L 22/03

Type-II matrices

A clique in $\Omega^b(d)$ is a flat $d \times d$ matrix. We used them to construct homomorphisms. There is an interesting generalization.

If W is an $m \times n$ matrix over $\mathbb{C}$ with no zero entries, we define the $d \times d$ $W^{(-)}$ by

$$(W^{(-)})_{i,j} := W_{i,j}^{-1}.$$

We call it the Schur inverse of W.

An $n \times n$ matrix W over $\mathbb{C}$ is a type-II matrix if $W W^{(-)T} = nI$. So if W is a type-II matrix, it is invertible and

$$W^{(-)T} = nW^{-1}$$

W $\qquad$ $W^{(-)T}$

$w_1 \cdots w_n$ $\qquad$ w_1^{-1}

w_2^{-1}

$\vdots$

Examples

(a) $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

(b) If $w^3 = 1$ & $w \neq 1$,

$$\begin{bmatrix} 1 & 1 & w \\ w & 1 & 1 \\ 1 & w & 1 \end{bmatrix}$$

(c) For any non-zero complex number t

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & t & -t \\ 1 & -1 & -t & t \end{bmatrix}$$

Any flat unitary matrix is a (scaled) type-II matrix. In fact there is a stronger claim.

Theorem For any complex $n \times n$ matrix W, the following are equivalent:

(a) W is type-II (b) $n^{-1/2} W$ is unitary

(c) $|W_{i,j}| = 1 \;\forall i, j$.

Note: any Hadamard matrix is type-II

We can also get flat unitary matrices from continuous quantum walks. Recall that, starting from a state D, we have local uniform mixing at time t if $U(t)DU(-t)$ has constant diagonal.

We are interested in the case where the initial state is a vertex state $D_a = e_a e_a^T$. Then

$$U(t) e_a e_a^T U(-t) = U(t) e_a (U(t) e_a)^*$$

and the diagonal is constant if & only if $U(t) e_a$ is flat, i.e., the a-column of $U(t)$ is flat.

We have uniform mixing at time t if $U(t)$ itself is flat. There is uniform mixing on the d-cube at time $\pi/4$.

Operations & Equivalence

If W is type-II, ~~so~~ so is W^T. The Kronecker product of type-II matrices is type-II.

Assume D_1 & D_2 are invertible diagonal matrices of order $n \times n$ and let P_1, P_2 be $n \times n$ permutation matrices. If W_1 is $n \times n$ and $W_2 = D_1 P_1 W_1 D_2 P_2$ then W_1 & W_2 are equivalent type-II matrices. Note: in general, W and W^T are not equivalent.

The Kronecker product $W_1 \otimes W_2$ of two type-II matrices is type-II, Two algebras

Let W be a type-II matrix of order $n \times n$.

Define

$$W_{i/j} := We_i \circ (We_j)^{(-)} \qquad 1 \le i,j \le n$$

and define the Nomura algebra $\mathcal{N}_W$ of W to be the set of $n \times n$ matrices for which each vector $W_{i/j}$ is an eigenvector. Note that $\mathcal{N}_{W^{(-)}} = \mathcal{N}_W$.

If $M \in \mathcal{N}_W$ then $\Theta(M)$ is the $n \times n$ matrix

"matrix of eigenvalues of M"

such that

$$M W_{i,j} = \Theta(M)_{i,j} W_{i,j}$$

Lemma Assume W is invertible and Schur-invertible. Then W is a type-II matrix if and only if $J \in \mathcal{N}_W$.

Thus if W is type-II, then $\mathcal{N}_W$ contains $\operatorname{span}\{I, J\}$.

Example Let θ be a primitive n-th root of unity and let V_n be the $n \times n$ Vandermonde matrix

$$(V_n)_{i,j} = \theta^{(i-1)(j-1)}$$

$$= \begin{bmatrix} 1 & 1 & & \\ 1 & \theta & & \\ \vdots & & \cdots & \\ 1 & \theta^{n-1} & & \end{bmatrix}$$

Then $V_n^{(-)} = \overline{V_n}$ & $V_n^{(-)T} = V_n^*$.

Further $V_n V_n^* = nI$ and so $\frac{1}{\sqrt{n}} V_n$ is a flat unitary matrix. We have

$$V_n e_i \circ (V_n e_j)^{(-)} = \left[\theta^{(i-j)h}\right]_{h=1}^{n}$$

It follows that the Nomura algebra of V_n is the algebra of $n \times n$ circulants.

Let η be a primitive $(2n)$-th root of unity and define the matrix W by

$$W_{ij} = \eta^{(i-j)^2} \qquad \eta^{(i-j)^2} = \eta^{i^2} \eta^{-2ij} \eta^{j^2}$$

Show that W & V_n are equivalent and have the same Nomura algebra. Show that W lies in its Nomura algebra.

Lemma If W is type-II, then $\mathcal{N}_W$ is commutative.

Proof. Since W is invertible, its columns We_j $(j=1,\ldots,n)$ are linearly independent. If D_1 is the diagonal matrix with $(D_1)_{i,i} = (We_1)_i$, then D_1 is invertible and the vectors $D_1^{-1} We_j = W_{j/1}$ are linearly independent. So, relative to the basis $W_{j/1}$ $(j=1,\ldots,n)$ all matrices in $\mathcal{N}_W$ are diagonal. $\square$

It follows that if $M, N \in \mathcal{N}_W$, then

$$Ⓝ(MN) = Ⓝ(M) \circ Ⓝ(N).$$

So $Ⓝ(\mathcal{N}_W)$ is a Schur-closed algebra. We note that $Ⓝ(I) = J$ & $Ⓝ(J) = nI$. We will see that $Ⓝ(\mathcal{N}_W)$ is closed under matrix multiplication.

We define some idempotents. Assume W is $n \times n$ type-II and define, for each i & j:

$$Y_{i,j} = \frac{1}{n} W_{i/j} (W_{j/i})^T$$

This is a rank-1 matrix. You may verify that

$$Y_{a,i} Y_{a,j} = \delta_{i,j} Y_{a,i}$$

and

$$\sum_i Y_{a,i} = I.$$

Lemma If $M \in \mathcal{N}_W$ then

$$M = \sum_i \mathcal{H}(M)_{a,i} Y_{a,i}$$

Proof

$$M Y_{a,j} = \frac{1}{n} M W_{a|j} W_{j/a}^T = \frac{1}{n} \mathcal{H}(M)_{a,j} W_{a|j} W_{j/a}^T$$

$$= \mathcal{H}(M)_{a,j} V_{a,j}$$

and summing this over j yields the claim. $\square$

The next result is critical.

Theorem If $M \in \mathcal{N}_W$ then

$$\Theta_W(M)\,(W^T)_{a/b} = n M_{a,b}\,(W^T)_{a/b}.$$

Hence $\Theta_W(M) \in \mathcal{N}_{W^T}$ and $\Theta_{W^T}(\Theta_W(M)) = n M^T \in \mathcal{N}_W$.

Corollary Θ_W and Θ_{W^T} are invertible.