2. Descriptive Set Theory

Main reference for this lecture: Moskowakis;

Definition. A subset A of X has the perfect set property if either A is countable or A has a perfect subset.

2.1 Theorem. $\mathbf{\Sigma^1_1}$ *sets are universally measurable, have the property of Baire and the perfect set property.*

Definition. A pointclass is a collection of pointsets in various spaces.

Definition. A pointclass $\mathbf{\Gamma}$ has the separation property if any pair of disjoint $\mathbf{\Gamma}$ sets can be separated by a $\mathbf{\Delta}$ set, where $\mathbf{\Delta} = \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$ ($\check{\mathbf{\Gamma}} =$ complements of $\mathbf{\Gamma}$ sets)

2.2 Theorem. $\mathbf{\Sigma^1_1}$ *and* $\mathbf{\Pi^1_2}$ *have separation property;* $\mathbf{\Pi^1_1}$ *and* $\mathbf{\Sigma^1_2}$ *do not.*

Remark. The axiom $V = L$ implies that there is a $\mathbf{\Delta^1_2}$ well-ordering of $\mathbb{R}$.

2.3 Theorem. $V = L$ *implies:*

(i) *There is a* $\mathbf{\Sigma^1_2}$ *set which is not Lebesgue measurable and does not have the Baire property.*

(ii) *There is a* $\mathbf{\Pi^1_1}$ *set violating the perfect set property.*

(iii) *For* $n \geq 2$, $\mathbf{\Pi^1_n}$ *has separation,* $\mathbf{\Sigma^1_n}$ *does not.*

Definition. By Projective Determinancy (PD) we abriviate the principle that all games $G(\mathbb{N}, X)$ where X is projective, are determined.

2.4 Theorem.

(i) *PD implies that all projective sets are universally measurable, have the Baire property and have the perfect set property.*

(ii) *The pointclasses* $\mathbf{\Sigma^1_1}$, $\mathbf{\Pi^1_2}$, $\mathbf{\Sigma^1_3}$, $\mathbf{\Pi^1_4}$, $\mathbf{\Sigma^1_5}$. *etc. have separation, the others do not.*

Remark. Recall that a subset A of ω^ω determines a game of perfect information between two players:

I	a_0		a_2		a_4		$\dots$	
II		a_1		a_3		a_5		$\dots$

• $\mathbf{\Pi^1_n}$ - determinancy: For any $\mathbf{\Pi^1_n}$ set $A \subset \omega^\omega$, the game A is determined, i.e. either player I or player II has a winning strategy. Note that $\forall n$ $\mathbf{\Sigma^1_n}$ determinancy $\iff$ $\mathbf{\Pi^1_n}$ determinancy.

• PD $\iff$ $\forall n$ $\mathbf{\Pi^1_n}$ determinancy.

Definition. A $*$-game for $A \subset 2^\omega$

I	s_0		s_1		s_2		$\dots$	
II		i_0		i_1		i_2		$\dots$

The player I plays $s_n < 2^{<\omega}$ (i.e. finite sequence from 2); II plays $i_n \in \{0, 1\}$; After ω moves we get $x = s_0{}^\frown i_0{}^\frown s_1{}^\frown i_1{}^\frown s_2{}^\frown i_2 \ldots \in 2^\omega$. We say that I wins the round of the game iff $x \in A$.

Remark. This is a game on a countable set.

2.5 Theorem.

(i) *If* I *has a winning strategy, then* A *has a perfect subset.*

(ii) *If II has a winning strategy, then A is countable.*

Proof.

(i) Let σ be a winning strategy for I. Define $f\colon 2^\omega \to 2^\omega$ as follows: $f(y)$ is the outcome of the game, when I follows σ and II plays $(i_0, i_1, i_2, \ldots) = y$; f is one-to-one and continuous. Since σ is a winning strategy, $\mathrm{Im}(f)$ is in A. So A has a perfect subset.

(ii) Let τ be a winning strategy for II. Let $p =< s_0, i_0, \ldots, i_n >$ be a position in the game in which it is I's turn to move. Let $x \in 2^\omega$. We say that τ rejects x at p if:

• p is an initial segment of x;

• for any s_{n+1} played by I, τ calls for II to play an i_{n+1} such that $p^\frown s_{n+1}{}^\frown i_{n+1}$ is not an initial segment of x;

For any $x \in A$, there is a position p such that τ rejects x at p - if not I could play so that the outcome is x, thus defeating the allegedly winning strategy τ. There are countably many positions. Two points cannot be rejected at the same position. So A is countable. □

Remark. Note that $\mathbf{\Gamma}$ determinancy implies that $\mathbf{\Gamma}$ sets have the perfect set property. Furthermore: $\mathbf{\Gamma}$ determinancy implies that $\exists^{\omega^\omega}\mathbf{\Gamma}$ sets have the perfect set property. For example $\mathbf{\Pi^1_n}$ determinancy implies that $\mathbf{\Sigma^1_{n+1}}$ sets have the perfect set property, (closed set) - determinancy implies that $\mathbf{\Sigma^1_1}$ sets have the perfect set property.

Unfolding the quantifiers: $A(x) \iff (\exists y \in \omega^\omega) B(x, y)$

I	$s_0, y(0)$	$s_1, y(1)$	$s_2, y(2)$	$\ldots$
II	i_0	i_1	i_2	$\ldots$

At the outcome we get $x = s_0{}^\frown i_0{}^\frown s_1{}^\frown i_1 \ldots$. Player I wins iff $(x, y) \in B$. Determinancy of the game imlpies that A has the perfect set property (see the proof of 2.5).

Effective methods

• Recursion theory - computability theory; recursive partial functions on the integers $f\colon \omega^n \to \omega$; recursively presented Polish spaces;
• This is an alternative approach to recursively presented Polish spaces; work with finite products of ω, ω^ω, 2^ω (all Polish spaces are Borel isomorphic);
• Consider a canonical bases:$\{B_n\}$ (determined by finite sequences) and recall that the open sets (or the $\mathbf{\Sigma^0_1}$ sets) are sets of the form $\cup B_{f(n)}$, for some function $f\colon \omega \to \omega$. The effectivley open sets ($= \Sigma^0_1$ sets) are sets of the form $\cup B_{f(n)}$, where $f\colon \omega \to \omega$ is a recursive function. Furthermore one defines the lightface (or Kleene) classes in the following way:

$$\Sigma^1_0 = \text{ all effectivley open sets,}$$

$$\Sigma^0_{n+1} = \exists^\omega \neg \Sigma^0_n,$$

$$\Pi^0_n = \neg \Sigma^0_n,$$

$$\Delta^0_n = \Sigma^0_n \cap \Pi^0_n.$$

Similarly

$$\Sigma_1^1 = \exists^{\omega^\omega} \Pi_1^0,$$

$$\Sigma_{n+1}^1 = \exists^{\omega^\omega} \neg \Sigma_n^1,$$

$$\Pi_n^1 = \neg \Sigma_n^1,$$

$$\Delta_n^1 = \Sigma_n^1 \cap \Pi_n^1.$$

Note the analogy with the bold face (i.e. Borel and Luzin) pointclasses:

$\boldsymbol{\Pi_0^1} =$ complement of $\boldsymbol{\Sigma_1^0}$	$\Pi_0^1 =$ complement of Σ_1^0
$\boldsymbol{\Sigma_0^2} = \exists^\omega \boldsymbol{\Pi_1^0}$	$\Sigma_0^2 = \exists^\omega \Pi_1^0$
$\boldsymbol{\Sigma_1^1} = \exists^{\omega^\omega} \boldsymbol{\Pi_1^0}$	$\Sigma_1^1 = \exists^{\omega^\omega} \Pi_1^0$

Hyperarithmetic = Effectively Borel

• The bold face classes are ω^ω- parametrized: i.e. there exists a pointset $U \subset \omega^\omega \times X$, which is universal for $\boldsymbol{\Sigma_n^1} \lceil X$.
• The lightface classes are ω-parametrized.

2.6 Theorem. *Let Γ be Σ_n^0, Π_n^0, Σ_n^1 or Π_n^1. Let Y be a recursively presented Polish space. Then there exists $U \subset \omega \times Y$ such that U is in Γ and every Γ subset of Y is a vertical section of U.*

Consider Σ_1^0 - the effectivley open sets in ω^ω. Define $U(n,x) \iff \exists i[\varphi_n(i)\downarrow \& \varphi_n(i) < x]$, where φ_n is a recursive partial function from ω into $\omega^{<\omega}$ with index n. If $U(n,x,y)$ is universal for $\Pi_1^0 \lceil (X \times Y)$ then $V(n,x) \iff \exists y U(n,x,y)$ is universal for $\Sigma_1^1 \lceil X$. This is a good parametrization (have recursive $S-m-n$ functions).

Claim. If A and B are Σ_1^1, so is $A \cap B$.

Furthermore this holds uniformly. That means that if $U(n,x)$ are universal sets there is a recursive function $f\colon \omega \times \omega \to \omega$ such that $U_{f(i,j)} = U_i \cap U_j$ for all i,j.

Definition. The Gandy-Harrington topology (on a recursively presented Polish topology) is the topology whose basis is the Σ_1^1 sets.

Two important properties:

(i) It is second countable, strong Choquet and finer than the usual topology.

(ii) The bases (i.e. the Σ_1^1 sets) have a good parametrization.

2.7 Theorem. *There exists a Gandy-Harrington comeager subset of ω^ω which is a Polish space in its relative topology.*

Separation: Two disjoint $\boldsymbol{\Sigma_1^1}$ sets can be separated by a $\boldsymbol{\Delta_1^1}$ set.

Effective version: Two disjoint Σ_1^1 sets can be separated by a Δ_1^1 set.

Effective Theory

In $\mathbb{R}$ or ω^ω, or 2^ω the points become interesting! A Δ_1^1-point $x \in \omega^\omega$: $\{x\}$ is $\Delta_1^1 \iff \{x\}$ is $\Sigma_1^1 \iff$ (x is a Δ_1^1 subset of ω^2).

2.8 Theorem. *(Effective Perfect Set Theorem) If A is Σ^1_1, then either A has a perfect subset or all members of A are Δ^1_1 points.*

2.9 Theorem. Π^1_1 *is closed under quantification of the form $(\exists x \in \Delta^1_1(y))$. That is, if $A(x,y)$ is Π^1_1 and $B(y) \iff (\exists x \in \Delta^1_1(y))A(x,y)$, then B is Π^1_1.*

From Theorem 2.9 one can deduce that $\exists$!Borel is Π^1_1. If it is unique, it is Δ^1_1.

Ordinal Codes

Identify every point $x \in 2^\omega$ with a binary relation $\leq_x$ on ω, namely $\leq_x = \{(n,m) \in \omega^2 : x(<n,m>) = 1\}$. Then let $WO = \{x : \leq_x$ is a well-ordering $\}$. Note that WO is Π^1_1. For every $x \in WO$ dnote by $|x|$ the corresponding ordinal.

Definition. If $\alpha < \omega_1$, α is a recursive ordinal if there is a recursive $x \in 2^\omega$ with $|x| = \alpha$.

Remark. Δ^1_1 ordinals are recursive ordinals and form a countable initial segment of ω_1.

Definition. Denote by ${\omega_1}^{\mathrm{CK}}$ the least non-recursvie ordinal and by ${\omega_1}^X$ the least non-recursive in X ordinal.

Remark. Every nonempty Π^0_1 subset of 2^ω has a Δ^1_1 element. The statement is flase for ω^ω.

2.10 Theorem.

(i) $\forall \boldsymbol{\Pi^1_1}$ *set A there is a continuous function $f\colon X \to 2^\omega$ such that $f^{-1}[WO] = A$.*

(ii) $\forall \Pi^1_1$ *set $A \subset X$ there is a recursive function $f\colon X \to 2^\omega$ such that $f^{-1}[WO] = A$.*

2.11 Theorem. *(Gandy Basis Theorem) Any nonempty Σ^1_1 set contains a point X such that $\omega_1^X = {\omega_1}^{CK}$.*

2.12 Claim. There exist a Π^1_1 set $P \subset \omega^\omega$, a Π^1_1 set $Q \subset \omega \times \omega^\omega$ and a Σ^1_1 set $R \subset \omega \times \omega^\omega$ such that:

(i) $\forall i \in P$, $Q(i,x) \iff R(i,x)$.

(ii) $\forall \Delta^1_1$ set $D \subset X$ there exists $i \in P$ such that $D(x) \iff Q(i,x)$.