L27 C3 Open Questions

Continuous walks

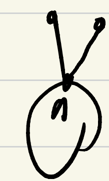
1. Is there a graph with perfect state transfer from a to b such that the sum of the eigenvalue support of u (or v) is not zero?

If $a \in V(X)$, the eigenvalue support of u is the set of zeros of the polynomial

$$\frac{\varphi(X,t)}{\gcd(\varphi(X,t), \varphi(X \setminus a,t))}$$

(Note that the zeros of this polynomial are all simple.)

2) Let Y be the graph obtained from X by adding two new vertices, each adjacent to the same vertex a. Find examples of Y, with $|V(Y)| \geq 4$, where there is perfect state transfer between the two new vertices.



Y

The two new vertices are strongly cospectral iff $\mathrm{mult}(0, X \setminus a) \leq \mathrm{mult}(0, X)$. (If $X = K_1$, then $Y = K_{1,2}$ which does admit perfect state transfer between the vertices of valency one.)

3) Characterize the cubelike graphs with perfect state transfer at time $\pi/8$.

Bernasconi, Godsil, Severini

We have characterizations for $\pi/2$ and $\pi/4$

Cheung, Godsil

Severini has asked the following:

4) Characterize the cubic graphs with perfect state transfer

Severini proved that the 3-cube is the only periodic cubic graph with perfect state transfer. (arxiv:1001.6074v2)

(If you solve this, you might be asked about graphs with maximum valency three.)

Note that a vertex-transitive graph with perfect state transfer is necessarily periodic (so we know what happens for vertex-transitive graphs).

5) Are there infinitely many graphs X whose average mixing matrix has rank two?

Ihringer & Betten have a graph on 64 vertices with

$\mathrm{rk}(\hat{M}_X) = 2$.

$$\hat{M}_X = \sum_r E_r^{\circ 2}$$

6) Is there a graph with uniform mixing which is not regular & not a Cartesian power of $K_{1,3}$?

$U(t)$ is flat

7) Is there uniform mixing on C_9? On C_{15}?

Natalie Mullin proved the uniform mixing does not occur on C_p (p prime) when $p \geq 7$. The case C_5 was ruled out by Tamon et al; C_3 does admit uniform mixing. Natalie showed that, C_4 aside, no even cycle admits uniform mixing.

Two conjectures due to Mullin:

8) If a graph admits uniform mixing at time t, then e^{it} is a root of unity.

9) If $n \geq 5$, no connected Cayley graph for $\mathbb{Z}_n^d$ admits uniform mixing.

Natalie Mullin's Ph.D. thesis and Hanmeng Zhan's M. Math thesis contain a lot of information about uniform mixing.

10) Which cubelike graphs admit uniform mixing?

Among cubelike graphs with perfect state transfer at time τ, which admit uniform mixing at time $\tau/2$?

The d-cubes admit perfect state transfer at time $\pi/2$ and uniform mixing at time $\pi/4$.

Discrete walks

You can find more information about discrete walks in H. Zhan's Ph.D. thesis (link on webpage). She helped draft the next three questions.)

11) Find examples of perfect state transfer in two-reflection walks $U=(2P-I)(2Q-I)$ where the initial state does not lie in $\mathrm{col}(Q)$.

(12) Is there uniform mixing on the arc-reversal walk, starting on the state $D_t^T e_a e_a^T D_t$?

There is for K_n. No other examples are known.

13] In the shunt decomposition walk, is there an initial state that has uniform average mixing?

Quantum homomorphisms...

14) Find a quantum isomorphism between two Hadamard graphs.

15) Find a non-commuting quantum automorphism of a Hadamard graph.

16) Characterize the local switchings that are quantum automorphisms/isomorphisms.

This is likely not a fair question. But just having more examples would be interesting.

17) A quantum homomorphism $P: X \xrightarrow{q} Y$ is an **epimorphism** if, for any two quantum homomorphisms Q_1 & Q_2 from Y to Z, if $Q_1 P = Q_2 P$ then $Q_1 = Q_2$.

Characterize the quantum epimorphisms.

(For the classical case, epimorphism = surjection)