ set $A \subset X$ there is a continuous $f: X \to 2^\omega$ such that $A = f^{-1}[WO]$.

(ii) For all $\Pi^1_1$ sets $A \subset X$ there is a recursive function $f: X \to 2^\omega$ such that $A = f^{-1}[WO]$.

**Remark.** True if $X$ is $\omega^\omega$.

**Descriptive Set Theory and Equivalence Relations.**

• Hjorth, Greg; Kechris, Alexander S.: New dichotomies for Borel equivalence relations. Bulletin of Symbolic Logic; 3(1997), p.329

Next week: Polish group actions and DST;

(i) Becker - Kechris: The DST of Polish Group Actions

(ii) Hjorth: Classification and Orbit Equivalence Relations;

Context: $X$ - Polish space ( wlog $X = \omega^\omega$ ); $E$ - an equivalence relation on $X$; $E \subset X \times X$ ( pointset in the product space ); So $E$ is Borel, $\mathbf{\Sigma}^1_1$, etc. ( as a pointset in $X \times X$).

**Remark.** $xEy$, $A \subset X$, $[A]_E$ = saturation of $A = \{y | \exists x \in A\ xEy\}$; $[x]_E$ - equivalence class of $E$. $A$ is invariant if $A = [A]_E$. Sometimes we consider $E\lceil A$.

**Remark.** Invariant DST - prove invariant versions of regular DST - theorems;
Invariant version of 2.10:
Suppose that $E$ is $\mathbf{\Sigma}^1_1$, $A \subset X$, $\mathbf{\Pi}^1_1$ and invariant. Then there exists Borel function $f: A \to 2^\omega$ with $f^{-1}[WO] = A$ and if $x, y \in A$ and $xEy$, then $|f(x)| = |f(y)|$.

**Invariant separation**

**3.1 Theorem.** *Suppose $E$ is $\Sigma^1_1$ equivalence relation on $X$ and $A$, $B$ disjoint, $E$-invariant and $\Sigma^1_1$. There is an invariant $\Delta^1_1$ set $D$ such that $A \subset D$, $D \cap B = \emptyset$.*

• Given an equivalence relation "how many" equivalence classes there are?

**Definition.** $A$ has perfectly many $E$-equivalence classes if there is a continuous $1-1$ function $f: 2^\omega \to A$ such that if $x \neq y$ then $(f(x), f(y)) \notin E$.

**Example.** Space $2^\omega$. Define $xEy \iff (x, y \in WO\ \&\ |x| = |y| \text{ or } x, y \notin WO)$. Then $E$ is an $\Sigma^1_1$-equivalence relation. Suppose that there were perfectly many equivalence classes. There is a $\Sigma^1_1$ well-ordering of that perfect set and this violates Fubini's Theorem.

**3.2 Theorem.** *( Silver's Theorem or the 1st Dichotomy Theorem) Suppose that $E$ is a $\mathbf{\Pi}^1_1$ equivalence relation, and $A$ is a $\mathbf{\Sigma}^1_1$ set. Then either $A$ has countably many or perfectly many $E$-equivalence classes.*

Note that if $E$ is "=", then $\mathbf{\Sigma}^1_1$ sets have the perfect set property ( there was given a proof of this fact in the summer course).

**Remark.** A $\Pi^1_1$ equivalence relation, restricted to a $\Pi^1_1$ - invariant set, may violate the dichotomy:

$$xEy \iff [x = y \text{ or } (x, y \in WO\ \&\ |x| = |y|] .$$

Then $E\lceil WO$ violates the dichotomy.

**3.3 Theorem.** *Let $E$ be a Borel equivalence relation on $\omega^\omega$; $A \subset \omega^\omega$;*

(i) *If $\boldsymbol{\Pi^1_1}$-sets have the perfect set property, then for any $\boldsymbol{\Pi^1_1}$ ( or $\boldsymbol{\Sigma^1_2}$ ) set $A$, $A$ has countably many ( or perfectly many ) classes.*

(ii) *PD implies that for any projective $A$, $A$ has countably many or perfectly many classes.*

**3.4 Theorem.**

(i) *A $\boldsymbol{\Pi^1_2}$ equivalence relation restricted to a $\boldsymbol{\Sigma^1_2}$ set has countably many ; $\aleph_1$ and not perfectly many, or perfectly many equivalence classes.*

(ii) *Similarly for a $\boldsymbol{\Sigma^1_2}$ equivalence relation, restricted to any projective set.*

**Remark.** There are 3 types: countable, $\aleph_1$, perfectly many;

**3.5 Theorem.** *(PD) For any projective equivalence relation $E$ and any projective set $A$, either $A$ has perfectly many $E$-equivalence classes, or the $E$-equivalence classes admit a projective well-ordering.*

**Remark.** So far we have seen $\aleph_0$, $\aleph_1$ many equivalence classes; What about $\aleph_2$? Assume $\boldsymbol{\Pi^1_1}$ - determinancy. Here is a $\Delta^1_3$ equivalence relation $E$: Define $f\colon \omega^\omega \to Ord$ by $f(x) =$ successor in $L(x)$ of $\aleph^v_1$ and $xEy \iff f(x) = f(y)$. It is consistent with large cardinals that there exists $\aleph_2$ classes! But it is still an open question about $\aleph_3$.

**Proof of Silver's Theorem:** Work with $\omega^\omega$, $E$ a $\Pi^1_1$ equivalence relation; Consider two topologies on $\omega^\omega$: $u$ = usual topology, $t$ = Gandy - Harrington topology.

**Proposition.** Let $x \in \omega^\omega$. TFAE:

(i) The $t$ interior of $[x]_E$ is nonempty.

(ii) There exists $\Sigma^1_1$ set $S \subset \omega^\omega$ with $S \subset [x]_E$.

(iii) There exists $\Delta^1_1$ set $D \subset \omega^\omega$ such that $D \in [x]_E$.

**Proof.** ii implies iii: $y \in [x]_E \iff \forall z[z \in S \to yEz]$, which is $\Pi^1_1$. So $\omega^\omega \backslash [x]_E$ is $\Sigma^1_1$, $S$ is $\Sigma^1_1$; separate by a $\Delta^1_1$ set. □

**Definition.** An $E$-equivalence class is big if i – iii of the above proposition hold.

Notice: If $E$ is $=$, a big class is a $\Delta^1_1$ point.

**3.6 Theorem.** *( Main Theorem) $E$ a $\Pi^1_1$ - equivalence relation, $S \subset \omega^\omega$, $S$ is $\Sigma^1_1$. Either $S$ has perfectly many equivalence classes, or every $E$-equivalence class that intersects $S$ is big.*

**Proof.** We make use of the following Lemmas:

**Lemma** (A). $X$ - Polish, $E$ - arbitrary equivalence relation on $X$; If $E$ is meager, there are perfectly many classes.

$\omega^\omega$ has 2 topologies - $u$, $t$ = Gandy-Harrington. Thus 3 - topologies on $\omega^\omega \times \omega^\omega$: $u \times u$, $t \times t$, $\widehat{t}$ = Gandy- Harrington topology on $(\omega^\omega)^2$. Note that $\widehat{t}$ is finer then $t \times t$.

**Lemma** (B). The projection maps are $\widehat{t}$ - to - $t$ open.

**Lemma** (C). For any $\Sigma^1_1$ set $\widehat{S} \subset \omega^\omega$, either $\widehat{S}$ has perfectly many $E$ - equivalence classes, or $\widehat{S}$ interesct at least one big class.

Outline of a Proof of Lemma (C):

Let $\tilde{S} = \widehat{S} \cap \{x : \omega_1^X = \omega_1^{\text{CK}}\}$ ( the set $\{x : \omega_1^X = \omega_1^{\text{CK}}\}$ is dense open with respect to $t$). Note that $t$ is Polish on $\widehat{S}$ and hence also on $\tilde{S}$. Now $E\lceil\widehat{S}$ is $\mathbf{\Sigma_1^1}$ with respect to $u \times u$, hence with respect the finer topology $t \times t$. $\mathbf{\Sigma_1^1}$ sets in Polish spaces have the property of Baire. So $E\lceil\tilde{S}$ has the Baire property with respect to $t \times t$. That means we have two cases:

Case 1: $E\lceil\tilde{S}$ is $t \times t$ - meager.

Case 2: There exists $t \times t$ open neighborhood $A \times B$ ( $A$, $B$ are $\Sigma_1^1$, such that $E\lceil\widehat{S} = t \times t$ - comeager in $A \times B$.)

Proof of Case 1: By Lemma A, there are perfectly many classes in $\tilde{S}$, hence in $\widehat{S}$.

Proof of Case 2: Using Lemma B and strictly topological arguments, $E$ is also $\widehat{t}$ - comeager in $A \times B$. But $E$ is $\Pi_1^1$, hence $\widehat{t}$ - closed. So $A \times B \subset E$, and $E$ is an equivalence relation, so $A$ is in 1-equivalence class. It is big.

Proof of 3.6 ( Main Theorem)

$S$ is $\Sigma_1^1$. To show perfectly many or all big: Let $P = \{x : \exists a \Delta_1^1 D \subset \omega^\omega,\ x \in D \text{ and } D \subset [x]_E\}$. Now if $S \backslash P = \emptyset$ - done ( all classes are big). So assume $S \backslash P \neq \emptyset$.

Claim: $P$ is $\Pi_1^1$.

Assume the claim and apply Lemma C to $\widehat{S} = S \backslash P$, which is $\Sigma_1^1$ and nonempty. $\widehat{S}$ intersects no big classes, so it has perfectly many classes.

Proof of the Claim:

By Theorem 2.12 from last class, there exists: $K \subset \omega$ which is $\Pi_1^1$, $L \subset \omega \times \omega^\omega$ which is $\Pi_1^1$ and $M \subset \omega \times \omega^\omega$ which is $\Sigma_1^1$, such that:

(i) $\forall i \in K\ L_i = M_i$

(ii) $\forall \Delta_1^1 D$ there exists $i \in K$ such that $D = L_i = M_i$.

$x \in P \iff \exists i[i \in K,\ (i, x) \in L\ \&\ \forall y, z((i, y) \in M \text{ and } (i, z) \in M \to yEz)].$ □

**Remark.** For the rest of the lecture we will consider only Borel equivalence relations on all of $\omega^\omega$.

**Definition.** $E$ - equivalence realtion is smooth if $\exists$ a countable collection $B_i$ of $E$-invariant Borel sets, such that $\forall x, y : xEy \iff \forall i(x \in B_i \leftrightarrow y \in B_i)$.

**Remark.** $\{B_i\}$ is called a separating family. Equivalently: smooth means that there exists a Borel function $f\colon \omega^\omega \to 2^\omega$ such that $xEy \iff f(x) = f(y)$. The quotient space, i.e. the set of equivalence classes, is $\Sigma_1^1$ subset of $2^\omega$.

**Definition.** $E_0$ equivalence relation on $2^\omega$: $xE_0y \iff (\exists n \forall m > n)(x(m) = y(m))$.

**Remark.** Lebesgue measure is $E_0$ - ergodic, i.e. all Borel invariant sets have measure 0 or 1.

**Remark.** A smooth equivalence relation has no non-trivial Ergodic measure.

**3.7 Theorem.** *( The Second Dichotomy Theorem - Glimm-Effros Dichotomy). If $E$ is a Borel equivalence relation, then either $E$ is smooth or there exists $f\colon 2^\omega \to \omega$ such that for all $x, y \in 2^\omega$ $xE_0y \iff f(x)Ef(y)$.*

**Definition.** If $E$ and $F$ are equivalence relations on $X$, $Y$ then write $E \leq_B F$ if there exists a Borel $f\colon X \to Y$ such that $xEy \iff f(x)Ef(y)$.

**Remark.** Let $E$ be a Borel equivalence relation, $n$ a Borel equivalence relation with $n$ classes, $\aleph_0$ Borel equivalence relation with $\aleph_0$ classes, and $\Delta$ - equality on $2^\omega$. Then:
The 1st Dichotomy: Either $E \leq_B \aleph_0$ or $\Delta \leq_B E$.
The 2nd Dichotomy: Either $E \leq_B \Delta$ or $E_0 \leq_B E$.

**Remark.** Initial segments of $\leq_B$: $1 \leq_B 2 \leq_B \ldots \leq_B \aleph_0 \leq_B \Delta \leq_B E_0 \leq_B$ ?
Is there a 3rd Dichotomy? The answer is - No! Above $E_0$ it is a mess!

**3.8 Theorem.** *There are $2^{\aleph_0}$ pairwise - incomparable ( with respect to $\leq_B$ ) Borel equivalence relations.*

- Restrict attention to special types of Borel equivalence relations and try to get a structure!

**Definition.** $E$ is a countable equivalence relation, if every $E$-equivalence class is countable;

**3.9 Theorem.** *There exists a $\leq_B$ - largest countable Borel equivalence relation.*