1. Descriptive Set Theory

First four classes:

- Classical Descriptive Set Theory
- Effective Methods and Strong Axioms
- Equivalence relations
- Polish Group Actions

Literature:

- Kechris, Classical Descriptive Set Theory
- Moschovakis, Descriptive Set Theory

Polish spaces = separable completely metrizable spaces

Example. $\mathbb{R}$

Proposition.

(i) Finite or Countable product of Polish spaces is Polish.

(ii) A subspace of a Polish space is Polish iff it is G_δ.

Examples.

- $[0,1]$, $[0,1]^\omega = [0,1]^{\mathbb{N}} =$ Hilbert cube
- $\omega = \mathbb{N}$ (countable discrete)
- $\omega^\omega =$ Baire Space
- $2^\omega =$ Cantor space
- Any separable Banach space.
- Suppose K is compact metrizable and X is Polish. Then the space $C(K, X)$ of all continuous functions from K into X, with metric $d(f,g) = sup_{y \in K} d_X(f(y), g(y))$ is Polish. In case $X = \mathbb{R}$, we simply refer to the above space as $C(K)$.
- For any Polish space X - the hyperspace $K(X)$ whose points are the compact subsets of X and has the Hausdorff metric $d(K, L) = \sup_{x \in K, y \in L}\{d_X(x, L), d_X(y, K)\}$ is Polish. Note that the Hausdorff metric is compatible with the Vietoris topology.

Definition. It T is a tree on A, then $f \in A^\omega$ is called an infinite branch through T if $f|n \in T$ for all $n \in \mathbb{N}$.

Definition. For any set A, a tree on A is a set of finite sequences from A, closed under initial segments.

Example. The full binary tree.

Definition. For any tree T, denote by $[T]$ the body of T, i.e. the set of all infinite branches of T.

Definition. A tree T is called well-founded if $[T] = \emptyset$ (i.e. it does not have infinite branches).

Definition. Given a well-founded tree T one can consider recursively defined functions on T. In particular, we refer to $rk_T : T \to \text{Ord}$, where $rk_T(\sigma) = min\{\alpha : \alpha > rk_T(\mu) \quad \text{for all} \quad \mu < \sigma\}$ as a rank-function on T.

Example. For any set $A \neq \emptyset$, considered with the discrete topology, one can consider the space A^ω. The closed subsets of A^ω have the form $[T]$, where T is a tree on A.

Example. The Polish space Tr of trees on $\mathbb{N}$: note that any tree on $\mathbb{N}$ can be viewed as a subset of $\mathbb{N}^{<\mathbb{N}}$ and so identified with its characteristic function. The set of all trees on $\mathbb{N}$ - Tr is a closed subset of $2^{\omega^{<\omega}}$, where $\omega^{<\omega}$ is the set of all finite sequences from ω.

Definition. Let A be a set; infinite game of perfect information on A:

$I \quad a_0 \quad a_2 \quad \dots$

$II \quad a_1 \quad a_3 \quad \dots$

$a_i \in A$; get $(a_0, a_1, ...) \in A^\omega$;

A game on A is a subset B of A^ω. We say that I wins a run of the game if $(a_0, a_1, ...) \in B$, otherwise II wins.

Since we work on topological spaces, it is natural to consider closed, borel, etc. games.

Definition. For Y - a topological space, consider the *Choquet game* of Y:

$I \quad U_0 \quad U_2 \quad \dots$

$II \quad U_1 \quad U_3 \quad \dots$

The sets U_i's are open and decreasing $U_0 \supset U_1 \supset U_1 ...$ We say that II wins if $\cap_i U_i \neq \emptyset$.

Definition. A topological space Y is called a *Choquet space*, if II has a winning strategy.

Definition. Given a topological space Y consider the Strong Choquet Game:

$I \quad (U_0, x_0) \quad (U_2, x_2) \quad \dots$

$II \quad U_1 \quad U_3 \quad \dots$

The sets U_i are open and decreasing; $x_i \in U_i$ and $x_i \in U_{i+1}$ for every even i. We say that II wins a if $\cap_i U_i \neq \emptyset$.

Definition. A topological space is called *Strong Choquet space* if the second player has a winning strategy.

Example. All nonempty completely metrizable spaces are strong Choquet.

Proposition.

(i) Any strong Choquet space is Choquet.

(ii) Products of strong Choquet spaces are strong Choquet.

complete metric spaces $\subset$

Strong Choquet $\subset$ Choquet $\subset$ Baire

compact T_2 $\subset$

1.1 Theorem. *(Choquet) Y a metrizable space. TFAE:*
(i) *Y is strong Choquet.*
(ii) *Y is completely metrizable.*

1.2 Corollary. *Y a topological space. TFAE:*
(i) *Y is Polish.*
(ii) *Y is T_1, regular, second countable and strong Choquet.*

Remark. Descriptive set theory is the study of definable sets and functions in Polish spaces. Except otherwise specified we always work with Polish spaces.

Definition. (The Borel Hierarchy) For every $\alpha : 1 \leq \alpha < \omega_1$ recursively define the the classes $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$:

$$\begin{aligned}
\mathbf{\Sigma}^0_1 &= \text{ open sets} \\
\mathbf{\Pi}^0_\alpha &= \text{ complements of } \mathbf{\Sigma}^0_\alpha \\
\mathbf{\Sigma}^0_\alpha &= \{A : A = \cup_{i\in\omega} B_i,\ \forall i \in \omega\ B_i \in \mathbf{\Sigma}^0_\beta \text{ for some } \beta < \alpha\} \quad (\forall \alpha > 1) \\
\mathbf{\Delta}^0_\alpha &= \mathbf{\Sigma}^0_\alpha \cap \mathbf{\Pi}^0_\alpha
\end{aligned}$$

Remark. Borel $= \cup_\alpha \mathbf{\Sigma}^0_\alpha$.

1.3 Theorem. *For any Polish spaces X and Y, such that X is uncountable and any $\alpha < \omega_1$, there is a set $U \subset X \times Y$ such that U is $\mathbf{\Sigma}^0_\alpha$ and a subset of Y is $\mathbf{\Sigma}^0_\alpha$ iff it is U_x $(= \{y : U(x,y)\})$ for some $x \in X$.*

Remark. According to the above theorem there exists a X-universal set for $\mathbf{\Sigma}^0_\alpha(Y)$. Similarly one can show that there exists a X-universal set for $\mathbf{\Pi}^0_\alpha(Y)$. Note that the classes $\mathbf{\Delta}^0_\alpha$ cannot have X-universal sets.

Proposition. The class $\mathbf{\Sigma}^0_\alpha$ is closed under finite unions, intersections countable unions, $\exists^\omega$, continuous preimages, but not under $\forall^\omega$, $\forall^{\mathbb{R}}$, $\exists^{\mathbb{R}}$, continuous image.

A main method for proving that a set is $\mathbf{\Sigma}^0_\alpha$ or $\mathbf{\Pi}^0_\alpha$ is the Tarski - Kuratowski algorithm, which idea is sipmly - write down a formula and count quantifiers. For example consider $\mathbb{R}^\omega$. We claim that the set of all convergent sequences $< x_i >$ is true $\mathbf{\Pi}^0_3$ (i.e. $\mathbf{\Pi}^0_3$, not $\mathbf{\Delta}^0_3$). Really the sequence

$$\begin{array}{lllllll}
< x_i > & \text{converges} & \iff & (\forall \varepsilon \in \mathbb{R}^+) & (\exists N) & (\forall i,j > N) & (|x_i - x_j| < \varepsilon)\ , \\
 & & \iff & (\forall \varepsilon \in \mathbb{Q}^+) & (\exists N) & (\forall i,j > N) & (|x_i - x_j| < \varepsilon)\ , \\
 & & \iff & (\forall \varepsilon \in \mathbb{Q}^+) & (\exists N) & (\forall i,j > N) & (|x_i - x_j| \leq \varepsilon)\ . \\
 & & & \mathbf{\Pi}^0_3 & \mathbf{\Sigma}^0_2 & \mathbf{\Pi}^0_1 & \mathbf{\Pi}^0_1
\end{array}$$

1.4 Theorem.
(i) *The one-to-one image of a Borel set under a continuous (or Borel measurable) function is Borel.*
(ii) *Borel set is the one-to-one continuous image of a Polish space (in fact, a closed subspace of ω^ω).*

1.5 Corollary. *Let A be a pointset in a Polish space X. TFAE:*
(i) *A is Borel.*
(ii) *There exists a finer Polish topology on X in which A is open.*
(iii) *There exists a finer topology on X, in which A is open, such that the space is regular, second countable, strong Choquet.*

Remark. For almost every countable collection $\mathcal{B}$ of Borel sets (i.e. for a club family of countable collections), $\mathcal{B}$ is a basis for a Polish topology.

1.6 Theorem. *Any two uncountable Polish spaces X and Y are Borel-isomorphic, i.e. there exists a bijection $f : X \to Y$ such that f, f^{-1} are Borel measurable.*

Definition. (Projective or Luzin Classes)

$$\begin{aligned} \mathbf{\Sigma^1_0} &= \mathbf{\Sigma^0_1} = \text{ open sets} \\ \mathbf{\Sigma^1_1} &= \exists^{\omega^\omega} \mathbf{\Pi^0_1} = \exists^{\mathbb{R}} G_\delta \\ \mathbf{\Sigma^1_{n+1}} &= \exists^{\omega^\omega} \neg \mathbf{\Sigma^1_n} \\ \mathbf{\Pi^1_n} &= \neg \mathbf{\Sigma^1_n} \\ \mathbf{\Delta^1_n} &= \mathbf{\Sigma^1_n} \cap \mathbf{\Pi^1_n} \end{aligned}$$

Note that we have the following picture of the Projective hierarchy:

$$\begin{array}{cccccccc} & & & \mathbf{\Sigma^1_1} & & \mathbf{\Sigma^1_2} & & \ldots \\ \text{Borel} & \mathbf{\Delta^1_1} & & & \mathbf{\Delta^1_2} & & \mathbf{\Delta^1_3} & \\ & & & \mathbf{\Pi^1_1} & & \mathbf{\Pi^1_2} & & \ldots \end{array}$$

where each class belongs to the class rigth of it.

1.7 Theorem. *$\mathbf{\Sigma^1_n}$ and $\mathbf{\Pi^1_n}$ have universal sets.*

Proposition. The classes $\mathbf{\Sigma^1_n}$ are closed under $\exists^\omega$, $\forall^\omega$, $\exists^X$ (X Polish), projections, Borel image or preimage, but not closed under $\forall^{\mathbb{R}}$.

Remark. $\mathbf{\Pi^1_1} = \neg\exists\text{Borel} = \neg\exists\neg\text{Borel} = \forall\text{Borel}$;

Example. The pointset of all well-founded trees in Tr is $\mathbf{\Pi^1_1}$. Really T is well-founded if and only if $\forall x \in \omega^\omega \exists n [x|n \notin T]$.

Example. The set $\{f : f \text{ differentiable }\}$ is $\mathbf{\Pi^1_1}$ in $C[0,1]$.

Proposition. $\mathbf{\Sigma^1_1}$ =projection of a Borel set =continuous image of a Borel set

Remark. $\mathbf{\Sigma^1_1}$ sets are called analytic sets. Equivalently analytic sets can be defined as a projection of a Borel set, or a continuous image of a Polish space.

Proposition. All analytic sets in ω^ω have a representation of the form

$$A(x) \iff \exists y \in \omega^\omega (< x, y > \in [T])$$

where T is a tree on ω^2.

Proposition. If $A \subset X$ is $\mathbf{\Sigma}^1_1$, then A has the form $A(x) \iff \exists y B(x,y)$, where B is $\mathbf{\Pi}^0_1$ (closed) subset of $X \times \omega^\omega$ (or B is $\mathbf{\Pi}^0_2$ (G_δ) subset of $X \times$(any uncountable Polish space)).

1.8 Theorem. *Let A be a pointset in a Polish space X. TFAE:*
(i) *The set A is $\mathbf{\Sigma}^1_1$ (analytic).*
(ii) *There exists a finer topology $\mathcal{T}$ on X, in which A is open, such that $\mathcal{T}$ is second countable and strong Choquet.*

Remark. For almost every countable collection of $\mathbf{\Sigma}^1_1$ sets, $\mathcal{B}$ is a basis for a Strong Choquet topology.

1.9 Theorem. *The class $\mathbf{\Pi}^1_1$ has the prewellordering propery - a $\mathbf{\Pi}^1_1$ set can be written as a union of $\aleph_1$ Borel sets, in a simply definable way.*

Let $P \subset X$ be $\mathbf{\Pi}^1_1$. Then there is a function $\varphi : P \to \omega_1$ such that there are $\mathbf{\Sigma}^1_1$ relations $\leq_\Sigma$, $<_\Sigma$ and $\mathbf{\Pi}^1_1$ relations $\leq_\Pi$, $<_\Pi$ such that for any $x \in P$

$$\{y : y \in P \text{ and } \varphi(y) < \varphi(x)\} = \{y : y <_\Sigma x\} = \{y : y <_\Pi x\}$$

Similarly for $\leq$.

1.10 Theorem. *Any two disjoint $\mathbf{\Sigma}^1_1$ sets can be separated by a Borel set, i.e. if A_1 and A_2 are $\mathbf{\Sigma}^1_1$ and $A_1 \cap A_2 = \emptyset$, there exists a Borel set B with $A_1 \subset B$, $B \cap A_2 = \emptyset$.*

1.11 Corollary. $\mathbf{\Delta}^1_1$ *= Borel*

Remark. Note that $\mathbf{\Pi}^1_1$ is not closed under $\exists^{\mathbb{R}}$

1.12 Theorem. *If $B \subset X \times Y$ is Borel and $A \subset X$ is defiend by $\exists! y B(x,y)$, then A is $\mathbf{\Pi}^1_1$.*

Example. The set $\{f : f'(x)$ exists except at exactly 1 point $\}$ is $\mathbf{\Pi}^1_1$ in $C[0,1]$.

1.13 Theorem. *Let $A \subset \omega^\omega$. If A is $\mathbf{\Sigma}^1_2$, then A is a projection of a tree on $\omega \times \omega_1$.*