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#_of vxs

Theorem Let X & Y be graphs. A subset S of $V(X) \times V(Y)$ is the graph of a homomorphism from an induced subgraph of X of size $|S|$ to Y if and only if it is a coclique in $X \ast Y$.

To motivate the proof, we offer a corollary.

Corollary $X \to Y \Leftrightarrow \alpha(X \ast Y) = |V(X)|$. $\square$

$(u,v) \sim (x,y)$ if $u = x$ or $u \sim x$ & $v \not\sim y$

Reminder: $A(X \ast Y) = A(X \square K_y) + A(X \times \bar{Y})$

Proof First, suppose f is a homomorphism from an induced subgraph X_0 of X to Y. If u & v are adjacent vertices in X_0, then $f(u) \underset{Y}{\sim} f(v)$ and so $(u, f(u))$ and $(v, f(v))$ are not adjacent in $X \square K_Y$ or in $X \times \bar{Y}$.

For the converse, assume $S \subseteq V(X \times Y)$ that is a coclique in $X \square K_Y$ and in $X \times \bar{Y}$. Define the domain $D(S)$ of S to be the set of vertices x in $V(X)$ such that $(x, y) \in S$ for some y in $V(Y)$, If $u \in D(S)$ and $y, z \in V(Y)$, then (u,y) & (u,z) are adjacent vertices in $X \square K_Y$. Hence if $u \in D(S)$, there is exactly one vertex y such that $(u,y) \in S$. So S is the graph of a function, g say, from $D(S)$ to $V(Y)$.

We show that g is a homomorphism. If (u,y) & (v,z) lie in S and $u \underset{X}{\sim} v$, then since S is a coclique in $X \square K_Y$, $y \neq z$. Since (u,y) & (v,z) are not adjacent in $X \times \bar{Y}$, we have $y \underset{Y}{\sim} z$.

Therefore g is a homomorphism from X to Y. $\square$

Lemma We have $X \xrightarrow{q} Y$ if and only if $\alpha_q(X \times Y) = |V(X)|$. $\square$

(The proof, in Roberson's thesis, is complicated)

Corollary If $X \xrightarrow{q} Y$ and $X \not\to Y$, then

exercise

$$\alpha(X \times Y) < \alpha_q(X \times Y) = |V(X)|. \quad \square$$

We have seen an inertia bound on $\alpha(X)$.

Following Wocjan & Elphick, we show this is also an upper bound on $\alpha_q(X)$.

Theorem If W is a Hermitian weighted adjacency matrix for X then

$$\alpha_q(X) \leq \min\{n - n^+(A),\ n - n^-(A)\}$$

Some preparation is needed.

A d-dimensional projective packing of a graph X is a homomorphism from X to the orthogonality graph on $d \times d$ projections. If ψ is such a homomorphism, its value is $\frac{1}{d} \sum_{v \in V(X)} \mathrm{rk}(\psi(v))$.

The projective packing number is the supremum of the values of the projective packings of X. It is denoted $\alpha_p(X)$.

Reminder: a quantum coclique in X is a $|V(X)| \times s$ matrix $P = (P_{ij})$ such that

(a) for each vertex u in X, $\sum_{u \in V(X)} P_{u,i} = I_d$.

(b) $P_{ui} P_{vj} = 0$ if u & v are equal or adjacent.

Lemma For any graph X, we have $\alpha_p(X) \le \alpha_q(X)$. typo fixed here, had p & q swapped

Proof Let $P_{u,i}$, $u \in V(X)$, $1 \le i \le s$ be a set of projections defining a quantum coclique size α. Set $P_u = \sum_{i=1}^{s} P_{u,i}$ for u in $V(X)$. These sums are orthogonal projections. Now

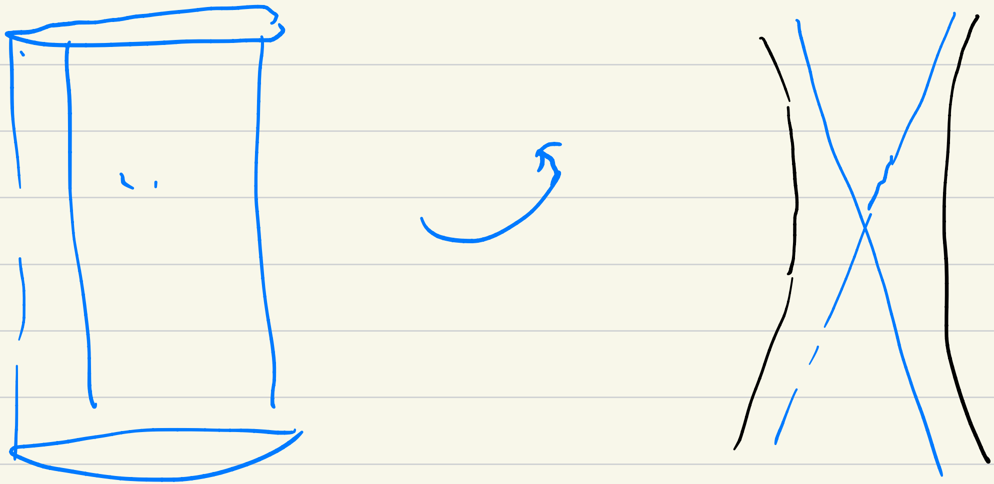
$$\langle P_u, P_v \rangle = \sum_{i,j} \langle P_{u,i}, P_{vj} \rangle = \sum_{i \neq j} \langle P_{ui}, P_{vj} \rangle + \sum_i \langle P_{u,i}, P_{v,i} \rangle = 0$$

and

$$\sum_{u \in V} rk(P_u) = \sum_i \sum_u rk(P_{u,i}) = ds$$

Consequently $\alpha_p(X) \le s = \alpha_q(X)$. $\square$

We need another proof of the inertia bound (as motivation). Assume W is a Hermitian weighted adjacency matrix for X. The lines spanned by vectors x such that $x^* W x = 0$ are the points of a projective quadric. A subspace U of $\mathbb{C}^n$ is isotropic if $u^* W v = 0$ $\forall u, v$ in U. If S is a coclique in X, then $\operatorname{span}\{e_u : u \in S\}$ is isotropic



hyperbolic quadric

It is known that the maximum dimension of an isotropic subspace is $\min\{n - n^+(W), n - n^-(W)\}$.

We turn to the proof of the theorem.

Proof (of inertia bound).

Recall that $P_u := \sum_{i=1}^{s} P_{u,i}$, and is a projection.

Let $\psi_{u,1}, \dots, \psi_{u,\mathrm{rk}(P_u)}$ be an orthonormal basis for $\mathrm{col}(P_u)$ and define vectors $\Psi_{u,i} := e_u \otimes \psi_{u,i}$.

We have

$$\langle \Psi_{u,i}, \Psi_{v,j} \rangle = \delta_{u,v}\, \delta_{ij}$$

$$\Psi_{u,i}^*(W \otimes I_d)\,\Psi_{v,j} = 0$$

So the vectors $\Psi_{u,i}$ $(u \in V(X),\ i = 1,\dots,rk(P_u))$ span an isotropic subspace on the quadric given by $W \otimes I_d$ and therefore

$$r \le \min\{d(n - n^+(W)),\ d(n - n^-(W))\}$$

Accordingly

$$q_p(X) \leq \frac{r}{d} \leq \min\{n - n^+(W), n - n^-(W)\}. \qquad \square$$

For a long time, we had no examples of graphs where the weighted inertia bound was not equal to $\alpha(X)$. The first such examples were found by Sinkovic. But now any graph with $\alpha(X) < q_p(X)$ works.

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Example Let $\mathcal{C}$ be the set of elements of order two in $\mathrm{Sym}(4)$, (so $|\mathcal{C}|=9$) and let X be the Cayley graph $X(\mathrm{Sym}(4), \mathcal{C})$. Then $\alpha(X)=5$ and $\alpha_q(X)=6$.

The permutations $\{(12)(34), (13)(24), (14)(23), (1)\}$ are a subgroup of order four, inducing a 4-clique. The six cosets of this subgroup partition $V(X)$ into 4-cliques.

Suppose $\mathcal{P}$ is a quantum n-colouring of K_n. Then $\mathcal{P}$ is an $n \times n$ matrix over $Mat_{d\times d}(\mathbb{C})$, such that each row & column sums to I_d. Then $\mathcal{P}\mathcal{P}^* = I_{nd}$, and so $\mathcal{P}$ is an $nd \times nd$ unitary matrix.

Classically, n-colourings of K_n correspond to projections.

We define a quantum permutation to be an $n \times n$ matrix of $d \times d$ projections, such that all row & column sums equal I_d

Coherent algebras

"Recall" A coherent algebra $\mathcal{B}$ is an algebra of $n \times n$ matrices such that

(a) $\mathcal{B}$ is closed under transpose & complex conjugation

(b) $\mathcal{B}$ contains J and is closed under the Schur product

Simplest example: $\mathcal{B} = \text{span}\{I, J\}$; $\dim(\mathcal{B}) = 2$.

Second simplest: A is the adjacency matrix of a strongly regular graph, $\mathcal{B} = \text{span}\{I, A, J-I-A\}$.

Tie for 2nd simplest: $\mathrm{Mat}_{n\times n}(\mathbb{C})$. Unlike the previous two examples, this is not commutative; its dimension is n^2, the maximum possible.

For more interesting examples:

Theorem. The commutant of a set of $n\times n$ permutation matrices is a coherent algebra.

Proof. If $\mathcal{C}_1$ & $\mathcal{C}_2$ are coherent subalgebras of $\mathrm{Mat}_{n\times n}(\mathbb{C})$, their intersection is a coherent algebra. So we only need to prove the theorem for a single permutation, P say. The commutant of P is a matrix algebra and contains J.

Assume $M, N \in \mathrm{Comm}(P)$. Then

$$P(M \circ N) = PM \circ PN = MP \circ NP = (M \circ N)P$$

and we're done. $\square$

The commutant of a set of permutations is equal to the commutant of the group they generate.

Because the intersection of coherent algebras is a coherent algebra, any set of matrices generates a coherent algebra — the intersection of the coherent algebra that contain the set. Our favourite example is generated by A & J.

We derive one of the most important facts about coherent algebras:

Theorem A coherent algebra has a unique basis of pairwise Schur orthogonal 01-matrices.

Proof Define the k-th Schur power $M^{\circ k}$ to be the matrix $\underbrace{M \circ \cdots \circ M}^{k \text{ terms}}$ and if $a(t) = a_0 t^m + \cdots + a_m$, define $a \circ M$ to be $\sum_{r=0}^{m} a_r M^{\circ(m-r)}$. It is a Schur polynomial in M.

Let $\mathcal{B}$ be a coherent algebra and assume $M \in \mathcal{B}$.

Let $\mu_1, \dots, \mu_d$ be the distinct entries of M. Then

$M = \sum_{i=1}^{d} \mu_i N_i$, where N_i is a 01-matrix. Let p_j be the

unique polynomial of degree d such that $p_j(\mu_i) = \delta_{j,i}$.

Then $N_i = p_i(M)$ lies in $\mathcal{B}$, for $i = 1, \dots, s$.

If K & L are two 01-matrices in $\mathcal{B}$, then $K \circ L \in \mathcal{B}$.