L2013

Cospectral graphs

Assume $\mathcal{P}$ is a quantum permutation of index d.

Then

$$\mathrm{Comm}(\mathcal{P}) = \{ M : M \otimes I_d \ \& \ \mathcal{P} \text{ commute} \}$$

is a coherent algebra. It contains J, in fact it contains the coherent algebra generated by A.

If we have a set of $n \times n$ quantum permutations, the commutant of the set is the intersection of the commutants of its elements.

Note that any coherent algebra of $n \times n$ matrices has dimension at most n^2. If $\mathcal{C}, \mathcal{D}_1, \mathcal{D}_2$ are coherent algebras and $\mathcal{C} = \mathcal{D}_1 \cap \mathcal{D}_2$ then

$\dim(\mathcal{C}) \leq \dim(\mathcal{D}_1), \dim(\mathcal{D}_2)$ (true for subspaces)

So if X is a graph and Q is its quantum automorphism group, the commutant of Q is a coherent algebra. This algebra is a subalgebra of the coherent algebra generated by $A(X)$. (Remark: the 'smallest' coherent algebra is $\mathrm{span}\{I, J\}$.)

If X & Y are quantum isomorphic via the quantum permutation $\mathcal{P}$ of index d, then

$$\mathcal{P}(A(X) \otimes I_d) = (A(Y) \otimes I_d)\mathcal{P}$$

Since $\mathcal{P}$ is unitary, this implies that the matrices $A(X) \otimes I_d$ & $A(Y) \otimes I_d$ are similar.

So X & Y are cospectral, but more is true.

For any two $n \times n$ matrices M & N,

$$MN \otimes I_d = (M \otimes I_d)(N \otimes I_d)$$

$$(M \circ N) \otimes I_d = (M \otimes I_d) \circ (N \otimes I_d)$$

Thus

$$\begin{aligned} P(A(x)^2 \otimes I_d) &= P(A(x) \otimes I_d)(A(x) \otimes I_d) \\ &= (A(y) \otimes I_d) P(A(x) \otimes I_d) \\ &\vdots \\ &= (A(y)^2 \otimes I_d) P \end{aligned}$$

It follows that, if $\mathcal{C}(X)$ denotes the coherent algebra generated by $A(X)$ and X & Y are quantum isomorphic via $\mathcal{P}$, then

$$\mathcal{P}(\mathcal{C}(X))\mathcal{P}^{-1} = \mathcal{C}(Y)$$

(this is not a type of isomorphism listed before!)

Hence $\mathcal{C}(X) \cong \mathcal{C}(Y)$. This implies X & Y are cospectral and $\bar{X}$ & $\bar{Y}$ are cospectral.

Let $\mathcal{C}_q(X)$ denote the commutant of the quantum automorphism group of X. This is a coherent algebra that contains $\mathcal{C}(X)$.

Group rings

Let G be a finite group. The group ring over the ring R may be defined as the set of formal sums

$$\sum_{g\in G} a_g\, g \qquad\qquad a_g \in R.$$

Only finitely many terms in this sum are not zero.

We have a product: $\qquad R^G$

$$\sum_{g\in G} a_g\, g \sum_{h\in G} b_h\, h = \sum_{c\in G} \Big(\sum_{gh=c} a_g b_h \Big) c \qquad \text{convolution}$$

To be rigourous, the elements of the group ring are functions from G to R with finite support. Addition and scalar multiplication are defined in the usual way. The product of functions f & g is denoted $f * g$. and

$$(f*g)c = \sum_{xy=c} f(x)\, g(y)$$

And now for what we'll actually do. Assume $|G| = n$. Then we can represent the elements of G by permutation matrices, where the map from G to $Mat_{n \times n}(\mathbb{R})$ is an isomorphism. Now the group ring is a subalgebra of $Mat_{n \times n}(\mathbb{R})$. This subalgebra is a coherent algebra with the permutation matrices as canonical basis.

The centre $Z(R)$ of the ring R is

$$\{x \in R : xy = yx \ \forall y \text{ in } R\}.$$

We are interested in the centre of the group ring.

Let us view subsets of G as sums of permutation matrices

Suppose $S \subseteq G$ & $x \in G$. If $xS = Sx$ and $a \in S$, there is $b \in S$ such that $xa = bx$, and so $b = xax^{-1}$

Thus if $a \in S$, then $xax^{-1} \in S$ & thus $xSx^{-1} = S$.

This implies that S is central if & only if it is a union of conjugacy classes, and hence the conjugacy classes form a basis for the centre of the group ring.

Therefore the centre of the group ring is a coherent algebra, and the conjugacy classes are the canonical basis. This coherent algebra is homogeneous & commutative — it is an association scheme.

Example Sym(4)

conj classes / eigenspaces	C_0 (1)	C_1 (12)(34)	C_2 (12)	C_3 (1234)	C_4 (123)
1	1	3	6	6	8
1	1	3	-6	-6	8
2	1	3	0	0	-4
3	1	-1	2	-2	0
3	1	-1	-2	2	0

form of the character table of Sym(4)

	involutions	derangements
	$C_1 + C_2$	$C_1 + C_3$
1	9	9
1	-3	-3
2	3	3
3	1	-3
3	-3	1

Some quantum isomorphic graphs

Examples

a) The two Cayley graphs on 24 vxs — part of a series

b) Graphs related to the E_8 root graph on 120 vertices.

c) Hadamard graphs

First we introduce the Hadamard graphs

A ==Hadamard graph== is an $n \times n$ ± 1-matrix H such that

$$HH^T = nI$$

e.g.

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}, \cdots$$

$1 \; 1 \; \cdots \; 1$

$1 \; {-1} \; 1 \; {-1} \; \cdots \; {-1}$ $\qquad$ n is even

==Lemma== If an $n \times n$ Hadamard matrix exists, either $n = 2$ or $4 \mid n$. $\square$

Converse? Who knows.

If H_1 & H_2 are Hadamard matrices, so is $H_1 \otimes H_2$.

Lemma Let H be an $n \times n$ Hadamard matrix. If

(H is regular) $H\underset{\sim}{1} = \alpha \underset{\sim}{1}$ (for some α) then $n = \alpha^2$.

Proof If $H\underset{\sim}{1} = \alpha \underset{\sim}{1}$, then $\underset{\sim}{1}^T H \underset{\sim}{1} = n\alpha$. But $H\underset{\sim}{1} = \alpha\underset{\sim}{1}$

then $\alpha H^T \underset{\sim}{1} = H^T H \underset{\sim}{1} = n\underset{\sim}{1}$ and $\alpha \underset{\sim}{1}^T H^T \underset{\sim}{1} = n^2$. So

$$\alpha^{-1} n^2 = \underset{\sim}{1}^T H^T \underset{\sim}{1} = \underset{\sim}{1}^T H \underset{\sim}{1} = n\alpha. \qquad \square$$

Hadamard graphs. Suppose H is an $n \times n$ Hadamard matrix. Then we can write $H = H(+) - H(-)$ where $H(+)$ & $H(-)$ are 01-matrices and $H(+) + H(-) = J$. Define

$$N := H(+) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + H(-) \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and set

$$A := \begin{pmatrix} 0 & N \\ N^T & 0 \end{pmatrix}$$

This is the adjacency matrix of a bipartite graph.

Set $L = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$. Then $L^T L = I_2$ and $L^T\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}L = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$,

$L^T\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}L = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ and consequently

$$(I_n \otimes L^T)\, N\, (I_n \otimes L) = H(+) \otimes \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + H(-)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

Therefore the matrices

$$N, \quad \begin{bmatrix}\overbrace{H(+)+H(-)}^{J} & 0\\ 0 & \underbrace{H(+)-H(-)}_{H}\end{bmatrix} = \begin{bmatrix}J & 0\\ 0 & H\end{bmatrix}$$

permutation $P^2 = I$, $P = P^T$,

are similar, in fact $N = (I_n \otimes L^T)\, P \begin{pmatrix}J & 0\\ 0 & H\end{pmatrix} P\, (I_n \otimes L)$

We have (finally)

$$A^2 = \begin{pmatrix} 0 & N \\ N^T & 0 \end{pmatrix}^2 = \begin{pmatrix} NN^T & 0 \\ 0 & NN^T \end{pmatrix}$$

$$N^T N = (I_n \otimes L^T) P \begin{pmatrix} \sigma^2 & 0 \\ 0 & H^T H \end{pmatrix} P (I \otimes L^T)$$

From this we conclude that the spectrum of A is

mult	1	n	$n-1$	n	1
θ	$-n$	$-\sqrt{n}$	0	$\sqrt{n}$	n

Theorem (Mančinska & Roberson) Two graphs X and Y are quantum isomorphic if & only if, for each planar graph Z, the number of homomorphism $Z \to X$ equals the number of homomorphism $Z \to Y$. $\square$

Theorem (Chan & Martin; Gromada)

Hadamard graphs of the same order are quantum isomorphic. $\square$

We have no actual example of a pair of Hadamard graphs and a quantum isomorphism between them.

Nor do we know the quantum automorphism group of any Hadamard graph.

Lemma If two trees are quantum isomorphic, they are isomorphic.

$P_{ik} P_{jk} = 0$

$P_{ik} P_{il} = 0$

want $P_{ik} P_{jl} = 0$ if $i \sim j$, $k \nsim l$

$P^T (A \otimes I)$

$$\sum_{r \sim j} P_{ri} = \sum_{s \sim i} P_{js}$$

$$\sum_r P_{ir} A_{rj} = \sum_s A_{is} P_{sj}$$

$$\sum_{r \sim j} P_{ir} = \sum_{s \sim i} P_{sj}$$

, P $r \sim j$ $\quad P_{ir} = \sum_{s \sim i} P_{sj} P_{ir}$

$P_{ri} \quad \sum_{s \sim i} P_{js} P_{ri}$

$i \; r \sim j \qquad i \sim s \; j$

	0	1	2	3	4
0	.	.	.	.	.
1	.				
2	.				
3	.				
4	.				

	0	2
1	.	.
2	.	.

0 1 2 3 4

0 1 0 0 1

1 0 1 0 0

0 1 0 1 0

0 0 1 0 1

1 0 0 1 0

$= P_{01} + P_{02}$

$P_{10} + P_{12}$