21/3/03

Gleason's theorem

We work with the Hilbert space $\mathbb{R}^3$. A state is given by a projection xx^* ($\|x\|=1$) and a measurement is given by the standard basis e_1, e_2, e_3. The outcome of a measurement is i (in $\{1,2,3\}$) with probability $\langle xx^*, e_ie_i^*\rangle$

So each state determines a function f on the unit sphere s.t. that:

(a) $f \geq 0$

(b) If x, y, z is an orthonormal basis, then

$f(x) + f(y) + f(z) = 1$.

Certainly the function $y \mapsto \langle xx^*, yy^* \rangle$ has these properties. Are there alternatives?

We call f as above a frame function. Do all frame functions arise from inner products?

Theorem. (With notation as above.) Yes. $\square$

Gleason's argument proceeds in two steps.

I: If $d \geq 3$ and frame function exists, it is continuous.

II: If $d \geq 3$ and f is a continuous frame function, there is a matrix $M \succeq 0$ such that $f(x) = x^* M x$ $\forall x$.

$$x^* M x = \langle xx^*, M \rangle$$

Only the first claim matter to us.

Theorem The chromatic number of $\Omega_{\mathbb{R}}(3)$ is greater than three.

Proof Assume we are given a 3-colouring of $\Omega_{\mathbb{R}}(3)$ and let S be one of the three colour classes.

If $\{e_1, e_2, e_3\}$ is an orthonormal basis then $|S \cap \{e_1, e_2, e_3\}| = 1$. The characteristic function of S is a frame function, but is not continuous. $\square$

A Kochen-Specker set is a union of orthonormal bases from $\Omega(d)$. If the subgraph of $\Omega(d)$ induced by a Kochen-Specker set is d-colourable, it contains a coclique that meets each basis in exactly one vertex. If X is such a subgraph and no such coclique exists, then

$$\xi(X) < \chi_q^{(1)}(X).$$

It follows from Gleason's theorem, by compactness, that there are finite Kochen-Specker graphs with no d-colouring.

In $\mathbb{R}^3$, this means we have graphs X with $\xi(X)=3$ and $\chi(X)\geq 4$. If $\chi_q^{(1)}(X)=3$, then $\chi(X)=3$ and hence $\xi(X)<\chi_q^{(1)}(X)$.

There is a ratio bound on $\xi(X)$ (following Elphick & Wocjan):

Theorem For any graph X, we have $\xi(X) \geq 1 - \frac{\theta_1}{\theta_{min}}$. $\square$

Vector colourings

Let $\Omega(d, \alpha)$ be the graph with unit vectors in $\mathbb{R}^d$ as its vertices, with vectors x & y adjacent if $\langle x, y\rangle \leq \alpha$.

(In practice, $-1 \leq \alpha \leq 0$). The vector chromatic number χ_v of X is

$$\inf\{1 - \tfrac{1}{\alpha} : X \to \Omega(d, \alpha),\ \alpha < 0\}.$$

If $\alpha = -1$, then X is bipartite.

$$(a+bi) \longrightarrow \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

There is a related parameter. Let $\Omega^{=}(d,\alpha)$ be the graph with unit vectors in $\mathbb{R}^d$ as vertices, vectors adjacent if $\langle x,y\rangle = \alpha$. The strict vector chromatic $\chi_{sv}(X)$ is $\inf\{1-\frac{1}{\alpha} : X \to \Omega^{=}(d,\alpha),\ \alpha<0\}$.

Clearly $\chi_{sv}(X) \geq \chi_v(X)$. We also note that $\chi_{sv}(X)$ is the Lovász ϑ-number of $\bar{X}$.

(We note that $\chi_{sv}(X)$ is Lovász-ϑ of the complement of X.)

Mančinska & Roberson also holds for χ_v

Theorem If $X \xrightarrow{q} Y$ then $\chi_{sv}(X) \leq \chi_{sv}(Y)$.

Proof Suppose $X \xrightarrow{q} Y$ and $Y \to \Omega^{=}(d,\alpha)$ for some d.

We want to show $X \to \Omega^{=}(e,\alpha)$ for some e.

Since $Y \to \Omega^{=}(d,\alpha)$, there exist unit vectors $\psi(y)$ for y in $V(Y)$ such that $\psi(y)^* \psi(z) = \alpha$ when $y \underset{Y}{\sim} z$. We claim that entries of the quantum homomorphism $X \xrightarrow{q} Y$ can be assumed to be real. Assume

$\mathcal{P} = (P_{x,y})$ $(x \in V(X), y \in V(Y))$.

If $x \in V(X)$, define vectors $\varphi(x)$ by

$$\varphi(x) = \frac{1}{\sqrt{d}} \sum_{y \in V(Y)} \psi(y) \otimes P_{x,y}$$

$\langle A \otimes B, C \otimes D \rangle$

$= \langle A, C \rangle \langle B, D \rangle$

Assume $\langle u \otimes P, v \otimes Q \rangle = u^T v \langle P, Q \rangle$. Then

$$\langle \varphi(w), \varphi(x) \rangle = \frac{1}{d} \left\langle \left(\sum_y \psi(y) \otimes P_{wy} \right), \left(\sum \psi(z) \otimes P_{x,z} \right) \right\rangle$$

$$= \frac{1}{d} \sum_{y,z \in V(Y)} \langle \psi(y), \psi(z) \rangle \langle P_{wy}, P_{z,x} \rangle$$

Since $\langle P_{wy}, P_{wz}\rangle = 0$ when $y \neq z$, we get

$$\langle \varphi(w), \varphi(w)\rangle = \frac{1}{d} \sum_{y,z \in V(Y)} \langle \psi(y), \psi(y)\rangle \langle P_{wy}, P_{wy}\rangle$$

$$= \frac{1}{d} \operatorname{tr}\left(\sum_y P_{wy}\right)$$

$$= \frac{1}{d} \operatorname{tr}(I_d)$$

$$= 1$$

Hence $\|\varphi(w)\| = 1$ and we need only check the value of $\varphi(w)^T \varphi(z)$

$w \underset{X}{\sim} x$.

Now $\langle P_{wy}, P_{xz} \rangle = 0$ when $y \neq z$ and so ??

$$\varphi(w)^T \varphi(x) = \frac{1}{d} \sum_{y \sim z} \langle \psi(y), \psi(z) \rangle \langle P_{wy}, P_{xz} \rangle$$

$$= \frac{\alpha}{d} \sum_{y \sim z} \langle P_{wy}, P_{xz} \rangle$$

$$= \frac{\alpha}{d} \left\langle \sum_{y \in V(Y)} P_{wy}, \sum_{z \in V(Y)} P_{xz} \right\rangle$$

$$= \alpha . \qquad \square$$

We point out that we can compute $\chi_{sv}(X)$ (semidefinite programming), and so we can use this theorem to prove that there is no quantum homomorphism from X to Y.

The **quantum clique number** of a graph X is $\max\{n : K_n \xrightarrow{q} X\}$.
It is denoted by $\omega_q(X)$. The **quantum independence/coclique number** is $\omega_q(\bar{X})$, and is denoted by $\alpha_q(X)$. (We will offer another definition later.)

For practice, we start with a simple result.

Lemma If X is vertex transitive, $\alpha(X)\omega_q(X) \leq |V(X)|$.

clique-coclique bounds

Proof Assume $m = \omega_q(X)$. Then $K_m \xrightarrow{q} X$; assume this is given by an $m \times |V(X)|$ matrix $\mathcal{P}$ of index d, with all projections of rank r. (So $|V(X)|r = d$.).

Let S be a coclique in X and consider the submatrix $\mathcal{P}(S)$ of $\mathcal{P}$ formed from the columns indexed by vertices in S. Then the entries in each column of $\mathcal{P}(S)$ are pairwise orthogonal. Further since K_m is complete and S is a coclique, projections in different columns of $\mathcal{P}(S)$ are orthogonal. Summing up, any two distinct projections in $\mathcal{P}(S)$ are orthogonal. Hence their sum is a projection of rank $m\alpha(X)r$, and thus

$$m\alpha(X)r \leq d = r\,V(X).$$

So $w_q(X)\alpha(X) \leq |V(X)|$. $\square$

If $K_n \to X$, then $K_n \xrightarrow{q} X$ and so $w_q(X) \geq w(X)$. It follows that $\alpha_q(X) \geq \alpha(X)$.

We offer a second definition of a quantum coclique.

We say a quantum coclique of size s in X is given by a $|V(X)| \times s$ matrix $\mathcal{P} = (P_{ij})$ such that:

(a) for each vertex u in X, $\sum_{u \in V(X)} P_{u,i} = I_d$.

(b) $P_{ui} P_{vj} = 0$ if u & v are equal or adjacent.

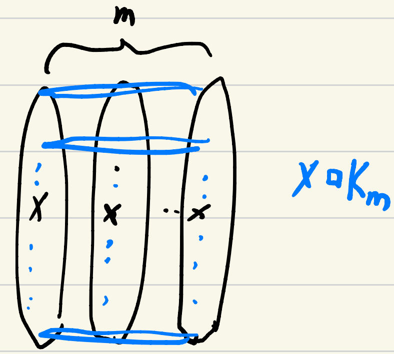
The homomorphic product

We are going to present a construction due to Nešetřil & Hell.

First we present a special case.

Lemma $\alpha(X \square K_m)$ is the size of the largest subgraph of X that is the union of m cocliques.

Proof (by picture).



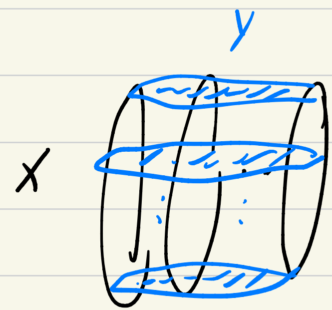
Corollary $\chi(X) = m \Leftrightarrow \alpha(X \square K_m) = |V(X)|$. □

The homomorphic product $X \ltimes Y$ is the graph with vertex set $V(X) \times V(Y)$, with distinct vertices (u,v) & (x,y) adjacent if either $u=x$, or $u \sim x$ and $v \nsim y$.

We have

$$A(X \ltimes Y) = A(X \,\square\, \underbrace{K_Y}_{K_{|V(Y)|}}) + A(X \times \bar{Y})$$

$$\left(= A(X) \otimes I + I \otimes (J_Y - I) + A(X) \otimes A(\bar{Y}) \right)$$



The graph of a function $f: V(X) \to V(Y)$ is $\{(x, f(x)) : x \in V(X)\}$.